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Some Applications of Group Actions in Complex Geometry

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Abstract: In this article, we give a further survey of some progress of the applications of group actions in the complex geometry after my earlier survey around 2020, mostly related to my own interests.

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§1. Introduction

Let $M = G/H$ be a complex manifold, with G a real finite dimensional Lie group, H a closed Lie subgroup such that the complex structure on M is invariant under the action of G. We call M a complex homogeneous space.

We have discussed this kind of spaces and their related topics extensively in our earlier survey article [52]. It is clear that the Lie group actions play a critical role around this area. Also, we see that the co-homogeneity one actions play a further interesting role in related analysis and geometry. Therefore, we shall give a further survey related to applications of group actions in complex geometry.

This is an extension of my talk on April 7, 2023 in the conference of geometry and physics in Henan University.

In the Spring 1992, we proved in [33, 60] that on completions of certain $\mathbf{C}^*$ bundle over some compact Kähler manifolds, there exist Calabi extremal metrics. Later on, on the way of searching for Hermitian-Einstein metrics, we found another kind of standard metrics on these manifolds. Around the beginning of this century, LeBrun, in his study of Einstein-Maxwell

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equations on compact Riemannian four manifolds, introduced a concept of Einstein-Maxwell metrics. It turns out that under the condition being strongly Hermitian, his definition and its generalization to the higher dimension is equivalent to ours. This concept has been generalized considerably in the mathematical society.

In this article, we first give some survey of the existence for the k generalized Maxwell-Einstein metrics defined by Futaki and Ono conformally related to a metric in a given Kähler class for any $k \geq 2$ and some related problems in Hermitian-Einstein metrics, including Kähler-Einstein metrics.

In the same time we deal with the co-homogeneity one compact Kähler manifolds, which is obviously related to the actions of Lie groups.

We then spent some pages on hyperkähler manifolds, which have caught great attention recently. Then we deal with the existing problem of the complex structure on S^6, related to a $SO(6)$ gauge. We shall touch the co-homogeneity one version of the Yau's conjecture on complete Kähler metrics with positive bisectional curvatures and the CP conjecture for the projective manifolds with nef tangent bundles. This is an ongoing project I am working with some students and faculties in Henan university. Here, we take this chance to thank Professors Feng and Han for their supports, and Professor Z. J. Liu for the invitation to the conference of geometry and physics in Henan University. We also thank Ms. J. Chen, X. Duan, S. Jing, N. Li, Mr. M. Liang, and Ms. Tang as well as Professor Z. Wang for their cooperations.

Somehow, we leave out the co-homology group of the compact solvmanifolds that we dealt with in [44] and the k-Calabi metrics, which is related to the convergence of a metrics flow similar to the Calabi flow, for examples, in [45].

§2. Generalized Maxwell-Einstein metrices

A. Futaki and H. Ono discussed the existence of the Einstein-Maxwell metrics in [27–29], they also gave a definition of the generalized Einstein-Maxwell metrics in [29].

In [54], we proved the existence of the Maxwell-Einstein metrics in every Kähler class on certain completions of some $\mathbf{C}^*$ Bundles. We also, earlier in a series of papers, solved the Kähler-Einstein problem on compact co-homogeneity one manifolds. See [33, 34, 43, 46–49, 51, 55, 58, 61], for examples.

In last decade, C. LeBrun gave some non-Einstein Einstein-Maxwell metrics in [69–71] on compact real four dimensional Riemannian manifolds.

In fact, LeBrun found many Maxwell-Einstein metrics on Hirzebruch surfaces in [71]. This is a special case from [54] in which the base manifold is CP^1.

We notice that Futaki and LeBrun used Einstein-Maxwell while we use Maxwell-Einstein in our papers. That is the difference from the Physics and the Mathematics. From [69] we see that the original Einstein-Maxwell equations were quite different from ours. They were the equation for the Faraday differential 2-form F coupling with the Einstein field equation on the real four dimensional Lorentz manifolds. The Faraday differential 2-form satisfies a couple equations $dF=0$ and $d{*}F=0$. There was no necessary even a complex structure on it. Therefore, our simple concept of Maxwell-Einstein metrics was only a very special case of their physic concept of Einstein-Maxwell metrics. That was one of the reasons that I did not pay much attention to LeBrun's work earlier.

Actually, LeBrun's definition [70,71] for a Einstein-Maxwell metric is following: Let (M,h) be a connected, oriented Riemannian 4-manifold. We will say that h is an Einstein-Maxwell metric if there is a 2-form F on M such that the pair (h,F) satisfies the Einstein-Maxwell equations $dF=d*F=0$ and $[r+F\circ F]_0=0$. Here r is the Ricci curvature of h and the subscript $[\ ]_0$ indicates the trace-free part with respect to h. In [70,71], LeBrun proved that: Einstein-Maxwell + strongly Hermitian = Maxwell-Einstein. For example, see [71] Proposition 5.

Question 1. *Are there any Einstein-Maxwell metrics other than our Maxwell-Einstein metrics?*

Since the classification of compact complex homogeneous spaces are almost fully understood and they did not provide enough information for geometrical analysis on compact complex manifolds, we study a bigger class of compact complex manifold called compact almost homogeneous complex manifolds.

A compact complex manifold M is called an **almost homogeneous manifold** if the holomorphic automorphism group has an open orbit on M. First we consider a general classification of compact almost homogeneous manifolds. Let M be a compact complex almost homogeneous manifold and O be an open orbit, then $M-O$ is a lower dimensional subvariety. The following result was well-known, see, e. g., [1,65].

Theorem 2.1. *Let M be a projective almost homogeneous manifold under a complex Lie group G and G has an open orbit, then $M-O$ has at most two connected components. We call the manifolds compact complex almost homogeneous manifold of one or two ends, according to the number of the components of $M-O$ being one or two. In particular, if M has two ends, then M has three G orbits O, E_0 and E_∞, O is a $\mathbf{C}^*$ bundle over a projective homogeneous space Q.*

For example, if $Q=\mathbf{C}P^1$ and M is a completion of a complex line bundle by adding the infinity section. In this case, we call M a Hirzebruch surface.

Theorem 2.2. *([54]) For any Kähler class on a compact almost homogeneous manifold with two ends, there is at least one Maxwell-Einstein metrics in the given Kähler class.*

Now, we give our definition of a Maxwell-Einstein metric:

Definition 2.1. *For a given Kähler class, if there is an Hermitian h metric of a constant scalar curvature conformally related to the Kähler metric g in this class such that $h=u^{-2}g$ with a function u such that its gradient is a holomorphic vector field, then we say that h is a Maxwell-Einstein metric related to this Kähler class.*

Now, we give a simple definition of a special class of Futaki-Ono's k-generalized Maxwell-Einstein metrics. They are more geometric to us. And therefore, it is nature to ask the existence of this kind of metrics in this class.

Definition 2.2. *For a given Kähler class, if there is an Hermitian metric h of a constant scalar curvature conformally related to the Kähler metric g in this class such that $h = u^{-k}g$ with a function u of which its gradient is a holomorphic vector field, then we say that h is a Futaki-Ono k generalized Maxwell-Einstein metric related to this Kähler class.*

Theorem 2.3. *For any Kähler class on a compact almost homogeneous manifold with two ends, there is at least one Futaki-Ono k generalized Maxwell-Einstein metrics in the given Kähler class for any $k \geq 2$ being an integer.*

The proof is much harder. Therefore, after the first effort we strategically withdrew from the general case and finished the special cases in which the two ends are complex hyper-surfaces in [13].

The difficulty for this first paper was the proving of the positivity of the solution, which we called the positive Lemma in [54] and also earlier, e. g., in [62]. The original proof did not work at all and we applied a complete new method. [13] was submitted.

When one of the ends is not a hyper-surface, the problem reduced to an inequality for any complex dimension n of the base and k, as well as the the co-dimensions of the ends. We eventually reduced the inequality to an inequality of polynomials. Then with a help of computer, we are able to prove the existence for each positive integer $k > 2$ in [11] for $k \leq 11$ one by one.

Then we were eventually able to solve the general cases first for $k > 11$ and $n > 13$ for k being an even integer with some analysis and computer help, then for the cases in which $n \leq 13$ in [12]. The same worked for k being odd.

§3. Co-homogeneity one Kähler-Einstein metrics and equivariant Mabuchi moduli spaces

It is very natural to classify almost homogeneous Fano manifold of complex dimension three with a reductive Lie group to find possible Kähler-Einstein metrics as we explained in [53] section 4. This was done in [37, 39, 40].

Even so, it is still quite difficult to find Kähler-Einstein metric for the almost homogeneous manifolds. We call the projective almost homogeneous manifolds with two ends the type III co-homogeneity one Kähler manifolds.

Recall that a Riemannian manifold is **co-homogeneity one** if its isometry group has a real hyper-surface orbit.

Proposition 3.1. *Any compact co-homogeneity one Kähler manifold is almost homogeneous.*

Obviously, there are many non-type III compact co-homogeneity one Kähler manifolds, see [1,65] for example. One could also consult with [58]. As professor Huckleberry once mentioned later on, some people wanted to do it, but none knew how to do it.

Eventually, we found some simple series of examples N_n and M_n in [55]. They are compact co-homogeneity one Kähler manifolds of type II but N_n behave as type I manifolds.

In the Kähler geometry, a Kähler metric gives a cohomology class. Given a Kähler metric

ω_0, any Kähler metric in the same cohomology class can be written as $\omega=\omega_0+\partial\bar{\partial}F$. That is, the Kähler metrics in the same cohomology class consist of an infinite dimension vector space, with the smooth functions as the tangent vectors. Mabuchi gave a Riemannian metric

$$g(f,h)=\int_M fh\omega^n.$$

An interesting question is: Are there smooth geodesics as in the finite dimensional case?

In 1987, Mabuchi [75] calculated the curvature for the moduli space of the Kähler metrics:

$$R(f,h,k,l)=-\frac{1}{4}g(f,\{h,\{k,l\}\}),$$

where $\{f,h\}=\omega^*(df,dh)$.

In 1991, Semmes [83, 84] rediscovered this and noticed that the geodesics satisfy a homogeneous complex Monge-Ampére equation $\det\partial\bar{\partial}F(t)=0$, where we regard t as the real part of a complex coordinate z_0.

In 1997, Donaldson [20] rediscovered this again and conjectured that it is a complete symmetric space under certain group.

In [50] we proved that:

Theorem 3.1. *Along any curve, the parallel transformation of a given vector always exists.*

Theorem 3.2. *We always have* $\nabla R=0$.

In the case in which the metric is co-homogeneity one, we have that $\{f,h\}=0$ for any two functions which are invariant under the group action. Therefore,

Theorem 3.3. *For a cohomgeneity one manifold, the equivariant Mabuchi moduli space is flat.*

Theorem 3.4. *For a cohomgeneity one manifold, any vector at any point of the equivariant Mabuchi moduli space can be extended to a global parallel vector.*

In 1987, Mabuchi also gave the Mabuchi functional

$$M(\omega_1,\omega_2)=\int_0^1\int_M \partial_t F(t)(R-HR)\omega_t^n.$$

And proved that it is independent of the choice of the path. It is convex up along the geodesic. That is, the second derivative is positive. Given a smooth geodesic $L:t\in[0,T)\to L(t)$, one can define the generalized Futaki invariant

$$F_L=\lim_{t\to T}\partial_t M.$$

Theorem 3.5. *On a cohomgeneity one manifold, for any two points in the equi-variant Mabuchi moduli space there is a smooth geodesic connecting them. And for any vector at any point there is a smooth geodesic starting at this point with this vector as the initial tangent vector. The maximal geodesic ray is infinite if and only if this vector is geodesic convex.*

This result gave a proof of the **uniqueness** for the Kähler-Einstein metrics and metrics with constant scalar curvatures. This fact was pointed out by Donaldson. It also gave a negative answer to the completeness and the symmetric-ness conjectures from Donaldson.

Theorem 3.6. *On a cohomgeneity one manifold, the generalized Futaki invariant is independent of the maximal geodesic but only depend on the global parallel vector field. And it is infinite if the maximal geodesic ray is finite. There is a metric of constant scalar curvature if and only if the Kähler class is uniformly stable.*

A Kähler class is stable if $F_f > 0$ unless f is a generating function of a holomorphic vector field.

It is uniformly stable if $F_f > C\|f\|_s$. Here $\|f\|_s$ is a given semi-norm with C a constant. Uniformly stable is equivalent to the negativity of a topological integral.

Kähler-Einstein case is the same to say that the Kähler class is a multiple of the Ricci class. In the Calabi extremal metric case, we just replace the Mabuchi functional by the modified Mabuchi functional.

Unfortunately, in general, it is not as fortunate as in [101] for the Kähler-Einstein metrics with the first Chern class being zero or negative, the geodesic connecting two points could only be $C^{1,1}$ in the real sense [17]. See [18, 19, 79]. These made our results in [38, 49, 51, 61] and the co-homogeneity one case special. There was also some basically failed efforts for proving the uniqueness, for example, in [16] according to [79].

Fortunately, the uniqueness was eventually solved in [5].

For the existence, Yau's originally and Donaldson's modified conjectures seemly are both failed. Only Tian's pseudo-original conjecture seems ok so far. It tends very technical. I mean, with both the arguments and the statements.

§4. Compact complex cohomogeneity one manifolds

The type III case with the Einstein problem was done by Sakane and Koiso [68]. They gave the first nonhomogeneous examples of compact Fano manifolds with Kähler-Einstein metrics. Their results are seemly even earlier than the examples given by Siu, Tian-Yau on the Fano surfaces. They used Kobayashi's construction of the Riemannian metrics on the S^1 bundles.

Theorem 4.1. *A Fano almost homogeneous manifold with two ends admits a Kähler-Einstein metric if and only if the Futaki invariant corresponding to the S^1 action is zero.*

Most works that took care of the compact almost homogeneous manifolds with two ends were done on related metrics when I was a Ph. D. graduate student around 1992. See [52] for a detail survey.

Recently, we apply the type III methods in [54] to the Maxwell-Einstein metrics; in [22, 56, 57] to the calculation of the holomorphic bisectional curvatures; and in [14, 15] finally connecting to the co-homogeneity one domains in complex Euclidean spaces.

Somehow, we did not get any new Kähler-Einstein metric at that time. Obviously, there are many non-type III compact co-homogeneity one Kähler manifolds, see [1, 65] for example.

Let H be a real hypersurface orbit, then $H = K/L$, and assuming that $A = \{k \in K|_{kh=hk \text{ for any } h\in L}\}$ be the centralizer of L in K, A is of real dimension either one or three. If

A is one dimensional, we say that the compact co-homogeneity one Kähler manifold M is **of type I** if it is not a type III manifold. If M is not of type I or type III, we say that M is **of type II**. See [61].

Basically, we proved the following:

Theorem 4.2. *For any Kähler class on a compact cohomogeneity one manifold with zero first Betti number which is not of type III, there is a Kähler metric in the given Kähler class with a constant scalar curvature if and only if certain topological integral is smaller than zero.*

This integral is the negative of the generalized Futaki invariant related to the exceptional closed orbit. We notice that since it is not of type III, the exceptional orbit is connected and homogeneous.

In our case, the necessary condition here is stronger than the semi-stable property of the closed orbit in the work of Professors Ross and Thomas [80].

The proof of Theorem 4.2 for the type I manifolds was published in [49]. The sufficient part for the type II manifolds was published in [43,46,47]. However, the necessary part for the type II manifolds is more complicated and is to be published later on. See [51], for example.

Therefore, we obtained infinite many Kähler-Einstein manifolds as well as infinite many Fano manifolds without any Kähler-Einstein metric.

Let us go into the details of these two orbits manifolds. First, consider the complexification of the co-homogeneity one Lie group action K. That is, the Lie algebra of K consist of holomorphic vector fields. We denote it by a real subspace $\mathcal{K}$. Let $V=\mathbf{C}\mathcal{K}$. By a Theorem of Mongomery, V is the Lie algebra of a complex Lie group G. And G is a subgroup of the automorphism group.

Theorem 4.3. *If G is not semi-simple, then M is a completion of a $\mathbf{C}^*$-bundle over a projective rational homogeneous space.*

That is, if G is not semi-simple, then M is of type III.

Therefore, we only need to take care of the case in which G is semi-simple.

There is a special case of the type II manifolds. If the open orbit is a $\mathbf{C}^k$-bundle (might not be a vector bundle) over a projective rational homogeneous manifold, we call M an affine type manifold (not to be confused with the closed complex submanifolds of $\mathbf{C}^m$).

Theorem 4.4. *Let M be a compact complex almost-homogeneous manifold with one hypersurface end and a complex semisimple Lie group G action, then M is a fiber bundle over a rational projective homogeneous space Q such that the fiber is an S standard fiber. Here $Q=G/P$ with P a parabolic subgroup. S is the semi-simple part of P.*

A list of the **Standard fibers** can be found in [1] page 67 and 68.

The list in page 67 are the standard fibers such that the open orbits are affine in the classical algebraic geometry, which is the same that the isotropic subgroup is reductive. While the list in page 68 are the standard fibers which are affine manifolds we defined above. All of these standard fibers are homogeneous except the last one in the list in page 68.

In page 67, we have 1. $O=A_n/(\mathbf{C}^*\times A_{n-1})$, $F=\mathbf{C}P^n\times(\mathbf{C}P^n)^*$;

2. $O=SO(n,\mathbf{C})/SO(n-1,\mathbf{C})$, $F=\mathbf{Q}^{n-1}$;
3. $O=SO(n,\mathbf{C})/S(O(1,\mathbf{C})\times O(n-1,\mathbf{C}))$, $F=\mathbf{C}P^{n-1}$;
4. $O=C_n/(C_1\times C_{n-1})$, $F=Gr(2n,2)$; 5. $O=F_4/B_4$, $F=E_6/P(\omega_{\alpha_1})$; 6. $O=G_2/A_2$, $F=\mathbf{Q}^6$;
7. $O=G_2/Norm(A_2)$, $F=\mathbf{C}P^6$; 8. $O=B_3/G_2$, $F=\mathbf{Q}^7$; 9. $O=SO(7,\mathbf{C})/G_2$, $F=\mathbf{C}P^7$.

We see that all these night standard manifolds are actually homogeneous themselves.

While in page 68, we have 1. $O=B_n/U$, $U=P_{D_n}(\omega_{\alpha_n})$; 2. $O=C_n/U$, $U=B_{A_1}\times C_{n-1}$; 3. $O=G_2/U$, U contains the Cartan sublgebra and the root vectors $\pm\alpha_2$, $2\alpha_1+\alpha_2$, $3\alpha_1+\alpha_2$, $3\alpha_1+2\alpha_2$.

Somehow, the description in page 69 of the item 1. in page 68 is a little bit misleading. We denote it by $F(B_n)$. Then $F(B_n)$ is not a fiber bundle over $\mathbf{C}P^{2n}$ but over $\mathbf{Q}^{2n}$. The reason is that the set of the isotropic n dimensional subspaces in $\mathbf{C}^{2n}$ with the standard rank $2n$ quadric on $\mathbf{C}^{2n}$ has two connected components according to the last chapter of [31].

We also denote the second standard manifold in this list $F(C_n)$ and the third $M(G_2)$. We notice that $F(B_n)$ and $F(C_n)$ are homogeneous and $M(G_2)$ is not.

Theorem 4.5. *Let M be a compact complex almost-homogeneous manifold with one hypersurface end and a complex semisimple Lie group G action, if G is strictly larger than S, then the identity compnent of the automorphism group is G and M is not homogeneous. Consequently, all these complex manifolds are bi-holomorphically different from each other.*

§5. Compact complex cohomogeneity one manifolds and Fano manifolds with nef holomorphic tangent bundles

After solving the co-homogeneous version of the existence of the Kähler-Einstein metrics, it is very natural to apply these kind of manifolds to other hard elliptic problems in complex geometry. There is a famous question called CP conjecture [9, 66]. It conjectured that every Fano manifold with a nef tangent bundle is homogeneous.

Question 2. *If a compact almost homogeneous manifold has a nef tangent bundle, is it homogeneous?*

To make the things simpler, we call a Fano manifold a CP manifold if it has a nef tangent bundle.

First, we know that Fano manifold has a lower positive bound for the Ricci curvature. This implies that the manifold is simply connected. As in the Kähler-Einstein case, we start with the type III cases. Even so, one can not solve this immediately. Therefore, we pay attention to the cases wth hyper-surface ends first. One can easily get

Theorem 5.1. *Let M be a Fano co-homogeneity one CP manifold with two hyper-surface ends, then it is a flat $\mathbf{C}P^1$ bundle over a projective rational homogeneous space.*

The proof is the following: Consider a rational curve on the base manifold Q generated by an action of an $sl(2,\mathbf{C})$, say C. First we claim that the fiber bundle over C is a product. A vector bundle over a manifold is nef, then it is nef over any smooth curve on it. Therefore, the tangent bundle of M is nef over C on both the zero and the infinite sections. We denote these two curves by C_s and C_n. But, also any quotient of a nef bundle is again nef. This implies that the restriction of the normal bundle of these two sections on the curves are nef. We notice that one is $O(k)$ and the other is $O(-k)$. Therefore, $k=0$. Let p be a point over C, then $Q=G/P$ with P a parabolic subgroup of G, and $p_s=\pi p$ be the corresponding point in the zero section. That is, P be the isotropic subgroup at p_s. Let H be the isotropic subgroup at p. P/H acts on $\pi^{-1}p_s$ holomorphically, and is a subgroup of the automorphism group of $\pi^{-1}C$. But, $\pi^{-1}C$ is a product. This implies that $H=P$. Therefore, M is also a product.

Now, let M be a compact almost homogeneous manifold with an open orbit G/H with G a complex Lie group and H a subgroup containing a maximal unipotent subgroup of G, then we call M a **horo-spherical manifold**.

One has [72]:

Theorem 5.2. *Any compact horo-spherical manifold CP manifold is homogeneous.*

Actually, he proved that

Theorem 5.3. *Any CP G-horo-spherical manifold is a product of a G-projective rational homogeneous space with a product of a series of homogeneous G-type III manifolds of second Betti number* 1.

If M is of type III, then O is a $\mathbf{C}^*$ bundle over a projective rational homogeneous space $Q=G/P$. Apply P action through H, we get $P/H=\mathbf{C}^*$. In particular, H contains a maximal unipotent subgroup of P and therefore, contains a maximal unipotent subgroup of G. Therefore,

Theorem 5.4. *Any type III co-homogeneity one CP manifold is homogeneous.*

More precisely, by the co-homogeneity property, one has

Theorem 5.5. *Any G-type III co-homogeneity one CP manifold is a product of a G-projective rational homogeneous space with a homogeneous G-type III manifold of second Betti number* 1.

In particular,

Theorem 5.6. *In the type III case, if a CP manifold is a G-equi-variant fibration, then it is a product.*

A similar result for type II manifolds is not true. One has

Theorem 5.7. *In the type I and the type II cases with a hyper-surface end, if a CP manifold is a G-equi-variant fibration which is not a product and the second Betti number is* 2*, then it is either $F(B_k)$ or $F(C_k)$.*

§6. Compact complex manifolds with holomorphic symplectic or hyper-kähler structures

It is very important to understand the classification of compact complex surface, that is, compact complex manifold of complex dimension two. And it is also very important to understand compact Riemannian manifold with a special holonomy group. In general, a real n dimensional compact orientable Riemannian manifold might have a holonomy group $SO(n)$.

When it is a complex m dimensional compact Kähler manifold, the holonomy group reduced to $U(m)$. When the holonomy group is a subgroup of $SU(m)$, the Kähler metric has a zero Ricci curvature. In the later case, as we mentioned earlier that the interesting cases are the Calabi-Yau manifolds and the hyperkähler manifolds. Hyperkähler manifolds have even complex dimension since a nondegenerate 2-form can only have an even rank, and the manifolds have holonomy group $Sp(l)$.

One of the Kodaira's conjectures was that (proven in [86])

Theorem 6.1. *A compact complex surface S is Kähler if and only if the first Betti number of S is even.*

The proof finally reduced to the K3 surfaces, which are simply connected compact complex surfaces with non-degenerate holomorphic volumes (or holomorphic symplectic forms). So, the question is: Are all K3 surfaces Kähler surfaces? Professor Todorov published a paper in Inventiones [91] and claimed the proof. But Professor Siu pointed out some defect in the Mathematical Review and published another paper in Inventiones [86] proving the conjecture.

The question left: Are all simply connected compact complex manifolds with holomorphic symplectic structures Kähler manifolds?

Professor Todorov wrote another long paper and claimed that all of them were also Kähler.

After a long time, none could say that he was correct or not. But most people believed that this was true. It was similar to the situation for the Bogomolov-Todorov-Tian unobstructed Theorem for the Calabi-Yau manifolds, for which Professor Tian eventually gave a clean solution in [90].

When I was a Ph. D. student in University of California at Berkeley, Professor Todorov gave a talk. Then I started producing compact complex homogeneous spaces with holomorphic symplectic structures in [32]. That is, $M=G/H$. We constructed more examples of solvmanifolds even without requiring that G to be complex. See [32]. Then we came to the simple examples of Kodaira-Thurston surfaces. A Kodaira-Thurston surface is a compact complex nonkähler surface with a holomorphic symplectic structure. By the solution of the Kodaira conjecture that it has an odd first Betti number. And it is a quotient of a real nilponent Lie group. Actually, its universal covering as a real Lie group has a right invariant complex structure on G.

Theorem 6.2. *Let S be a compact complex surface with a holomorphic symplectic structure, consider $S^{(k)}$ being the symmetric product of S. It consists of k points in S without order. Then there is a smooth complex manifold $S^{[k]}$ over $S^{(k)}$ with induced holomorphic symplectic structure.*

Here we say that a complex manifold M is **over** N if there is a holomorphic map $\pi: M \to N$ such that $\pi^{-1}(n)$ is a point for a dense open set of N. We call $S^{[k]}$ the k-th Hilbert scheme of S.

Basically, there are only three kind of such S: a complex two dimensional torus, a K3 surface, a Kodaira-Thurston surface.

$S^{[k]}$ are Kähler if S is either a complex torus or a K3 surface.

$S^{[k]}$ is not simply connected when S is a Kodaira-Thurston surface and is not Kähler.

A Kodaira-Thurston surface is a Lagrange torus fibration over a torus. Moreover, the center of G acts on the fiber as a holomorphic complex torus action C. The C action lifted to an action on $S^{[k]}$. Let $a\colon S\to T$ be the fibering map. Then there is a map $a\colon \prod_i S_i\to T$, with $S_i=S$ such that $a(s_1,\cdots,s_k)=a(s_1)+\cdots+a(s_k)$. This induces a map $a\colon S^{(k)}\to T$. Locally, we can regard a as a holomorphic function on $S^{[k]}$. Then da is a holomorphic 1-form. Using the holomorphic symplectic structure ω we lift da to a holomorphic vector field $X_a=\omega^*(\ ,da)$. Then X_a corresponds to the diagonal C action on $S^{[k]}$. One naturally have a holomorphic symplectic reduction $R_k=a^{-1}(0)//C$. However, R_k usually is only an orbifold but not a manifold. To get a manifold, a natural way is to get a covering $\pi:\tilde{R}_k\to R_k$ such that $\tilde{R}_k$ is a smooth manifold. This kind of covering is called a **good covering** and was studied by Thurston for real three dimensional manifold. To find this kind of good coverings we have to choose the right Kodaira-Thurston surface by adding a topological condition on the torsion part of the fundamental group of S.

By calculating a topological bilinear form on the second cohomology of $\tilde{R}_k$, we were able to prove in [35] that

Theorem 6.3. *For $k>2$, $\tilde{R}_k$ is a simply connected compact holomorphic symplectic manifold which, as a topological differentiable manifold, does not admit any Kähler structure.*

We actually proved in [35] that the Lefschets property for the Kähler manifold does not hold for $\tilde{R}_k$.

Theorem 6.4. *For $k>2$, the Lefschetz Theorem does not hold for $\tilde{R}_k$ with any closed 2 form.*

After we submitted the paper to Inventiones as Professor Todorov and Siu did earlier, Professor Gromov introduced me to Professor Bogomolov. Professor Bogomolov realized that one can just use a sub-manifold S in $\tilde{R}_k$ to prove the nonkählerness of our manifold. Since if the ambient manifold is Kähler, then the sub-manifold is also Kähler. But S is nonkähler, as Professor McDuff did in her construction [77] of a simply connected compact nonkähler real dimension ten real symplectic manifold.

Professor Bogomolov obtained

Proposition 6.1. *For $k>2$, $\tilde{R}_k$, as a complex manifold, is not Kähler.*

Therefore, he gave an alternative negative answer to Todorov's question with a simpler argument, to the nonkähler property.

After that, motivated by our construction, Professor Fernandez et al published a paper in Annals [25] constructing a simply connected compact real eight dimensional real manifold with a real symplectic structure which is not formal. Formality is a condition with which a compact Kähler manifold should satisfy.

Theorem 6.5. *There is a simply connected non-formal real eight dimensional manifold with a real symplectic structure.*

I was told about this result by Professor Fernandez and we proved that:

Theorem 6.6. *$\tilde{R}_3$ is not formal.*

The real dimension of $\tilde{R}_k$ is $4k-4$. For $k>3$ we have:

Conjecture 1: *$\tilde{R}_k$ is not formal for $k>2$.*

Further more, in a third paper for the examples of the nonkählerian holomorphic symplectic structure, we proved in [36] that

Theorem 6.7. *For our manifolds $M=\tilde{R}_k$, $H^2(M,\mathbf{C})=H^{0,2}(M)+H^{1,1}(M)+H^{2,0}(M)$. Moreover, there is a bilinear form on $H^2(M,\mathbf{R})$ with one dimensional kernel.*

Theorem 6.8. *Let M be a simply connected compact holomorphic symplectic manifold, if $H^2(M,\mathbf{C})=H^{2,0}(M)+H^{1,1}(M)+H^{0,2}(M)$, then M is unobstructed.*

We applied a symplectic geometry proof other than the earlier ones. Moreover, we proved:

Theorem 6.9. *For all the known simply connected compact holomorphic symplectic manifolds, it is deformation equivalent to a holomorphic symplectic manifold, which is a Lagrange complex torus fibration over a complex projective space.*

Therefore, one have:

Conjecture 2. *Any simply connected holomorphic symplectic manifold is deformation equivalent to a holomorphic symplectic manifold which is a holomorphic Lagrange torus fibration over a complex projective space.*

This was later on called as the SYZ conjecture for the holomorphic symplectic manifolds, especially by Professor Todorov.

When we talked about [35] with Professor Gromov in University of Maryland in 1994, he introduced me to Professors Bogomolov and Donaldson. He also showed me Salamon's paper [81] and asked me: Could you get bounds on the Betti numbers for compact hyperkähler manifolds?

The classification of compact hyperkähler manifolds was and is one of the central problems in Riemannian and complex as well as algebraic geometry. Motivated by his successful work on compact four manifolds, Professor K. S. Donaldson proposed in [21] with R. Thomas that they might use a similar gauge theory method to classify the compact hyperkähler complex four manifolds. Their program is seemly not very successful so far. However, we seemly accidently had a breakthrough in this direction.

At the end of 1999, we obtained some results for compact complex 4 dimensional simply connected hyper-kähler manifolds in [41]:

Theorem 6.10. *For a compact complex 4 dimensional simply connected hyper-kähler manifold M, we have $3\le b_2(M)\le 23$. Moreover, $b_3(M)\le \frac{1}{2}(b_2(M)+4)(23-b_2(M))$ and $b_4(M)=46+10b_2(M)-b_3(M)$. In particular, we have that if $b_2(M)=23$, then $b_3(M)=0$. The Hodge diamond of M is the same as that of Fujiki's first example.*

This was obtained by the Riemann-Roch formula from [81] and the representation of a Lie algebra $so(4,b_2(M)-2)$ on the cohomology ring of M in [92,93]. That is one of the reasons that this did not fit in our earlier survey [52] but this article. One might call this the *First Theorem of the Hodge diamond on compact complex four dimensional hyper-kähler manifolds.*

In the following, we simply use b_i for $b_i(M)$.

Theorem 6.11. *If $b_2 \neq 23$, then $3 \leq b_2 \leq 8$, and when $b_2 = 8$, we have $b_3 = 0$. Moreover, if $3 \leq b_2 \leq 7$, then $b_3 \leq \frac{4(23-b_2)(8-b_2)}{b_2+1}$.*

This result was obtained after seeing [64] through the referee's comments. This sharpers our most earlier results through the representation theory. After this, I eventually met Salamon and Hitchin in Durham in the summer of 2001.

Even though, our earlier representation argument produces following *Third Theorem* in [42]:

Theorem 6.12. *If $b_2 = 7$, then we have $b_3 = 0$ or 8. Moreover, if $b_3 = 8$, The Hodge diamond of M is the same as that of Beauvilles second example.*

We call Theorem 6.11 the *Second Theorem*, even it was actually proven after Theorem 6.12.

In Pisa in 2003, I gave two series of talks. One on cohomogeneity one Kähler-Einstein metrics before [2]. The other was the possible bounds on the Betti numbers of compact simply connected irreducible hyperkähler manifolds of complex six dimension. There was a difficulty of the representation of $so(4, b_2 - 2)$ on the cohomology group $H^3(M, \mathbf{R})$. This difficulty is still unsolved until today. In 2015, Professor Sawon posted a paper arXiv: 1511.09195 with a result on the bounds of b_2. He following the argument in [42, 61] without knowingly he assume that the representation on H^3 is the spinor representation. Therefore, his original argument has a flaw. I informed Salamon in a later email. Then, in the second version in 2021 he added this assumption, possible after seeing [67]. See also his published version [82].

The conjecture in [67] was checked for the known examples in [30]. In [67], they also obtained bounds for the Betti number b_2 under the assumption of the representation of $so(4, b_2 - 2)$. This is a very remarkable advance for the classification of compact hyperkähler manifolds.

Then in another preprint Sawon gave a "conjectured sign" of a series of Rosansky-Witten invariants, which was seemly known to us (that is, this was seemly true, and was possibly also known by Herrera). However we did not get any thing new from these inequality for the Betti numbers. From there he seemly was eventually able to prove a bound for b_2 for the higher dimensions. From the first glans, the proof of Theorem 8 there was basically from ours, while Corollary 10 was a new input. Because we are so busy recently, we do not get time look into the proof of the "conjectured sign" when we write up this article. And we hope that we, or some other people would be able to finish this in a near future.

Also, in a recent paper, Professor G. Tian with J. Streets in [87] proved that any symplectic structure close to a given hyperkähler structure under some very uncheck-able restricted condition is also Kähler with respect to another complex structure.

Actually, the restricted condition is unnecessary. First, if ω is close to ω_0 which is hyperkähler, by the continuity, $\omega^{2n} > 0$. Therefore, ω is in a cohomology class which is the Kähler class of some hyperkähler structure by the unobstructed Theorem. Now, by the Moser Theorem, we know that ω is different from a hyperkähler structure by a diffeo-morphism.

Therefore, one get:

Theorem 6.13. *Any symplectic structure close to a given hyperkähler structure is also Kähler with respect to possible another complex structure.*

§7. Compact homogeneous spaces with complex structures

It was well-known that (cf. [94])

Theorem 7.1. *Let G be an even dimensional compact Lie group, then on it, there is a G invariant complex structure.*

One could construct the complex structure through the complex-ification $G^{\mathbf{C}}$ of G. Let T be a Cartan subgroup of $G^{\mathbf{C}}$ and P be a parabolic subgroup containing T. Since $T\cap G$ is a real compact torus in G. One can regard T as $(\mathbf{C}^*)^k$, where k is the rank of G. $G^{\mathbf{C}}/P=G/(T\cap G)$ is a projective rational homogeneous space. By our assumption $k=2l$ is even. Let $\pi\colon\ \mathbf{C}^k\to(\mathbf{C}^*)^k$ be the natural universal covering. $\pi^{-1}(1)$ is generated by $\alpha_j=(a_{j1},\cdots,a_{jk})$, $a_{js}=0$ if $s\neq j$; $a_{jj}=2\pi i$, where $1\leq j\leq k$. One could choose a subspace $\mathbf{C}^l$ of $\mathbf{C}^k$ such that $\pi(\mathbf{C}^l)$ is closed in $(\mathbf{C}^*)^k$ and α_j are not in $(\mathbf{C}^*)^l$. For example, $\mathbf{C}^l$ has points of $(b_1,ib_1,b_2,ib_2,\ \cdots,\ b_l,\ ib_l)$. Let $P=TU$ with U the nil-radical of P and $H^{\mathbf{C}}=\pi(\mathbf{C}^lU)$, then $G^{\mathbf{C}}/H^{\mathbf{C}}=G$ has the natural complex structure induced from $G^{\mathbf{C}}$.

The same arguments works for the compact even dimensional homogeneous spaces G/H with G compact such that there is a subgroup J with G/J rational projective homogeneous and J/H being a torus. For example, the Hopf manifolds, and $S^{2n+1}\times S^{2m+1}$.

Question 3. *When does a compact homogeneous space G/H, with G compact admit a complex structure?*

It was well-known that S^{2n} does not admit any almost complex structure if n is not 1 or 3.

Conjecture 3. *S^6 does not exist any complex structure.*

Around 1986, Professor C. C. Hsiong, the initiator of the Journal of Differential Geometry gave a "proof". When he was giving a talk in the Institute of Mathematics, Academia Sinica, I was sitting there and I found his proof was wrong. Since Professor D. Z. Dong came late, I gave my "note" to him and told him that the proof was wrong.

Around 2003, Professor S. S. Chern had another effort before he passed away. Therefore, someone called it the Chern's Last Theorem [76]. See also [78]. We understood that the problem was still open.

Around 2005, Professor Etesi posted another **negative** "proof" in the arXiv saying that there was a complex structure on S^6, but quickly withdrew. Around 2011, he posted another "proof" of his claim in the same place, which was eventually published in 2015. See [23].

Two developments occurred right after that: 1. Professor Etesi wrote another paper and claimed that there is a conjugated G_2 orbit in G_2, which is diffeomorphic to S^6, and it is a complex three dimensional sub-manifold of G_2 with the same complex structure he published earlier [24]; 2. Professor Atiyah posted a "proof" of the conjecture in arXiv in 2016 [3], possibly with an effort of trying to disprove Professor Etesi's conclusion.

It seems that there is no general agreement in the mathematical society.

By using the homogeneous space theory, we were able to prove in [53] that Professor Etesi's second claim in [24] does not hold.

Theorem 7.2. *The conjugated G_2 orbit in G_2 mentioned by Professor Etesi is not a complex sub-manifold of G_2.*

With a proof of that the algebraic dimension of any complex structure on S^6 is zero (cf [7,8]), our result implies that the following conjecture is true:

Conjecture 4. *There is no compact complex threefold of S^6 type in G^2.*

Since our proof of Theorem 7.2 depends a lot on the abstract theory of compact homogeneous spaces and it did not describe any picture for the image of S^6 type orbit under the map $\pi: G_2 \to S^6$, in [59] we prove that:

Theorem 7.3. *The restriction of the map on the conjugated G_2 orbit in G_2 mentioned by Professor Etesi is a generically two to one map onto a closure of an open set in S^6.*

Therefore, if it is a complex sub-manifold, there is a holomorphic map from G_2 to a projective space, such that the image it is a sub set of complex 3 dimension. That is, the pull-back of the Kähler form of the projective space induces a nonzero H^2 class on S^6. This is impossible.

This gives a simpler proof for Theorem 7.2.

With Professor Z. H. Wang, we also checked the gauge used in [23] to construct his almost complex structure is $SO(6)$. This is comparable to the standard metric and therefore, the almost complex structure can not be integrable. We, with Professor Z. H. Wang and Dr. N. Li, also prove that there is no weakly Kähler complex structure on S^6.

Recently, Professor Tang, Z. Z. with W. J. Yan prove in [88]:

Theorem 7.4. *There is a complex structure on $S^1 \times S^7 \times S^6$.*

This is an interesting result.

One have:

Question 4. *Find (almost) complex structures on product of spheres?*

Of course, this is a special case of Question 3.

§8. Characteristic properties of complex Euclidean spaces as complex manifolds

For a Riemann surface, i. e., a complex manifold of complex dimension one, its universal covering is one of the three complex one dimensional homogeneous spaces: the Riemann sphere $\mathbf{C}P^1 = \mathbf{C} \cup \{\infty\} = S^2$; $\mathbf{C}$ and the unit ball $B = \{z \in \mathbf{C}|_{|z|<1}\}$.

In the higher dimension case, we also have three special homogeneous spaces: $\mathbf{C}P^n$; $\mathbf{C}^n$ and the unit ball $B^n = \{z \in \mathbf{C}^n|_{|z|<1}\}$.

There are also Hermitian symmetric spaces [63]; projective rational homogeneous spaces [6,89]; the Wolf compact homogeneous quaternion spaces and complex homogeneous contact spaces [95]; the quotient of the complex ball. We already discussed these in [52].

In this article, we emphasis on the complex plain $\mathbf{C}^n$.

There is a characteristic conjecture for $\mathbf{C}^n$ (cf. [10]).

Conjecture 5. *Any complete but noncompact Kähler manifold with positive bisectional curvature is bi-holomorphic to* $\mathbf{C}^n$.

Considerably works have been done, for example, see [73]. The $U(n)$ co-homogeneity one examples of $\mathbf{C}^n$ have been classified in [96]. See also, for example, [97, 98].

Question 5. *Does this Yau's conjecture (conjecture 5) holds for co-homogeneity one Kähler metrics?*

The simplest examples came from a compact co-homogeneity one Kähler manifolds by deleting one of the ends.

In [96], Professors H. H. Wu and F. Y. Zheng classified these kind of metrics on $\mathbf{C}^n$ and showed that the metrics in the special case in [73] is a very small subset in the set of these kind of metrics.

Next examples are the line bundles $O_n(k)$ over $\mathbf{C}P^n$. It was known that on $O_n(k)$, there is no $U(n+1)$ co-homogeneity one complete Kähler metrics with nonnegative bisectional holomorphic curvatures. Professor B. Chen asked, in January 2020, whether there is any co-homogeneity one complete Kähler metrics with positive sectional holomorphic curvatures on $O_n(k)$?

This question is not completely trivial. Therefore, it took us sometime to get some interesting examples. We found some examples in the summer 2020 and gave a series talks in Henan University in that Fall. Later on we found example 3.25 in [99] on $O_n(-1)$. It was quite different from ours. Our examples have more smoothness at least. Therefore, we invited Professor B. Yang at the end of 2020. Then in the Spring 2021, we constructed more examples, see [22]. We visited Xiamen University in the summer 2021. In a talk in March 2022, Professor B. Yang mentioned that, with Professor F. Y. Zheng, he obtained many more examples at the end of 2021, though it seems that they used a mildly different method; see [100]. Again, their examples are very different from ours. At least, ours have more smoothness.

Now, let us go into a little bit details in [22]. Let $z=(z_1,...,z_n)$ be the standard coordinate on $O^n=\mathbf{C}^n-\{0\}$. We write that

$$r=|z|^2=\sum_{i=1}^n |z_i|^2.$$

Our $U(n)$ invariant Kähler metric on O^n has a Kähler form $\omega=\frac{i}{2}\partial\bar\partial p(r)$, where $p\in C^\infty[0,+\infty)$. Under the coordinate z, the metric has components $g_{i\bar j}=f(r)\delta_{ij}+f'(r)\bar z_i z_j$. Here and from now on we write that $f(r)=p'(r)$, $h(r)=(rf)'$, and call the metric $ds^2(h)$ or $\omega(h)$. Both f and h are in $C^\infty[0,+\infty)$. Note that p is determined by f up to the addition of an arbitrary constant. It is easy to see that $\omega(h)$ is a metric and is complete if and only if $f>0$, $h>0$, and that

$$\int_0^{+\infty}\sqrt{\frac{h}{r}}dr=+\infty.$$

Now we let $\varphi=u^2=g(V,V)=hr$ as in [33]. Here V is the holomorphic vector field on O^n corresponding to the $\mathbf{C}^*$ action on the line bundle.

Then $dt=\sqrt{h}d|z|$, $U=\int_0^t udt=2^{-1}(rf-m)$ with m a positive number related to the metric on the zero section.

Theorem 8.1. *Let e_1 be the unit direction of V, then the holomorphic sectional curvature $A=R_{1\bar{1}1\bar{1}}=-\frac{\varphi_{UU}}{4}$ and $A>0$ if and only if φ is convex down with respect to U. And in the case of $\mathbf{C}^n$, the metric has positive bisectional holomorphic curvature if and only if $A>0$.*

In this case, we also have $U\in C^\infty[0,+\infty)$, and $\varphi(0)=0$, $\varphi_U\geq 0$.

Therefore, there are three cases for the asymptotic property of the function φ when U tends to the infinity: 1. φ does not have any asymptote; 2. φ has a horizontal asymptote; 3. φ has a slant asymptote.

Theorem 8.2. *For any integer k and any Kähler class K_m on an open almost homogeneous manifold $O_{n-1}(k)$ of complex dimension n, there is no $U(n)$ equivariant complete Kähler metrics with positive bisectional holomorphic curvature in the given Kähler class.*

Theorem 8.3. *For any integer $k>0$ and any Kähler class K_m on an open almost homogeneous manifold $O_{n-1}(k)$ of complex dimension n, there is no $U(n)$ equivariant complete Kähler metrics with positive sectional holomorphic curvature in the given Kähler class.*

Theorem 8.4. *For any integer $k>0$ and any Kähler class K_m on an open almost homogeneous manifold $O_{n-1}(-k)$ of complex dimension n, there are many $U(n)$ equivariant complete Kähler metrics with positive sectional holomorphic curvature and with a horizontal asymptote of φ with respect to U in the given Kähler class.*

This Theorem is one of the cores of [22].

In [22], we also prove that on $O_{n-1}(-1)$:

Theorem 8.5. *For any Kähler class K_m on an open almost homogeneous manifold $O_{n-1}(-1)$ of complex dimension n, there are many $U(n)$ equivariant complete Kähler metrics with positive sectional holomorphic curvature and without an asymptote of φ with respect to U, or with a slant asymptote of φ with respect to U in the given Kähler class.*

In this Theorem the case with a slant asymptote is very difficult and is one of the cores of [22].

We notice that it was pointed out by Professor Binglong Chen that a product metric would give a metric with positive holomorphic sectional curvature when $k=0$.

It turns out that for the case without an asymptote of φ with respect to U, or with a slant asymptote, the problem is much more difficult and our original method in [22] does not work at all.

Therefore and fortunately, in [56], which we obtained in the summer 2022, we are able to use a method similar to test the virus finally solve the problem.

Theorem 8.6. *For any integer $k>0$ and any Kähler class K_m on an open almost homogeneous manifold $O_{n-1}(-k)$ of complex dimension n, there are many $U(n)$ equivariant complete Kähler* *metrics with positive sectional holomorphic curvature and without an asymptote of φ with respect to U, or with a slant asymptote of φ with respect to U in the given Kähler class.*

Furthermore, in [100], they considered a special class of Kähler class on the affine quadric of complex dimension two $M_1=\{z_1^2+z_2^2+z_3^2=1\}$ and proved that they did not have nonnegative holomorphic sectional curvature. There was a mistake in their claim that all the co-homogeneity one Kähler metrics have their format. We informed Professor B. Yang many times but he seemly just ignore. Possibly, it was very difficult to explain that there are actually more co-homogeneity one Kähler metrics under the same group action. In [57], following the line in [47, 48], we explain the complex structure. After a long and a difficult journey, we finally are able to find a right bi-holomorphic map from the open orbit of $\mathbf{C}P^1\times\mathbf{C}P^1-\{(x,x)\}$ in [55] to the complex quadric, which gives us a peek into the possibility of a type II metric that is different from theirs. Obviously, there is an $SO(3)$ action on M_1. It turns out that it is a co-homogeneity one action.

Professor B. Yang gave a diffeo-morphism from M_1 to the tangent bundle $T(S^2)$ of two dimensional sphere S^2. Let r be the length of a tangent vector $v\in T(S^2)$. This diffeo-morphism is equi-variant with respect to this $SO(3)$ action. He also gave three vector fields X_i with $i=1,\ 2,\ 3$ as a basis of the Lie algebra of $SO(3)$ such that $[X_i,X_j]=-X_k$ for $(i,\ j,\ k)$ having the same order as that of $(1,\ 2,\ 3)$.

Let dr, ω_1, ω_2, ω_3 be the dual forms of ∂_r, X_1, X_2, X_3. Then $d\omega_i=\omega_j\wedge\omega_k$ if $(i,\ j,\ k)$ has the same order as that of $(1,\ 2,\ 3)$.

Let $Y_1=dr-i\omega_3$ and $Y_2=\omega_1-i\coth r\omega_2$. Then Y_1, Y_2 generically form a basis of $T^{1,0}M_1$. In [57] we let $U=b^2-c^2\tanh^2 r$ with b a function of r and c a constant, and we obtained:

Theorem 8.7. *A closed* $(1,\ 1)$ *form has a format*

$$\omega=b(r)Y_1\wedge\bar{Y}_1+b(r)\tanh rY_2\wedge\bar{Y}_2+ci\frac{\sinh r}{cosh^2 r}(Y_1\wedge\bar{Y}_2-Y_2\wedge\bar{Y}_1)$$

And, it is Kähler if and only if $U(0)=0$, $U>0$ for $r>0$.

When $c=0$, we get a special type of Kähler metrics given by [100]. But there are examples such that c is not zero. This fits quite well with [47].

This is only a small corner of a glacier mountain. However, from this example, one could see the futile part of the co-homogeneity one geometry.

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