On Some Recent Progress in Complex Geometry—the Area Related to Homogeneous Manifolds

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June 26, 2020

In this article, we give a survey of some progress of the complex geometry, mostly related to the Lie group actions on compact complex manifolds and complex homogeneous spaces in the last thirty years. In particular, we explore some works in the special area in Differential Geometry, Lie Group and Complex Homogeneous Space. Together with the special area in nonlinear analysis on complex manifolds, they are the two major aspects of my research interests.

1 Introduction

Let $M = G/H$ be a complex manifold, with G a real finite dimensional Lie group, H a closed Lie subgroup such that the complex structure on M is invariant under the action of G. We call M a complex homogeneous space.

American Mathematics Subject Classification. 53C15, 57S25, 53C30, 22E99, 15A75.
Key words and phrases: invariant structure, homogeneous space, complex torus bundles, Hermitian manifolds, reductive Lie group, compact manifolds, Ricci form, locally conformal Kähler manifolds.

Let M be a complex homogeneous space, h be a Hermitian metric, if h is also invariant under the action of G, we call (M, h) an Hermitian homogeneous space.

Some classical examples of Hermitian homogeneous spaces appeared as early as once mathematians understood the Riemann surfaces, i. e., the complex one dimensional manifolds. When $\dim_{\mathbf{C}} = 1$, we have $\mathbf{C} = \mathbf{R}^2 = \mathbf{C}$ (the third as an abelian Lie group); $\mathbf{C}P^1 = S^2 = PSL(2, \mathbf{C})/B$, where B is the subgroup of $PSL(2, \mathbf{C})$ which is corresponding to the upper triangular matrices (we notice that $B[1, 0] = [1,\ \mathbf{C}]$);

$$\begin{aligned}\mathbf{D} = \{z \in \mathbf{C}|_{|z|<1}\} &= \{(x, y) \in \mathbf{R}^2|_{x^2+y^2<1}\} \\ &= \{z \in \mathbf{C}|_{\mathrm{Im}z>0}\} = SL(2, \mathbf{R})i \\ &= \{\frac{ai+b}{ci+d}|_{ad-bc=1,\ a,\ b,\ c,\ d\in\mathbf{R}}\} \\ &= \{\frac{ac+bd+i}{c^2+d^2}|_{ad-bc=1,\ a,\ b,\ c,\ d\in\mathbf{R}}\}.\end{aligned}$$

Higher dimensional Hermitian homogeneous spaces appeared first in the E. Cartan's landmark classification of the symmetric Riemannian manifolds, which also led eventually to the classification of the semi-simple and reductive Lie groups. Hermitian symmetric space has three fundamental types: flat ones; spaces with positive sectional curvatures or the compact type; spaces with negative sectional curvature type or the noncompact type. There is a duality between the compact type and the noncompact type. All the noncompact type can be realized as a homogeneous bounded domain in $\mathbf{C}^n$. See [Hel].

It turns out that there are many more complex homogeneous manifolds other than Hermitian symmetric spaces.

There is a big class of Kähler homogeneous spaces (M, h). If h is an

Hermitian metric, i.e., $h(J\ ,J\ ) = h(\ ,\ )$, we let $\omega(\ ,\ ) = h(\ ,J\ )$, we call ω the Kähler form of h. We say that (M,h) is Kähler if $d(\omega) = 0$.

A compact complex homogeneous space with an invariant Hermitian structure was classified by H. C. Wang in [W], see also [HKo]. In fact, they classified the compact complex homogeneous space under compact Lie groups. A Hermitian manifold is a Riemannian manifold. The identity component of the Riemannian isometric group for a compact Riemannian manifold is a compact Lie group. So is the identity component of the Hermitian isometric group for a compact Hermitian manifold.

Therefore, we have:

Proposition 1. *If $M = G/H$ is a compact homogeneous Riemannian manifold with G connected, then G is a subgroup of a compact Lie group. In particular, both G and H are reductive with compact semi-simple parts.*

We then have (see [HKo] Theorem B):

Theorem 1. *Any compact Hermitian homogeneous manifold is a complex torus bundle over a rational (therefore simply connected) projective homogeneous space.*

In the case in that h is Kähler, the fibration is a product. Kähler homogeneous spaces with a Semisimple Lie group were classified by A. Borel in [Bo], Kähler homogeneous spaces with a reductive Lie group were classified by Matsushima in [Mat]. Eventually, Káhler homogeneous spaces were classified by Dorfmeister and Nakajima in [DN].

Theorem 2. *Any simply connected Kähler homogeneous space is, biholomorphically, a product of a rational projective homogeneous space, a $\mathbf{C}^n$ and a bounded homogeneous domain in $\mathbf{C}^m$.*

Therefore, the classification problem for the Kähler homogeneous spaces were completely solved.

After 1987, more attentions go to the classifications of non-Kählerian complex homogeneous spaces; compact complex manifolds with an open orbit, i.e., almost homogeneous spaces (cf. [HO]).

The rational projective homogeneous spaces are Kähler-Einstein. Therefore, the geometry on compact Kähler homogeneous spaces are relatively simpler. People were more interested in non-homogeneous geometrical manifolds.

The partial differential equations involved in the geometry usually are very hard to be understood. Many of them are nonlinear. In many earlier efforts, those equations were reduced into ordinary differential equations on many doable situations.

One of the major case is in which the metric is invariant under a Lie group action that has a real hyper-surface orbit. In this case, we call the manifold being **co-homogeneity one**.

Theorem 3. *Any compact cohomogeneity one complex manifold is almost homogeneous.*

There are also many interesting results and conjectures over the characteristics of certain classical homogeneous spaces.

Without a Kähler condition, the classification of compact complex homogeneous spaces will be addressed in the next section.

For noncompact case, there is a Kobayashi conjecture:

Conjecture 1. Let $M = G/H$ be a complex homogeneous space with a real Lie group G. Then the Kobayashi pseudo distance gives a holomorphic fibration over a bounded homogeneous domain in some $\mathbf{C}^m$ such that the Kobayashi pseudo distance on each fiber vanishes.

See, for example, [Wi1] and [St]. We notice that for any compact complex homogeneous manifold, the Kobayashi pseudo distance vanishes.

Complex homogeneous spaces of complex dimension three was classified by Winkelmann [Wi2], where he also proved the Kobayashi conjecture for complex dimension three.

2 Classification of Compact Complex Homogeneous Spaces

In the 1960's, there was a famous structure result for the compact complex homogeneous spaces given by an Abel prize winner J. Tits [Ti].

Let $M = G/H$ be a compact quotient of a complex Lie group G with H a complex Lie subgroup. Assume that H^0 is the identity component of H. $N = \{n \in G|_{nH^0n^{-1} \subset H^0}\}$. We notice that $H \subset N$. We call N the normalization of H^0 in G. Then H^0 is a normal subgroup of N. $N/H = (N/H^0)/(H/H^0)$. That is, H/H^0 is a cocompact discrete Lie subgroup of N/H^0. We say that N/H is parallelizable. The holomorphic tangent bundle of N/H is trivial, i. e., a product. That is (from [Ti]):

Theorem 4. *For any compact complex homogeneous space M, we can write $M = G/H$ with a complex Lie group G and a closed complex Lie subgroup H. Moreover, there is holomorphic fibration $G/H \to G/N$ such that G/N is a rational projective homogeneous space and N/H is a parallelizable complex homogeneous space.*

Therefore, if we really want to classify all the compact complex homogeneous spaces, we need to do two things: (1) Classify all the compact complex parallelizable complex manifolds. (2) Understand the bundle structure.

These were almost done in [Gu1]. First, let G be a complex Lie group, then there is a complex semi-simple Lie subgroup S such that $G = SR$ with R a complex solvable Lie normal subgroup of G. R is the maximal solvable Lie normal subgroup of G. We call R the **solvable radical** of G and S a **Levi subgroup** of G.

In [Gu1], we proved the following:

Proposition 2. *Let G/H be a parallelizable compact complex homogeneous space, S be a Levi subgroup of G. Assuming that S is simple and acts on R nontrivially. Then $S = A_l$ for a positive integer l.*

Therefore, we have Theorem A there:

Theorem 5. *Let G/H be a compact complex parallelizable homogeneous space. Then there is a holomorphic fibration $G/H \to G/HR$. Moreover, assume that S is a Levi subgroup of G, then G/HR is an S compact complex parallelizable homogeneous space. And any factor of S which acts on R nontrivially is of A_l type.*

We notice that $R/(H \cap R)$ is a solv-manifold. Therefore, it is natural to reduce the classification to the classification of the solv-manifolds and the irreducible representation of A_l on solv-manifolds.

Then it is natural that we have Theorem B, then Theorem C, D, G, H in [Gu1]. They are quite technical and therefore we do not state them here. The interested reader might just look at [Gu1]. This work was done at the end of last century. Some constructions in the last three paragraphs in the fifth section of [Gu1] was also appeared in [SW].

Lower dimensional compact complex solv-manifolds of complex dimension up to five was classified in [Gu2].

3 Pseudo-kählerian homogeneous spaces

Let $M = G/H$ be a complex homogeneous space, i. e., the complex structure is invariant under the action of G. Let ω be a differential $(1,1)$-form on M. We say that ω is pseudo-kähler if it is non-degenerate at every point. If ω is also invariant under G, we say that (M, ω) is a pseudo-kählerian homogeneous space. We notice that if ω is positive, then ω is Kähler. Therefore, pseudo-kähler is a generalization of Kähler. It can be also regarded as the semi-Riemannian version of generalization to the Riemannian geometry as for the Kähler geometry. Recall that a semi-Riemannian manifold (M, g) is a pair of a differential manifold and a differentiable nondegenerate symmetric 2-tensor g on its tangent bundle. If g is positive definite at every point, we have a Riemannian structure. For a pseudo-kähler manifold (M, ω), $g(\ ,\ ) = \omega(\ , J\ )$ introduces a natural semi-Riemanian structure on M.

After the completion of the classification of Kähler homogeneous space (cf. [DN]), it is natural to classify the pseudo-kählerian homogeneous spaces.

For the Kähler case, it is easy to see that H can be compact. Therefore H is reductive. In the case in which M is compact, we also have that G can be compact.

However, in the pseudo-kähler case, both H and G could be much more complicated. This made the classification much more complicated, even when M is compact. However, in [DG1], we proved:

Theorem 6. *If $(M = G/H, \omega)$ is a compact pseudo-kählerian homogeneous space, then G and H are compact. The Tits' fibration for the complexification $G^{\mathbf{C}}$ is a product of pseudo-kähler homogeneous spaces and the fibers are compact complex torus.*

In the proof, we used both Tits' fibration from [Ti] and Hano-Kobayashi fibration from [HKo]. Actually, we proved that

Theorem 7. *If $M = G/H$ is a compact complex homogeneous space with an invariant volume form, then the Tits' fibration and the HK fibration are the same.*

The Hano-Kobayshi fibration is actually **the Ricci form fibration**. Let $V = K dz \wedge d\bar{z}$ be the invariant volume form, then $\partial\bar{\partial} \log K$ is a global differentiable 2-form on M, which we call the Ricci form. The fibers of the Ricci form fibration are just the integrating sub-manifolds of the distribution given by the kernel of the Ricci form. One could also see in [Gu3] page 66, Remark for a detail understanding of this fibration.

In general, the classification of certain type of pseudo-kählerian homogeneous spaces is very difficult due to the complication of the isotropic subgroup H. Therefore, we tried the case that G is reductive. Fortunately, we succeeded. In [DG2, DG3, DG4], we proved the following:

Theorem 8. *If $(M = G/H, \omega)$ is a pseudo-kählerian homogeneous space with a reductive Lie group G. Then H is also reductive. Moreover, $G = C \times S$ with C abelian and S semisimple, and M is a product of a flat pseudo-kählerian homogeneous space $C/H \cap C$ and a Wolf space $S/H \cap S$.*

The Wolf spaces are the pseudo-kählerian homogeneous spaces found in [Wo] by Professor Wolf in the 1960's when he was working on infinite dimensional representation theory of semi-simple Lie groups. Let S be a semi-simple Lie group, T a compact torus in S. Let $H = C_S(T) = \{s \in S|_{st=ts, \text{ for any } t \in T}\}$ be the centralizer of T, then G/H is a Wolf space and is simply connected. If H is compact, then $M = G/H$ is Kähler. This gives Borel's result in [Bo] (resp. Matsushima's result in [Mat]) of the classification of semi-simple (resp. reductive) Kähler homogeneous spaces. In particular, if S is compact, G/H is a rational projective homogeneous space.

When G is reductive, we obtained a generalization of Theorem 2 in the pseudo-kählerian case in [DG4]:

Theorem 9 *Let $(M = G/H, \omega)$ be a pseudo-k"ahlerian homogeneous space with G reductive, then M is biholomorphically a product of rational projective homogeneous space, a flat pseudo-kählerian homogeneous space and a pseudo-kählerian homogeneous space of open Wolf type. Moreover, the bounded holomorphic function fibration is a trivial bundle over a bounded symmetric domain, with fibers of reductive pseudo-kählerian homogeneous space with only constant bounded holomorphic functions.*

Further efforts on the classification of compact complex homogeneous spaces with a (non-necessary invariant under the group action) pseudo-kählerian form were carried out in [Gu3, 4]. We shall address this in a later section.

Following [Wi2], one could also classify the homogeneous pseudo-kähler spaces up to complex dimension three, and check the Hano-Kobayashi fibration.

4 Compact cohomogeneity one manifolds

Since the classification of compact complex homogeneous spaces are almost fully understood and they did not provide enough information for geometrical analysis on compact complex manifolds, we study a bigger class of compact complex manifold called compact almost homogeneous complex manifolds.

A compact complex manifold M is called an **almost homogeneous manifold** if the holomorphic automorphism group has an open orbit on M.

In complex geometry, a major problem is to find Kähler-Einstein metrics on compact complex manifolds.

A compact complex manifold M is **Einstein** if there is an Hermitian metric h such that $Ric(h) = kh$ in any sense. However, in this survey article we only consider the case in which the Ricci curvature tensor is given by the Ricci form defined by the volume form in our earlier sections.

When k is a constant, we could assume that it is in three different situations: $k = -1$, $k = 0$, $k = 1$.

In the case in which $k = -1$, h must be Kähler, the problem was started by Calabi, who proved the uniqueness and the third order estimate [Ca1, 2, 3]. We called this the first Calabi problem as the special case of the first Calabi conjecture. Then the problem was fully resolved mild independently by Aubin [Au] and Yau [Ya] with handy proofs. To be fair, one has to say that Yau's solution of this conjecture was heavily depended on the earlier work of Calabi and Aubin. See the Math review of [Ya], for example. In this case, the Ricci curvature defines a negative $(1, 1)$ form in the first Chern class $C_1(M) \in H^{1,1}(M) \cap H^2(M, \mathbf{C})$. We have following Aubin-Yau Theorem:

Theorem 10. *If M has a negative first Chern class, then M admits a Kähler-Einstein metric.*

In the case in which $k = 0$, one has the generalized Calabi first problem:

Problem 1. Find all the compact Hermitian manifolds such that the metric h is Einstein, with the Ricci form defined earlier.

When h is Kähler, this is a special case of the second Calabi conjecture, which says that for any compact Kähler manifold and any 2-form α in the first Chern class, there is a Kähler metric such that α is the Ricci curvature form of the Kähler metric. This was proven by Professor Yau in [Ya] also, which, together with his other outstanding achievements in Riemannian geometry, garanteed him winning the Fields medal.

Theorem 11. *If M is a compact Kähler manifold with a Kähler class $\gamma \in H^{1,1}(M) \cap H^2(M, \mathbf{C})$, then for any $(1,1)$ form α in the first Chern class, there is a Kähler metric ω in γ such that $Ric(\omega) = \alpha$.*

In particular, if M is a Kähler manifold with a zero first Chern class, then there is a Ricci flat Kähler-Einsten metric in any given Kähler class. That is, the Ricci form of the solution Kähler metric is zero. In this case, we call (M, ω) a **Calabi-Yau manifold**.

We notice that the Calabi-Yau manifolds do not give all the Ricci flat compact complex manifolds. For example, a compact complex parallelizable manifold has an invariant holomorphic volume and a zero Ricci form.

Therefore, the existence problem for the compact Kähler manifold was completely solved when the first Chern class $C_1(M) \leq 0$. The solution is somehow more topological and is very easy to check.

When $k = 1$, the situation is quite different. If M has a positive first Chern class, we call M a **Fano manifold**. The following Theorem is due to Matsushima [Ma]:

Theorem 12. *If M is a Fano manifold and is Kähler-Einstein, then the automorphism group is reductive.*

It seems that this obstruction is good enough for $\dim_{\mathbf{C}} M = 2$. After long struggle through Professors Siu, Y. T. [Si1] and Tian, G.-Yau, S. T. [Tia1], [TY], [Tia2], the following result was obtained by [Tia2]:

Theorem 13. *Let M be a Fano surface, then M is Kähler-Einstein if and only if M has a reductive automorphism group.*

There are very few Fano surfaces. They are homogeneous surfaces $\mathbf{C}P^2$,

$\mathbf{C}P^1 \times \mathbf{C}P^1 = Q^2$; almost homogeneous manifolds: $\mathbf{P}l,\ l = 1,2,3$; and projective manifolds $\mathbf{P}l,\ l = 4,5,6,7,8$. Here $\mathbf{P}l$ are blow up of $\mathbf{C}P^2$ at l generic points. Except $\mathbf{P}l,\ l = 1,2$, all other Fano surfaces have reductive automorphism group. Tian's proof was very technical.

We have to say that this solution for the surfaces case depends heavily on the classification of Fano surfaces. And there was not any stability involved.

Indeed, it is possible:

Conjecture 2. For any Kähler class on a Fano surface, there is an Calabi extremal metric in the given Kähler class.

Here, a Kähler metric is Calabi extremal if its scalar curvature is a potential function of a holomorphic vector field.

One notice that when $C_1(M) < 0$, there is no holomorphic vector fields on M and when $C_1(M) = 0$, up to a finite covering, M is a product of compact complex torus, Calabi-Yau manifolds Y_i such that $H^{l,0}(Y_i) = 0$ when $l \neq 0, n_i = \dim_{\mathbf{C}} Y_i$ and $H^{0,0}(Y_i) = H^{n_i,0}(Y_i) = \mathbf{C}$; hyperkähler manifolds K_j such that $H^{l,0}(K_j) = 0$ when l is odd or bigger than $m_j = \dim_{\mathbf{C}} K_j$ and $H^{l,0}(K_j) = \mathbf{C}$ when l is even and $\leq m_j$. In the last case, $H^{*,0}(K_j)$ is generated by a holomorphic symplectic structure in $H^{2,0}(K_j)$. That is, all the holomorphic vector fields come from the first factor, the compact complex torus. Therefore, the Fano condition in Theorem 12 is artificial, unnecessary.

However, we see that the case in which the automorphism group is kind of rich, with its Lie algebra consisting of the holomorphic vector fields, is exactly in the Fano case.

Therefore, it is very natural to classify almost homogeneous Fano manifold of complex dimension three with a reductive Lie group. This was done in [Gu5, 6, 7].

Even so, it is still quite hard to find Kähler-Einstein metric for the almost homogeneous manifolds. First we consider a general classification of compact almost homogeneous manifolds. Let M be a compact complex almost homogeneous manifold and O be an open orbit, then $M - O$ is a lower dimensional subvariety. The following result was well-known, see, e. g., [Ah], [HSn].

Theorem 14. *Let M be a projective almost homogeneous manifold under a complex Lie group G and G has an open orbit, then $M-O$ has at most two connected components. We call the manifolds compact complex almost homogeneous manifold of one or two ends, according to the number of the components of $M - O$ being one or two. In particular, if M has two ends, then M has three G orbits O, E_0 and E_∞, and O is a $\mathbf{C}^*$ bundle over a projective homogeneous space Q.*

We see that the compact complex almost homogeneous manifolds with two ends are easier to be handled and the maximal compact subgroup of G has a real hypersurface as an orbit. We call the projective almost homogeneous manifolds with two ends the **type III cohomogeneity one Kähler manifolds**.

Recall that a Riemannian manifold is **cohomogeneity one** if its isometry group has a real hypersurface orbit.

Proposition 3. *Any compact cohomogeneity one Kähler manifold is almost homogeneous.*

This is also the Theorem 3 we mentioned earlier. The type III case with the Einstein problem was done by Sakane and Koiso [KS]. They gave the first nonhomogeneous examples of compact Fano manifolds with Kähler-Einstein metrics. Their results are seemly even earlier than the examples given by Siu, Tian-Yau on the Fano surfaces. They used Kobayashi's construction of the Riemann metrics on the S^1 bundles.

Theorem 15. *A Fano almost homogeneous manifold with two ends admits a Kähler-Einstein metric if and only if the Futaki invariant corresponding to the S^1 action is zero.*

For a general Kähler class $\gamma \neq C_1(M)$, we can not have a Kähler-Einstein metric. Therefore, it is interesting to find a Kähler metric with a constant scalar curvature. Even in the Ricci class, in the compact complex almost homogeneous manifold with two ends, there is an obstruction of the Futaki invariant (cf. [Fu]). Therefore, it is natural to find some other kind of canonical metrics.

In early 1980's Professor Calabi [Ca4, 5] defined the **Calabi extremal metrics** being the critical metrics for the functional:

$$\omega \to \int_M R^2 \omega^n$$

which is called the **Calabi functional**. Here, R is the scalar curvature of the metric.

Calabi prove that:

Theorem 16. *The Kähler metric g is extremal if and only if the lifting of dR is a holomorphic vector field.*

The following result was given by Professor Andrew Hwang (Cf. [Hw]), when we were trying to find Kähler-Einstein metrics, using a Ricci class method.

Theorem 17. *A Fano almost homogeneous manifold with two ends always admits a Calabi extremal metric.*

We were able to prove the following result by a calculation of the scalar curvature equation in [Gu8]. By our argument (but not the one in [Hw]) it is easy to see that the Yau-Tian-Donaldson conjecture generally does not hold for Kähler metrics with constant scalar curvature. It is about the positive lemma involved there. This was also pointed out many years later by an Inventiones paper [ACGT]. One notices that the admissible metrics construction there was very restricted and does not include the compact complex almost homogeneous with two ends. So did [Hw] do not eventually treat the compact complex almost homogeneous manifold with two ends but only some special cases of the admissible metrics.

Theorem 18. *For any Kähler class on a compact complex almost homogeneous manifold with two ends, there always exists a Calabi extremal metric in the given Kähler class.*

However, to apply the Ricci flow method to get some kind of generalization of the Kähler-Einstein metrics, it is natural to consider the **complex quasi-Einstein metrics** which is the fixed point of the generalized Ricci flow:

$$g_t = Ric(g) - HRic(g).$$

Here $HRic$ is the harmonic part of Ric.

A Kähler metric ω is a **generalized complex quasi-Einstein metric** if there is a function f such that let X be the vector field obtained by the lifting of df, then

$$L_X\omega = Ric(\omega) - HRic(\omega).$$

In [Gu9], we obtained:

Theorem 19. *X must be holomorphic. Moreover, for any Kähler class on a compact complex almost homogeneous manifold with two ends, there* *always exists a generalized complex quasi-Einstein metric.*

These works took care of the compact almost homogeneous manifolds with two ends, which were done when I was a Ph. D. graduate student. We shall mention the Maxwell-Einstein and the generalized Kähler Ricci-soliton metrics (somehow we would not treat some similar m-Calabi metrics) in our section 8.

Somehow, we did not get any new Kähler-Einstein metric at that time. Obviously, there are many non-type III compact co-homogeneity one Kähler manifolds, see [Ah], [HSn] for example. As professor Huckleberry once mentioned later on, some people wanted to do it, but none knew how to do it.

Eventually, we found some simple series of examples N_n and M_n in [GC]. They are compact co-homogeneity one Kähler manifolds of type II but N_n behave as type I manifolds. Let $P_n = \mathbf{C}P^n \times \mathbf{C}P^n$, then $SL(n+1, \mathbf{C})$ acts on P_n diagonally with two orbits: $D = \{(x, x)|_{x \in \mathbf{C}P^n}\}$ and $O = P_n - D$. M_n is just the blow up of P_n along D and the symmetric group $S_2 = \mathbf{Z}_2$ acts on P_n with D being the set of fixed points. Let $Q_2 = P_2/S_2$, then N_2 is just the blow up of Q_2 along D. Moreover, $SU(n+1)$ acts on P_n, M_n, N_n as a cohomogeneity one action, i. e., with real hypersurface orbits.

Let H be a real hypersurface orbit, then $H = K/L$, and assuming that $A = \{k \in K|_{kh=hk \text{ for any } h \in L}\}$ be the centralizer of L in K, A is of real dimension either one or three. If A is one dimensional, we say that the compact co-homogeneity one Kähler manifold M is **of type I** if it is not a type III manifold. If M is not of type I or type III, we say that M is **of type II**. See [Gu21].

Basically, we proved the following:

Theorem 20. *For any Kähler class on a compact cohomogeneity one manifold with zero first Betti number which is not of type III, there is a Kähler metric in the given Kähler class with a constant scalar curvature if and only if certain topological integral is smaller than zero.*

This integral is the negative of the generalized Futaki invariant related to the exceptional closed orbit. We notice that since it is not of type III, the exceptional orbit is connected and homogeneous.

In our case, the necessary condition here is stronger than the semi-stable property of the closed orbit in the work of Professors Ross and Thomas [RT].

The proof of Theorem 20 for the type I manifolds was published in [Gu10]. The sufficient part (corresponding to the Uhlenbeck-Yau part for the holomorphic vector bundle case, see [UY]) for the type II manifolds was published in [Gu11, 12, 13]. However, the necessary part (corresponding to the Kobayashi part, cf. [Kb1, 2]) for the type II manifolds is more complicated and is to be published later on. See [Gu14], for example.

Therefore, we obtained infinite many Kähler-Einstein manifolds as well as infinite many Fano manifolds without any Kähler-Einstein metric.

Basically, we used Kähler-Einstein metrics (or Kähler metrics with constant scalar curvature) smooth on the open orbit O but with conic singularities on the exceptional orbit $M - O$ to approach the required smooth metrics. The generalized Futaki invariant is naturally coming from the possible obstruction for this process.

Our approach also led to the following:

Conjecture 3. Any smooth complex hypersurface in $\mathbf{C}P^n$ is Kähler-Einstein.

5 Compact complex homogeneous spaces with invariant volume

When a complex homogeneous space admits an invariant volume form, one can define the Ricci form as earlier in section 3. Then the kernel of the Ricci form gives a distribution on the space. In [HKo], Professors Hano and Kobayashi proved that the distribution is complex and integrable. Therefore, it gives a real fibration with complex fibers. The fibration might not be holomorphic. Our Theorem 7 says that in the case in which $M = G/H$ is compact, the fibration is actually holomorphic. After the classification of the compact pseudo-kähler homogeneous space in [DG1], it is natural to consider the same question for compact complex homogeneous spaces with an invariant symplectic structure or even volume form.

The symplectic structure case was actually done in [Hu]. See also [Gu3].

Theorem 21([Gu3]). *Let $M = G/H$ be a compact complex homogeneous space such that there is a non-degenerate 2-form ω which is invariant under the action of G, then M is a product of a projective rational homogeneous space and a compact complex torus.*

This result implies that the compact complex homogeneous space with an invariant symplectic structure is not much different from a compact Kähler homogeneous space. This is obviously not true for compact complex homogeneous spaces with invariant volume forms. First, if $M = G/H$ is compact complex homogeneous with G compact as dealt with by H. C. Wang in [W] and Hano-Kobayashi in [HKo], then M admits a G invariant volume. We call this kind of homogeneous spaces the **Wang spaces**. Next, if $M = G/H$ is compact complex parallelizable with H a discrete co-compact subgroup, then G is unimodular and admits an invariant volume. Therefore, any product of a Wang space and a compact complex parallelizable homogeneous space admits an invariant volume. Fortunately enough, by our insight from Theorem 7, this is not very far from a classification.

In [Gu] we have:

Theorem 22. *Any compact complex homogeneous space with an invariant volume is a compact complex torus bundle over a product of a Wang space and and a compact complex parallelizable space.*

6 Compact complex manifolds with holomorphic symplectic or pseudo-kähler structures

It is very important to understand the classification of compact complex surface, that is, compact complex manifold of complex dimension two. And it is also very important to understand compact Riemannian manifold with a special holonomy group. In general, a real n dimensional compact orientable Riemannian manifold might have a holonomy group $SO(n)$. When it is a complex m dimensional compact Kähler manifold, the holonomy group reduced to $U(m)$. When the holonomy group is a subgroup of $SU(m)$, the Kähler metric has a zero Ricci curvature. In the later case, as we mentioned earlier that the interesting cases are the Calabi-Yau manifolds and the hyperkähler manifolds. Hyperkähler manifolds have even complex dimension $2l$ since a nondegenerate 2-form can only have an even rank, and the manifolds have holonomy group $Sp(l)$.

One of the Kodaira conjectures was that (proven in [Si2])

Theorem 23. *A compact complex surface S is Kähler if and only if the first Betti number of S is even.*

The proof finally reduced to the K3 surfaces, which are simply connected compact complex surfaces with non-degenerate holomorphic volumes (or holomorphic symplectic forms). So, the question is: Is any K3 surface a Kähler surface? Professor Todorov published a paper in Inventiones [To] and claimed the proof. But Professor Siu pointed out some defect in the Mathematical Review and published another paper in Inventiones [Si2] proving the conjecture.

The question left: Is any simply connected compact complex manifold with a holomorphic symplectic structure a Kähler manifold?

Professor Todorov wrote another long paper and claimed that all of them were also Kähler.

After a long time, none could say that he was correct or not. But most people believed that this was true. It was similar to the situation for the Bogomolov-Todorov-Tian unobstructed Theorem for the Calabi-Yau manifolds, for which Professor Tian eventually gave a clean solution in [Ti3].

When I was a Ph. D student in UCBerkeley, Professor Todorov gave a talk. Then I started producing compact complex homogeneous spaces with holomorphic symplectic structures in [Gu3]. I realized that they must be complex parallelizable and even must be a solvmanifold. That is, $M = G/H$ with G solvable. We constructed more examples of solvmanifolds even without requiring that G to be complex. See [Gu3]. Then we came to the simple examples of Kodaira-Thurston surfaces. A Kodaira-Thurston surface is a compact complex nonkähler surface with a holomorphic symplectic structure. By the solution of the Kodaira conjecture that it has an odd first Betti number. And it is a quotient of a real nilponent Lie group. Actually, its universal covering as a real Lie group has a right invariant complex structure on G.

Theorem 24. *Let S be a compact complex surface with a holomorphic symplectic structure, consider $S^{(k)}$ be the symmetric product of S. It consists of k points in S without order. Then there is a smooth complex manifold $S^{[k]}$ over $S^{(k)}$ with induced holomorphic symplectic structure.*

Here we say that a complex manifold M is **over** N if there is a holomorphic map $\pi : M \to N$ such that $\pi^{-1}(n)$ is a point for a dense open set of N. We call $S^{[k]}$ the k-th Hilbert scheme of S.

Basically, there are only three kind of such S: a complex two dimensional torus, a K3 surface, a Kodaira-Thurston surface.

$S^{[k]}$ are Kähler if S is either a complex torus or a K3 surface.

$S^{[k]}$ is not simply connected when S is a Kodaira-Thurston surface and is not Kähler.

A Kodaira-Thurston surface is a Lagrange torus fibration over a torus. Moreover, the center of G acts on the fiber as a holomorphic complex torus action C. The C action lifted to an action on $S^{[k]}$. Let $a : \; S \to T$ be the fibering map. Then there is a map $a : \; \prod_i S_i \to T$, with $S_i = S$ such that $a(s_1, \cdots, s_k) = a(s_1) + \cdots + a(s_k)$. This induces a map $a : \; S^{(k)} \to T$. Locally, we can regard a as a holomorphic function on $S^{[k]}$. Then da is a holomorphic 1-form. Using the holomorphic symplectic structure ω we lift da to a holomorphic vector field $X_a = \omega^*(\; , da)$. Then X_a corresponds to the diagonal C action on $S^{[k]}$. One naturally have a holomorphic symplectic reduction $R_k = a^{-1}(0)//C$. However, R_k usually is only an orbifold but not a manifold. To get a manifold, a natural way is to get a covering $\pi : \tilde{R}_k \to R_k$ such that $\tilde{R}_k$ is a smooth manifold. This kind of covering is called a **good covering** and was studied by Thurston for real three dimensional manifold. To find this kind of good coverings we have to choose the right Kodaira-

Thurston surface by adding a topological condition on the torsion part of the fundamental group of S.

By calculating a topological bilinear form on the second cohomology of $\tilde{R}_k$, we were able to prove in [Gu15] that

Theorem 25. *For $k > 2$, $\tilde{R}_k$ is a simply connected compact holomorphic symplectic manifold which, as a topological differentiable manifold, does not admit any Kähler structure.*

In the proof, we also corrected an error of Professor Fujiki on the topological bilinear form on the second co-homology group for the Kähler case.

We actually proved in [Gu15] that the Lefschets property for the Kähler manifold does not hold for $\tilde{R}_k$.

Theorem 26. *For $k > 2$, the Lefschetz Theorem does not hold for $\tilde{R}_k$ with any closed 2 form.*

After we submitted the paper to Inventiones as Professor Todorov and Siu did earlier, the referees claimed first that my construction did not work, then they said that my construction was ok but my proof was wrong, they gave an idea of their thought for the case of $k = 3$ in the referee report. Basically, the idea is, one find a covering of $S^{(k)}$ first, then apply the symplectic reduction. It is like, to go from the northwest corner of a block in a city to its southeast corner, you might go to the south first and then the east as I did, or go to the east first then the south. I worked out their "proof" for the complex dimension four and generalized it to all the cases with $k > 2$. Then, someone asked me where Professor Todorov was wrong. I said that it was not my obligation.

As such, Professor Gromov introduced me to Professor Bogomolov. Professor Bogomolov realized that one can just use a sub-manifold S in $\tilde{R}_k$ to prove the nonkählerness of our manifold. Since if the ambient manifold is Kähler, then the sub-manifold is also Kähler. But S is nonkähler, as Professor McDuff did in her construction [MD] of a simply connected compact nonkähler real dimension ten real symplectic manifold. Later on, simply connected compact nonkähler real six and four dimensional real symplectic manifolds were constructed by Professor Gompf in [Gf], possibly stimulated by our work. Our work was submitted in the summer 1993. A similar construction, generally called the Kummer construction, was used by Professor Joyce to construct the first compact simply connected Riemannian manifolds with special holonomy group G_2 (seven dimensional) and $Spin(7)$ (eight dimensional) in [Jo], possibly also stimulated by our work.

Anyway, Professor Bogomolov obtained

Proposition 4. *For $k > 2$, $\tilde{R}_k$, as a complex manifold, is not Kähler.*

Therefore, he gave an alternative negative answer to Todorov's question with a simpler argument, to the nonkähler property. Somehow, he had problem with the restriction of the fundamental group of the Kodaira-Thurston surface to be chosen. I informed him the choosing condition. I also talked to one of the referees of his paper. After that, with possibly helping from Professor Siu, both papers were accepted. As a matter of fact, in that year, 1995, Professor Bogomolov published three papers with a similar feature: gave a different proof for several important results, completely or partially.

After that, motivated by our construction, Professor Fernandez et al published a paper in Annals [FM] constructing a simply connected compact real eight dimensional real manifold with a real symplectic structure which is not formal. Formality is a condition with which a compact Kähler manifold should satisfy.

Theorem 27. *There is a simply connected non-formal real eight dimensional manifold with a real symplectic structure.*

I was told about this result by Professor Fernandez and we proved that:

Theorem 28. $\tilde{R}_3$ *is not formal.*

The real dimension of $\tilde{R}_k$ is $4k-4$. For $k>3$ we have:

Conjecture 4: $\tilde{R}_k$ *is not formal for* $k>2$.

Also, the classification of compact complex homogeneous spaces with (non-necessary invariant) pseudo-kähler structure were classified in a series of papers in [Gu3, 4, 16, 17]. For the Kähler case, it was proved by [BR] that M is a product of projective rational homogeneous space and a complex torus.

Theorem 29. *Any compact complex homogeneous space with a real symplectic structure is a product as a compact complex homogeneous space of a rational projective homogeneous space with a solv-manifold with a real symplectic structure. If M is pseudo-kählerm the second factor is also pseudo-kähler with an abelian nilradical. If M is Kähler, then the second factor is a complex torus. If M is holomorphic symplectic, then the first factor is a point.*

Further more, in a third paper for the examples of the nonkählerian holomorphic symplectic structure, we proved in [Gu29] that

Theorem 30. *For our manifolds* $M=\tilde{R}_k$, $H^2(M,\mathbf{C})=H^{0,2}(M)+H^{1,1}(M)+H^{2,0}(M)$. *Moreover, there is a bilinear form on* $H^2(M,\mathbf{R})$ *with one dimensional kernel.*

Theorem 31. *Let M be a simply connected compact holomorphic symplectic manifold, if*

$$H^2(M,\mathbf{C})=H^{2,0}(M)+H^{1,1}(M)+H^{0,2}(M),$$

then M is unobstructed.

We applied a symplectic geometry proof other than the earlier ones. Moreover, we proved:

Theorem 32. *For all the known simply connected compact holomorphic symplectic manifolds, it is deformation equivalent to a holomorphic symplectic manifold, which is a Lagrange complex torus fibration over a complex projective space.*

Therefore, one have:

Conjecture 5. Any simply connected holomorphic symplectic manifold is deformation equivalent to a holomorphic symplectic manifold which is a holomorphic Lagrange torus fibration over a complex projective space.

This was later on called as the SYZ conjecture for the holomorphic symplectic manifolds, especially by Professor Todorov.

7 Compact homogeneous spaces with complex structures

It was well-known that (cf. [W])

Theorem 33. *Let G be an even dimensional compact Lie group, then on it, there is a G invariant complex structure.*

One could construct the complex structure through the complexification $G^{\mathbf{C}}$ of G. Let T be a Cartan subgroup of $G^{\mathbf{C}}$ and P be a parabolic subgroup containing T. Since $T \cap G$ is a real compact torus in G. One can regard T as $(\mathbf{C}^*)^k$, where k is the rank of G. $G^{\mathbf{C}}/P = G/(T \cap G)$ is a projective rational homogeneous space. By our assumption $k = 2l$ is even. Let $\pi : \mathbf{C}^k \to (\mathbf{C}^*)^k$ be the natural universal covering. $\pi^{-1}(1)$ is generated by $\alpha_j = (a_{j1}, \cdots, a_{jk})$, $a_{js} = 0$ if $s \neq j$; $a_{jj} = 2\pi i$, where $1 \leq j \leq k$. One could choose a subspace $\mathbf{C}^l$ of $\mathbf{C}^k$ such that $\pi(\mathbf{C}^l)$ is closed in $(\mathbf{C}^*)^k$ and α_j are not in $(\mathbf{C}^*)^l$. For example, $\mathbf{C}^l$ has points of $(b_1, ib_1, b_2, ib_2, \cdots, b_l, ib_l)$. Let $P = TU$ with U the nil-radical of P and $H^{\mathbf{C}} = \pi(\mathbf{C}^l U)$, then $G^{\mathbf{C}}/H^{\mathbf{C}} = G$ has the natural complex structure induced from $G^{\mathbf{C}}$.

The same arguments works for the compact even dimensional homogeneous spaces G/H with G compact such that there is a subgroup J with G/J rational projective homogeneous and J/H being a torus. For example, the Hopf manifolds, and $S^{2n+1} \times S^{2m+1}$.

Question 1. When does a compact homogeneous space G/H, with G compact admit a complex structure?

It was well-known that S^{2n} does not admit any almost complex structure if n is not 1 or 3.

Conjecture 6. S^6 does not exist any complex structure.

Around 1986, Professor C. C. Hsiong, the initiator of the Journal of Differential Geometry gave a “proof”. When he was giving a talk in the Institute of Mathematics, Academia Sinica, I was sitting there and I found his proof was wrong. Since Professor D. Z. Dong came late, I gave my “note” to him and told him that the proof was wrong.

Around 2003, Professor S. S. Chern had another effort before he passed away. Therefore, someone called it the Chern’s Last Theorem [MN]. See also [Br]. We understood that the problem was still open.

Around 2005, Professor Etesi posted another **negative** “proof” in the arXiv saying that there was a complex structure on S^6, but quickly withdrew. Around 2011, he posted another “proof” of his claim in the same place, which was eventually published in 2015. See [Et1].

Two developments occurred after that: 1. Professor Etesi wrote another paper and claimed that there is a conjugated G_2 orbit in G_2, which is diffeomorphic to S^6, and it is a complex three dimensional sub-manifold of G_2 with the same complex structure he published earlier [Et2]; 2. Professor Atiyah posted a "proof" of the conjecture in arXiv in 2016 [At], possibly with an effort of trying to disprove Professor Etesi's conclusion.

It seems that there is no general agreement in the mathematical society.

By using the homogeneous space theory, we were able to prove in [Gu18] that Professor Etesi's second claim in [Et2] does not hold.

Theorem 34. *The conjugated G_2 orbit in G_2 mentioned by Professor Etesi is not a complex submanifold of G_2.*

With a proof of that the algebraic dimension of any complex structure on S^6 is zero (cf [CDP1], [CDP2]), our result implies that the following conjecture is true:

Conjecture 7. There is no compact complex threefold of S^6 type in G^2.

8 Compact complex manifolds with Maxwell-Einstein metrics and Generalized Ricci Solitons

8.1 Maxwell-Einstein metrics and so on

Definition 1. For any given Kähler class, there is a Maxwell-Einstein metric conformally related to the Kähler class if $\tilde{g} = u^{-2}g$ is an Hermitian metric with a constant scalar curvature such that u is the Hamiltonian function of a holomorphic vector field related to a Kähler metric g in the given Kähler class.

Here, again, we consider compact almost homogeneous manifolds with two ends. Consider that the metrics being invariant under the action of the maximal compact Lie group on the $\mathbf{C}^*$ bundle, it can be seen that if g is a Maxwell-Einstein metric, then $u = aU + b$ for some $a, b \in \mathbf{R}$, where U is the normalized potential function of the holomorphic vector field on the $\mathbf{C}^*$ bundle such that $U \in [-1, 1]$. That is, U achieves -1 on the zero section of the corresponding line bundle and 1 on the corresponding infinity section.

From [Dr], p.119, (6.1), or [Au], p.126, (1), we have that the scalar curvature of $\tilde{g}$ is:

$$\begin{aligned} \tilde{S} = -2\frac{2n-1}{n-1}v^{-\frac{n+1}{n-1}}\Delta v + Sv^{-\frac{2}{n-1}} \\ = \quad 2(2n-1)(u\Delta u - n|Du|^2) + Su^2 \end{aligned} \tag{1}$$

Here $v = u^{-n+1}$ and S being the scalar curvature of our Kähler metric g. Notice that here we have a different sign for the Laplacian.

We then in [Gu28] have:

Theorem 35. (cf. [KS], [Gu8]) *There is a Maxwell-Einstein metric related to any given Kähler class on a compact almost homogeneous manifold with two ends.*

Around twenty years ago, after we seeing [HSi1, 2] and [De], I came up with this kind of metrics and obtained some partial result of this Theorem. I told Professor Kobayashi about it. However, somehow, first we did not get further Hermitian-Einstein metric in the Riemannian sense and second, by the proof of Yamabe conjecture, every Kähler metric is conformally related to a Hermitian metric with a constant scalar curvature, just as every Hermitian metric has a smooth Riemannian scalar curvature, we did not pay much attention to these metrics. However, recently, after the publications of LeBruns two papers [LB1,2], it seems to us that Maxwell-Einstein metrics became a hot topic in the mathematical community.

Together with Calabi extremal metrics and the quasi-Einstein metrics, we have three virtual-Kähler-Einstein metrics on compact almost homogeneous manifolds with two ends now.

Moreover, there was another kind of virtual-Kähler-Einstein metrics on these kind of manifolds, which was called generalized Kähler Ricci solitons, defined by [Na].

The existence result was obtained in [Gu23]. Since it is difficult for the mathematics community to reach that article, we would like to copy some of the results in the next two subsections for the convenience of the readers. This material also shows the general strategy how did we approach a proof of our Theorem 35.

Above discussions shows that on $\mathbf{C}P^2$ blow up one point, there exists all these four kind of standard metrics in any given Kähler class. One can actually expect:

Conjecture 8. On $\mathbf{C}P^2$ blow up two points, there always exist all of these four kinds of canonical metrics in any given Kähler class.

8.2 Existence of the Generalized Extremal-solitons on Certain Completions of Line Bundles

In every Kähler class of a compact almost homogeneous manifold with two ends we found a unique Calabi extremal metric in [Gu8], [Gu22]. See Theorem 18. Recall that a compact Kähler manifold is almost homogeneous with two ends if and only if it is an equivariant completion of a homogeneous $\mathbf{C}^*$ bundle over a compact homogeneous Kähler manifold. Moreover, we found a unique extremal metric in a given Kähler class on certain completion of $\mathbf{C}^*$ bundle if the function Φ in [Gu8] or [Gu22] is positive. We realized in [Gu21] that this is equivalent to the geodesic stability of the Kähler class.

Therefore, we see how important is the positivity of Φ in [Gu8]. There are abundant evidences that the existence of the Calabi extremal metrics is very rare. For example, in our construction, if the base manifold is negatively curved, it is very difficult to get any Calabi extremal metric. In many cases, Φ will not be always positive on the given open interval. Also, there is another canonical metrics called the generalized quasi-Einstein metrics (cf. Theorem 19). What are the relations between the existence of these two kinds of metrics? In [Gu24], we defined a family of metrics called extremal solitons which connects these two kinds of metrics. Again, the existence of those metrics on compact almost homogeneous Kähler manifolds with two ends is heavily depended on the positivity of Φ, through the positive Lemma.

Different from working on the Ricci class in the earlier works, in [Gu8] we started to deal with the scalar curvatures directly. That made the breakthrough in [Gu8]. The positive Lemma then secured the existence. We told professors Kobayashi and Peter Li about our results and gave out our results in the Riverside geometric conference in the spring 1992.

It turns out later on this positive Lemma is critical in the solution of the existence of Kähler-Einstein metrics of cohomogeneity one [Gu11], [Gu13], [Gu21], [Gu25].

In the following of this section we shall define a certain class of completions of $\mathbf{C}^*$ bundles which we will consider and prove the existence of the generalized extremal soliton metrics on them. To interpolate the extremal metrics and those quasi-Einstein metrics in [Gu9], which are a kind of Kähler-soliton metrics as a generalization of Ricci-soliton metrics, as well as extremal-solitons, we define a big family of generalized extremal solitons which also take the generalized Kähler-Ricci soliton in [Na] as special examples. A Kähler metric is **extremal** if

$$R - HR = \phi$$

where R is the scalar curvature, HR is the average of the scalar curvature and ϕ a potential function of a holomorphic vector field. A Kähler metric is a quasi-Einstein metric or a **Kähler-soliton** if

$$R - HR = \Delta\phi$$

where $\Delta\phi$ is the Laplacian of a potential function of a holomorphic vector field. When the Kähler class is the Ricci class or the negative Ricci class we have exactly the Kähler Ricci-soliton. The Kähler Ricci-solitons were first studied by H. D. Cao (see [Co]), Koiso [Ki1], [Ki2] and Tian, which was motivated by Hamilton's similar work on the Ricci-solitons in the Riemannian case.

A Kähler metric is an **extremal-soliton** if we have

$$R - HR = \phi_1 + \Delta\phi_2$$

with two potential functions ϕ_1, ϕ_2 of holomorphic vector fields.

A Kähler metric is a **generalized extremal-solition** if we have

$$R - HR = \phi_1 + \Delta\phi_2 + (\nabla\phi_3, \nabla\phi_4) \qquad (*)$$

with four potential functions $\phi_i,\ i = 1, 2, 3, 4,$ of holomorphic vector fields in the holomorphic version, and with four parallel functions in the equivariant Mabuchi moduli space of the Kähler metrics in the parallel version (see [Gu26]).

We see that the parallel version can be much more general since the space of the potentials of holomorphic vector fields is only a finite dimensional vector space, while the space of the parallel infinitesimal Kähler potentials is an algebra.

In this subsection 8.2 we consider the existence of the generalized extremal-solitons. This generalized the results in [Gu8], [Gu9], [Gu24].

In particular we have:

Theorem 36. *There is a continuous family of generalized extremal-soliton metrics, defined by the equation (18), in every Kähler class on a compact almost homogeneous manifolds with two ends, whenever the characteristic equation of the corresponding homogeneous equation of (16) (which defining our metrics and is just the equation (*) in our circumstance) has two real roots, which interpolates the extremal metric and the generalized quasi-Einstein metric we obtained before, as well as the generalized Kähler-Ricci solitons defined in [Na].*

In our proof, we see that this family is defined by the equation (18) with above four functions. Therefore, the subset of the generalized extremal-solitons has a real dimension of three. The condition in our Theorem 32, i.e., with two real roots, is basically whenever our definition make sense.

In the process, we should prove a generalized version of the positive Lemma, the Lemma 10.

This does not give the existence of the generalized Kähler-Ricci soliton in [Na] 3.10 automatically. We need a little bit more work. A generalized extremal-soliton is a **generalized Kähler-Ricci soliton** if $\phi_3 = \phi_4$, $\phi_2 = 4\phi_3$, $\phi_1 = m\phi_2$ with a constant $m = \pi \frac{C_1 \wedge \omega^{n-1}}{\omega^n}$, where C_1 is the first Chern class of M. We notice that m is just the product of π and the average of the scalar curvature. In [Na] p.509 to 512, he gave an example. However, he was not able to solve the problem for compact almost homogeneous manifolds with two ends. See page 512 there. We calculate the formula of ϕ_1, which is determined by b in (10), in the Lemma 9 carefully, in subsection 8.3. There are three parts in the numerator and just one part in the denominator. We are able to prove that when the characteristic equation has repeated real roots, b is equivalent to the square (and the negative square) of the roots as the root turns to the infinity. That gives us:

Theorem 37. *There is always a generalized Kähler-Ricci soliton in any given Kähler class if the base manifold has constant and positive trace Ricci eigenvalues (see (4) after Lemma 1). In particular, if the base manifold is compact homogeneous, there are generalized Kähler Ricci solitons in any given Kähler class.*

Our results can be regarded as a continuation of [Gu24] (and [KS1], [KS2], [Ki2], [Gu22], [Gu8], [Gu9], [Gu21], [Gu27], [Na]). Thus we suggest that the reader be familiar with the Kähler geometry and the material in some of those papers. We state, without detailed proof, the Lemmas and Proposition 5 similar to those in the afforementioned papers as follows. The readers might take [Gu8], (Cf. [Gu9], [Gu27]) as the standard reference. Most Lemmas and Proposition 5 can be actually found in [Gu8]. Lemma 2 and 3 can be found in [Gu27] (although they are not really necessary for this result). Lemma 2 and 3 give us a better understanding of the construction. Another way to understand our construction is that U in the Lemma 1 is just the moment map of the $\mathbf{C}^*$ action on the line bundle.

Let $p: L \to M$ be a holomorphic line bundle over a compact complex Kähler manifold M and h a hermitian metric of L. Denote by L^0 the open subset $L - \{0\text{-section}\}$ of L and let $s \in C^\infty(L^0)_{\mathbf{R}}$ be defined by $s(l) = \log |l|_h$ $(l \in L^0)$, where $|\ |_h$ is the norm defined by h. Now we consider a function $\tau = \tau(s) \in C^\infty(L^0)_{\mathbf{R}}$ which depends only on s and is monotone-increasing with respect to s.

Let $\tilde{J}$ be the complex structure of L and J be the complex structure of M. Now we consider a Riemannian metric on L^0 of the form

$$\tilde{g} = d\tau^2 + (d\tau \circ \tilde{J})^2 + g, \tag{2}$$

where $g(l) = p^* g_{\tau(s(l))}(m)$ with $m = p(l) \in M$ and g_τ is a one parameter family of Riemannian metrics on M. Define a positive function u on L^0 depending only on τ by $u(\tau)^2 = \tilde{g}(H,\ H)$, where H is the real vector field on L^0 corresponding to the $\mathbf{R}^*$ action on L^0.

Lemma 1.(Cf [KS1], [Gu8] p. 2257) *Suppose that the range of* τ *contains* 0. *Then* $\tilde{g}$ *is Kähler if and only if* g_0 *is Kähler and* $g_\tau = g_0 - UB$, *where* B *is the curvature of* L *with respect to* h *and* $U = \int_0^\tau u(\tau) d\tau$

Throughout this section, we assume the following

(1) $\hat{L}$ is a compactification of L^0 and $\tilde{g}$ is the restriction of a Kähler metric of $\hat{L}$ to L^0.

(2) The range of τ contains 0.

(3) The eigenvalues of B with respect to g_τ are constant on M.

(4) The traces of the Ricci curvature r of g on each eigenvector space of B are constant.

Condition (4) here is much more general than as it is presented in [Gu18], [Gu22] in which we have:

(4)' the eigenvalues of r are constants.

Our results cover some results which appeared in later years. For example, if g has a constant scalar curvature and B has only one eigenvalue.

Abusing the language, we call the constants in (4) the *trace eigenvalues*.

Let $(z^1, ..., z^n)$ be a system of holomorphic local coordinates on M. where $n = \dim_{\mathbf{C}} M$. Using a trivialization of L^0, we take a system of holomorphic local coordinates $(z^0, ..., z^n)$ on L^0 such that $\partial/\partial z^0 = H - \sqrt{-1}\tilde{J}H$.

Here we notice that z^0 corresponds to w_1 in [Gu27] p.552, and s can be regarded as $\mathrm{Re}(z^0)$ near the point under consideration. So s is the x_1 in [Gu27] p.552. As in [Gu8], we let $\varphi = u^2$ as a function of U and we let F be the Kähler potential as in [Gu27] p.552. Then, by comparing [Gu8] Lemma 2 (or the Lemma 4 below) with [Gu27] p.552, we immediately have[1]

$$\frac{\partial^2 F}{\partial s^2} = \tilde{g}_{0\bar{0}} = 2\varphi.$$

This gives the following lemma.

Lemma 2. $2\varphi = \frac{\partial^2 F}{\partial s^2}$.

From $H = 2^{-1}\frac{\partial}{\partial s}$ we have $\frac{1}{4}(\frac{d\tau}{ds})^2 = \varphi$ and $\frac{d\tau}{ds} = 2u$. Hence

$$U = \int_0^\tau u d\tau = \int_{s(0)}^s 2u^2 ds = \int_{s(0)}^s \frac{\partial^2 F}{\partial s^2} ds,$$

that is, $\frac{\partial F}{\partial s} = y_1$ up to a constant as presented in [Gu27] p.552, i.e.,

Lemma 3. *U is the Legendre transformation of s.*

Here we use the Legendre transformation in [Gu27] instead of the moment map in [Gu8] since we need the new insight in the later sections.

Remark 1. We can see in [Gu24] that the function U here, the Legendre transformation in [Gu27] and the miraculous function U in [GC], [Gu11], [Gu21] are special cases of the parallel coordinates along the curves in the Mabuchi moduli space of Kähler metrics on compact almost-homogeneous manifolds with actions of reductive groups.

[1] The F we used in [Gu27] is the $\frac{1}{4}$ of the usual potential function in the Kähler geometry. The difference might cause a constant factor in the calculations, e.g., for Lemma 2, but does not affect our conclusions.

Let $\hat{X}_i,\ \hat{X}_{\bar{i}}$ $(0 \leq i \leq n)$ be the partial derivatives $\partial/\partial z^i,\ \partial/\partial \bar{z}^i$ on L^0 and $X_i,\ X_{\bar{i}}$ $(1 \leq i \leq n)$ be the partial derivatives $\partial/\partial z^i,\ \partial/\partial \bar{z}^i$ on M.

Lemma 4.(Cf [KS1], [KS2], [Gu8] Lemma 2) *We have that*

$$\tilde{g}_{0\bar{0}} = 2u^2, \quad \tilde{g}_{0\bar{i}} = 2u\hat{X}_{\bar{i}}\tau, \quad \tilde{g}_{i\bar{j}} = g_{i\bar{j}} + 2\hat{X}_i\tau \cdot \hat{X}_{\bar{j}}\tau \tag{3}$$

where $1 \leq i,j \leq n$. *At the point* $P \in L^0$ *considered, we can choose a local coordinate system around* $m = p(P) \in M$ *such that* $(\partial/\partial z^i)\tau = 0$ *at* m, *making* $\hat{X}_i\tau = \hat{X}_{\bar{j}}\tau = 0$ *at* P. *Then if* f *is a function on* L^0 *depending only on* τ, *we have*

$$\hat{X}_0\hat{X}_{\bar{0}}f = u\tfrac{d}{d\tau}\left(u\tfrac{df}{d\tau}\right), \quad \hat{X}_i\hat{X}_{\bar{0}}f = 0,$$

$$\hat{X}_i\hat{X}_{\bar{j}}f = -\tfrac{1}{2}uB_{i\bar{j}}\tfrac{df}{d\tau}, \tag{4}$$

if f *is a function on* L^0 *depending only on* t. *The Ricci curvature at this point is*

$$\tilde{r}_{0\bar{0}} = -u\tfrac{d}{d\tau}\left(u\tfrac{d}{d\tau}\log(u^2Q)\right), \quad \tilde{r}_{0\bar{i}} = 0,$$

$$\tilde{r}_{i\bar{j}} = p^*r_{0\ i\bar{j}} + \tfrac{1}{2}u\tfrac{d}{d\tau}\log(u^2Q) \cdot B_{i\bar{j}}, \tag{5}$$

where $Q = \det(g_0^{-1} \cdot g_\tau)$. *In particular, we have the scalar curvature*

$$\tilde{R} = \tfrac{\Delta}{Q} - \tfrac{1}{2Q}\tfrac{d}{dU}\left(\tfrac{d}{dU}Q\varphi\right) \tag{6}$$

where $\varphi = u^2$ *as a function of* U *and* $\Delta(U) = Q\sum_{i,j} r_{0\ i\bar{j}}g^{i\bar{j}}_{\tau(U)}$. *We also have* $\varphi'(\min U) = 2$ *and* $\varphi'(\max U) = -2$.

Lemma 5.(Cf. [FMS], [Mb], [Gu8] Lemma 3) *We can also regard* U *as a moment map corresponding to* $(\tilde{g},\ \tilde{J}H)$ *and* g_τ *as the symplectic reduction* *of $\tilde{g}$ at $U(\tau)$. Furthermore, $\tilde{g}$ is extremal if and only if $\tilde{R} = a + bU$ for some $a, b \in \mathbf{R}$.*

Let $M_0 = U^{-1}(\min U)$ and $M_\infty = U^{-1}(\max U)$. M_0 and M_∞ are complex sub-manifolds, since they are components of the fixed point set of $H - \sqrt{-1}\tilde{J}H$, which is semi-simple. Let D_0 be the codimension of M_0 in $\hat{L}$, D_∞ be the codimension of M_∞ in $\hat{L}$.

Lemma 6.(Cf. [Gu8] Lemma 4) *Suppose that there is another Kähler metrics $\tilde{g}^\vee$ on $\hat{L}$ in the same Kähler class, which is of form* (1) *on L^0. Let*

$$\tau^\vee,\ g^\vee,\ U^\vee,\ Q^\vee,\ \Delta^\vee,\ \varphi^\vee,\ u^\vee$$

be the corresponding metric and the corresponding functions of s. Then there is a unique corresponding $\tau^\vee$ such that $g_0^\vee = g_0$. In this case, $\min U^\vee = \min U$ (*or* $\max U^\vee = \max U$) *and $Q^\vee = Q$, $\Delta^\vee = \Delta$ hold. So we may write $D = \max U$ and $-d = \min U$. Then*

$$Q(U) = (1 + \tfrac{U}{d})^{D_0 - 1} Q_{-d}$$

$$(or\ \ = (1 - \tfrac{U}{D})^{D_\infty - 1} Q_D), \tag{7}$$

where Q_{-d} (or Q_D) is a polynomial of U such that $Q_{-d}(-d) \neq 0$ (or $Q_D(D)$ $\neq 0$) and

$$\Delta(U) = D_0(D_0 - 1)\tfrac{1}{d}(1 + \tfrac{U}{d})^{D_0 - 2} Q_{-d} \pmod{(1 + \tfrac{U}{d})^{D_0 - 1}}$$

$$(or\ \ = D_\infty(D_\infty - 1)\tfrac{1}{D}(1 - \tfrac{U}{D})^{D_\infty - 2} Q_D \pmod{(1 - \tfrac{U}{D})^{D_\infty - 1}}). \tag{8}$$

Proof: Let $\tilde{g} - \tilde{g}^\vee = i\hat{\partial}\bar{\hat{\partial}}\phi$, then

$$\tilde{g}^\vee_{i\bar{j}} = \tilde{g}_{i\bar{j}} + \tfrac{1}{2}u\tfrac{d\phi}{d\tau}B_{i\bar{j}} = (g_0)_{i\bar{j}} - (U - \tfrac{1}{2}u\tfrac{d\phi}{d\tau})B_{i\bar{j}}, \tag{9}$$

for $1 \le i, j \le n$. So at $\min U$ (or $\max U$), $\tilde g_{i\bar j} = \tilde g^{\vee}_{i\bar j}$, meaning there is a τ_0 such that $g^{\vee}_{\tau^{\vee}(\tau_0)} = g_0$. By choosing $\tau^{\vee}$ such that $\tau^{\vee}(\tau_0) = 0$, one sees that $\min U^{\vee} = \min U$ and $\max U^{\vee} = \max U$, as desired.

The last statement follows from the fact that the scalar curvature $\tilde R$ is finite on both M_0 and M_∞. Q. E. D.

In order to proceed, we will need normalization. By rescaling we have the following.

Lemma 7.(cf. [Gu9] Lemma 5) *For any given $a_1 \in \mathbf{R}$, $\tilde g$ is a generalized extremal-soliton if and only if $\tilde g^{\vee} = a_1^2 \tilde g$ is a generalized extremal solution. Furthermore, we can choose $U^{\vee} = a_1^2 U + a_2$ for any $a_2 \in \mathbf{R}$, allowing us to assume that* $\max U - \min U = 2$ *and* $\min U = -1$, *then* $\max U = 1$.

For example, if $\hat L = \mathbf{C}P^{n+1}$, then M_0 is a point, $M_\infty = M = \mathbf{C}P^n$. In this case $\hat L$ is the one point completion (compactification) of the hyperplane line bundle L over M with M_∞ as the zero section. The anticanonical line bundle is $(n+1)L$. Therefore $r_{0,ii} = n+1$ and $Q = (1+U)^n$. We have that the Kähler metric at $U = 0$ is the curvature of L and therefore $\Delta = n(n+1)(1+U)^{n-1}$.

From Lemma 5, it can be seen that, if $\tilde g$ is a generalized extremal-soliton metric, then

$$\tilde R = a + bU + c\tilde\Delta U + (\nabla(eU + f), \nabla(gU + h)) \tag{10}$$

for some $a, b, c, e, g \in \mathbf{R}$ with a $\phi_2 = cU + d,\ \phi_3 = eU + f, \phi_4 = gU + h$. We notice that here we have $\phi_1 = a + bU$ and the corresponding holomorphic vector field is determined by b.

By Lemma 4 we have that

$$\tilde\Delta f = \tilde g^{\bar\alpha\beta} \hat X_{\bar\alpha} \hat X_\beta f$$

$$
\begin{aligned}
&= \tilde{g}^{\bar{0}0}\hat{X}_{\bar{0}}\hat{X}_0 f + \tilde{g}^{\bar{a}0}\hat{X}_{\bar{a}}\hat{X}_0 f + \tilde{g}^{\bar{0}a}\hat{X}_{\bar{0}}\hat{X}_a f + \tilde{g}^{\bar{a}b}\hat{X}_{\bar{a}}\hat{X}_b f \\
&= \frac{1}{2u^2}(\hat{X}_{\bar{0}}\hat{X}_0 f) + 0 + 0 + \tilde{g}^{\bar{a}b}(\hat{X}_{\bar{a}}\hat{X}_b f) \qquad (11)\\
&= \frac{1}{2u^2}u\frac{d}{d\tau}(u\frac{d}{d\tau}f) + g_\tau^{\bar{a}b}(-2^{-1}u\frac{df}{d\tau}B_{b\bar{a}}) \\
&= 2^{-1}\frac{d}{dU}(\varphi(U)\frac{d}{dU}f) - 2^{-1}\varphi(U)(\frac{d}{dU}f)g_t^{\bar{a}b}B_{b\bar{a}} \\
&= 2^{-1}\frac{d}{dU}(\varphi\frac{d}{dU}f) + 2^{-1}\varphi(\frac{d}{dU}f)\frac{1}{Q}\frac{d}{dU}Q \\
&= \frac{1}{2Q}\frac{d}{dU}(\varphi Q\frac{d}{dU}f),
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla f_1, \nabla f_2) &= \tilde{g}^{\bar{\alpha}\beta}((\hat{X}_{\bar{\alpha}}f_1)(\hat{X}_\beta f_2) + (\hat{X}_{\bar{\alpha}}f_2)(\hat{X}_\beta f_1)) \\
&= \tilde{g}^{\bar{0}0}((\hat{X}_{\bar{0}}f_1)(\hat{X}_0 f_2) + (\hat{X}_{\bar{0}}f_2)(\hat{X}_0 f_1)) \qquad (12)\\
&= \frac{1}{2u^2}\cdot 2(u^2\frac{d}{dU}f_1)(u^2\frac{d}{dU}f_2) \\
&= \varphi f_1' f_2'.
\end{aligned}
$$

From these we get

$$
\begin{aligned}
\tilde{\Delta}\phi_2 + (\nabla\phi_3, \nabla\phi_4) &= \frac{1}{2Q}\frac{d}{dU}(\varphi Q\frac{d}{dU}(cU+d)) + eg\varphi \\
&= \frac{c}{2Q}\frac{d}{dU}(\varphi Q) + eg\varphi \\
&= \tilde{R} - (\int_{-1}^1 \tilde{R}QdU)/(\int_{-1}^1 QdU) - \phi_1 \qquad (13)\\
&= \frac{\Delta}{Q} - \frac{1}{2Q}\frac{d}{dU}(\frac{d}{dU}Q\varphi) - (a+bU).
\end{aligned}
$$

Let $m = \int_{-1}^1 \tilde{R}QdU / \int_{-1}^1 QdU$, $\alpha = \int_{-1}^1 QdU$ and $\beta = \int_{-1}^1 UQdU$. Then we have that

$$
\begin{aligned}
\int_{-1}^1 \tilde{R}QdU &= \int_{-1}^1 [\Delta - 2^{-1}\frac{d}{dU}(\frac{d}{dU}Q\varphi)]dU \\
&= \delta - 2^{-1}\frac{d}{dU}(Q\varphi)|_{-1}^1
\end{aligned}
$$

$$
\begin{aligned}
&= \delta - 2^{-1}[Q\frac{d}{dU}\varphi + \varphi\frac{d}{dU}Q]|_{-1}^{1} \\
&= \delta - 2^{-1}[Q(1)\cdot(-2) - Q(-1)\cdot 2] \\
&= \delta + Q(1) + Q(-1),
\end{aligned}
\tag{14}
$$

where $\delta = \int_{-1}^{1} \Delta dU$. Therefore, $m = (\delta + Q(-1) + Q(1))/\alpha$.

Hence we obtain

$$
c(\varphi Q)' + 2eg(\varphi Q) = -(Q\varphi)'' - 2(a + bU)Q + 2\Delta. \tag{15}
$$

Let $f = \varphi Q,\ F = 2\Delta - 2(a + bU)Q,\ l = 2eg$, then

$$
f'' + cf' + lf = F. \tag{16}
$$

Assume that $c^2 - 4l \geq 0$, then the corresponding homogeneous equation has a solution $f_h = e^{kU}$ with $k = \frac{-c-\sqrt{c^2-4l}}{2}$ or $\frac{-c+\sqrt{c^2-4l}}{2}$ to be one of the solutions of the characteristic equation. We notice that $k' = -k - c$ is the other solution.

Let $f = ge^{kU}$, then $e^{kU}(g'' + 2kg' + cg') = F$ and

$$
f = e^{kU}\int e^{-(2k+c)U}[\int e^{(k+c)U}FdU]dU.
$$

Let $U = -1$, we have $f(-1) = 0$ and therefore,

$$
f = e^{kU}\int_{-1}^{U} e^{-(2k+c)y}[\int_{-1}^{y} e^{(k+c)x}F(x)dx + c_1]dy.
$$

But

$$
(fe^{-kU})'(-1) = (\varphi' Qe^{-kU})(-1) = 2Q(-1)e^{k} = e^{2k+c}c_1.
$$

$$
2Q(-1)e^{-k-c} = c_1.
$$

Let $U = 1$, we have

$$
(\varphi' Qe^{-kU})(1) = -2Q(1)e^{-k} = e^{-2k-c}[\int_{-1}^{1} e^{(k+c)x}F(x)dx + c_1].
$$

$$-2Q(1)s = -2a\alpha_e - 2b\beta_e + 2\delta_e + 2Q(-1)s^{-1}.$$

Therefore,

$$a = [\delta_e + Q(1)s + Q(-1)s^{-1} - b\beta_e]/\alpha_e = m_e - b\beta_e/\alpha_e,$$

where

$$s = e^{k+c} = e^{-k'}, \alpha_e = \int_{-1}^{1} e^{-k'U} Q(U)dU, \beta_e = \int_{-1}^{1} e^{-k'U} UQ(U)dU,$$

$$\delta_e = \int_{-1}^{1} e^{-k'U} \Delta(U)dU,$$

$$m_e = \frac{\delta_e + Q(1)s + Q(-1)s^{-1}}{\alpha_e}.$$

All these coefficients are only depended on k'. We fix k'.

We have that

$$Q\varphi = e^{kU}[\int_{-1}^{U}[\int_{-1}^{y} e^{-k'x}[-2(a+bx)Q(x) + 2\Delta(x)]dx \\ +2Q(-1)s^{-1}]e^{(k'-k)y}dy]. \quad (17)$$

We denote the right side by $\Phi(U)$.

Let $U = 1$, we have the equation

$$0 = \int_{-1}^{1}[\int_{-1}^{y} e^{-k'x}[-2(a+bx)Q(x) \\ +2\Delta(x)]dx + 2Q(-1)s^{-1}]e^{(k'-k)y}dy. \quad (18)$$

Let

$$p(U) = \int_{-1}^{U} 2e^{-k'x}[\Delta(x) - (a+bx)Q(x)]dx + 2Q(-1)s^{-1}, \quad (19)$$

then by $P(1) = -2Q(1)s$ we have:

$$p(U) = \int_{U}^{1} 2e^{-k'x}[(a+bx)Q(x) - \Delta(x)]dx - 2Q(1)s. \quad (20)$$

By the last statement of Lemma 6, we know that $p(U)$ is nonnegative near -1 and non-positive near 1. Since the right side of the above equation (18) turns to $-\infty$ (or $+\infty$) when $-k$ turns to $+\infty$ (or $-\infty$), there is at least one solution c. We pick up the *smallest* c, and have

Lemma 8. *For any b, k', there is a solution c for* (18)*.*

Proposition 5.(cf. [KS1], [Gu8] Lemma 6) *There is a generalized extremal-soliton metric in the Kähler class of $\tilde{g}$ for a given b if $\varphi^0(U) = \Phi(U)/(Qe^{cU})$ is positive on $(-1,\ 1)$.*

If we let $b = 0$ we have the extremal part free metrics including those (generalized) quasi-einstein metric, as in [Gu9].

To obtain metrics with same $k = k'$ including extremal metrics and the generalized Kähler-Ricci soliton metrics in [Na] we just let $c = -2k$ and solve (18) to find a and b as we did in [Gu8] p.2259 (see Lemma 5 there). Let $k - k' = 0$, $\delta_{1e} = \int_{-1}^{1} e^{-k'x} x\Delta(x)dx, \gamma_e = \int_{-1}^{1} e^{-k'x} x^2 Q(x)dx$, then (18) becomes

$$\begin{aligned}
0 = \int_{-1}^{1}\int_{-1}^{y} e^{-k'x}((a+bx)Q(x) - \Delta(x))dxdy - 2Q(-1)s^{-1} \\
&= \int_{-1}^{1}\int_{x}^{1} e^{-kx}((a+bx)Q(x) - \Delta(x))dydx - 2Q(-1)s^{-1} \\
&= \int_{-1}^{1} e^{-kx}(1-x)((a+bx)Q - \Delta)dx - 2Q(-1)s^{-1} \\
&= a\alpha_e + b\beta_e - a\beta_e - b\gamma_e - \delta_e + \delta_{1e} - 2Q(-1)s^{-1} \\
&= m_e\alpha_e + \delta_{1e} - 2Q(-1)s^{-1} - a\beta_e - b\gamma_e - \delta_e \\
&= m_e\alpha_e + \delta_{1e} - 2Q(-1)s^{-1} - m_e\beta_e + \frac{b}{\alpha_e}(\beta_e^2 - \alpha_e\gamma_e) - \delta_e.
\end{aligned}$$

We notice that the coefficient of b can not be zero since

$$\alpha_e t^2 + 2\beta_e t + \gamma_e = \int_{-1}^{1} e^{-kU}(t+U)^2 Q dU > 0$$

for any t. Therefore, there is a unique solution of b. Similarly, for any different pair of k, k', we can also solve b uniquely.

Lemma 9. *For any pair of k, k', there is a solution b for (18).*

Lemma 10.(cf. [Gu8], [Gu9], [Gu22]) *If r has nonnegative trace eigenvalues, then for a given b we have:*

(1) Φ *as above is always positive on* $(-1,\ 1)$.

(2) *the solution c in Lemma 8 is unique.*

Proof: Since the derivative of $Q\varphi e^{-kU}$ is $p(U)e^{(k'-k)U}$, we have that

$$\frac{d}{dU}(e^{-(k'-k)U}\frac{d}{dU}(Q\varphi e^{-kU})) = 2e^{-k'U}(\Delta(U) - (a+bU)Q(U)). \qquad (21)$$

Diagonalizing B, we see that Q is a product of polynomials of degree 1. Let

$$-a_1^{-1} < ... < -a_p^{-1} < b_1^{-1} < ... < b_q^{-1},$$

denote the distinct roots of Q for which some corresponding Ricci curvature $r_{i\bar{i}}$ is nonzero, where a_i, b_j are positive. Let

$$S(U) = U\prod_{i=1}^{p}(1+a_iU)\prod_{j=1}^{q}(1-b_jU),$$

$$T(U) = UQ(U)/S(U)$$

and

$$\Psi(U) = \left(e^{k'U}\frac{d}{dU}(e^{-(k'-k)U}\frac{d}{dU}[(Q\varphi)(U)e^{-kU}])\right)/T(U).$$

Then Ψ is a polynomial of degree $p+q+1$ and $\Psi(a) = -r_aS'(a)$ for a a root of $S(U)/U$, where $r_a \in \mathbf{R}^+$ be the corresponding trace eigenvalue of the ricci curvature, since r is nonnegative. We can see that $S'(a) \neq 0$ and > 0 (or < 0) if and only if $S' < 0$ (or > 0) for the root before a and after a (if such exists). Because $S'(0) > 0$, we have $S'(-a_p^{-1}) < 0$ and $S'(b_1^{-1}) < 0$, that is, $\Psi(-a_p^{-1}) > 0$ and $\Psi(b_1^{-1}) > 0$. Now there are $p-1$ (or $q-1$) zero points of Ψ in $(-a_1^{-1}, -a_p^{-1})$ (or in (b_1^{-1}, b_q^{-1})) if $p,\ q$ are not zero (one may also check the case of $q = 0$ or $p = 0$). If φ has some nonpositive points in $(-1, 1)$, then in $(-1, 1)$, $Q\varphi$ has at least two maximal and one minimal points since $\varphi(-1) = \varphi(1) = 0,\ \varphi(-1+\epsilon) > 0,\ \varphi(1-\epsilon) > 0$ for ϵ small enough. So we get that there are at least 4 zero points of Ψ in $(-a_p^{-1}, b_1^{-1})$. Ψ has at least $(p-1)+(q-1)+4 = p+q+2$ zeros, a contradiction. Thus we have (1).

For (2) we only need to prove that the function $p(U)$ in (19) has only one zero point in $(-1, 1)$. If $p(U)$ has at least two zero points in $(-1, 1)$, then $p(U)$ has at least three zero points, since it is nonnegative near -1 and non-positive near 1. So Ψ has at least 4 zero points in $(-a_p^{-1}, b_1^{-1})$, which is a contradiction.

Q. E. D.

Corollary 1.(cf. [Gu8], [Gu9]) *For every Kähler class of a compact almost homogeneous manifold with two ends, there exists generalized extremal-soliton metric for any given pair b, k'. In particular, there is always an extremal metric and a (generalized) quasi-Einstein metric, as well as the extremal-soliton metric for any given b.*

Proof: Since every compact Kähler almost homogeneous space is a completion of a $\mathbf{C}^*$ bundle over a product of a torus A and a C–space N with two homogeneous Kähler spaces as two ends [HSn] Theorem 3.2), a maximal compact subgroup of the identity component of the automorphism group of this manifold is $G = A \times S \times S^1$, where A is also the Albanese torus and S is a maximal compact subgroup of the identity component of the automorphism group of N. For any Kähler metric g, $g_G = \int_{h\in G} h^* g dm$ is a Kähler metric of form (2), where m is the Haar measure on G; it is invariant under G. Also the Ricci curvature of $A \times N$ is nonnegative, the condition in our assumption follows from the well-known property of the invariant cohomology (1,1) classes for these manifolds (see [DG3] p. 326 proof of the Proposition).

Q. E. D.

In the case of $b = 0$ we can have a little bit more. We say that the trace eigenvalues (see three lines under the condition (4)') is *nonnegative at one side* if the trace eigenvalues are nonnegative for all $-a_i^{-1}$ or for all b_j^{-1} in the proof of the Lemma 9. We have:

Corollary 2.(cf. [Gu8], [Gu9]) *If the trace eigenvalues only change sign once and nonnegative at one side, then for any given k', there is an extremal part free (i.e., $b = 0$) generalized extremal-soliton metric. In particular, the completion of the Hodge line bundle over a Hodge manifold with a constant scalar curvature admits (generalized) quasi-einstein metric.*

Proof: In the proof of Lemma 10, if $b = 0$, then the polynomial $e^{k'x}p'$ is one degree lower. Therefore by ignoring the root at which the trace eigenvalue is negative and changing the sign, the proof still goes through. By the argument in the proof of the Theorem 5.4 in [KS1] p. 177, we have our Corollary.

Q. E. D.

This Corollary also says that there is more chance for the existence of the extremal free metrics than that of others.

8.3 The existence of the generalized Kähler-Ricci solitons

In the last section, Lemma 8, 9, 10 give the existence of metrics for many cases, once (18) holds. Lemma 9 further gives a direction in finding the metrics in [Na]. In this subsection, we should show that our argument in the last subsection does give the solution of the existence of the metrics in [Na], a five years question when [Gu23] was done according to the date he submitted his paper. Now we let $k = k'$. A calculation of b in Lemma 9 shows that

$$b = [(\alpha_e + \beta_e)Q(-1) + s(\beta_e\delta_e - \alpha_e\delta_{1e} + sQ(1)(\beta_e - \alpha_e)] \quad (22)$$
$$/ \quad [s(\beta_e^2 - \alpha_e\gamma_e)].$$

We then have:

Lemma 11. *When $-k$ turns to $+\infty$, b is dominated by k^2. That is, $\lim_{k\to-\infty} b/k^2 = 1$.*

Proof: We let

$$L_m = \int_0^1 e^{-kx}(1-x)^m dx.$$

Then when $-k$ turns to $+\infty$,

$$L_0 = \frac{s-1}{-k}$$

is equivalent to $s/(-k)$.

$$L_m = \frac{1}{-k}\int_0^1 (1-x)^m d(e^{-kx}) = \frac{1}{-k}[1 + mL_{m-1}]$$

is equivalent to $\frac{m}{-k}L_{m-1}$ or $\frac{m!}{(-k)^m}L_0$ and so $\frac{m!s}{(-k)^{m+1}}$.

Now, if $Q(1)$ is not zero, then $s^2(\beta_e - \alpha_e)Q(1)$ dominates the numerator. It is equivalent to $\frac{-s^3(Q(1))^2}{k^2}$. This can be done by express $Q(x)$ by its Taylor series around 1. $s(\beta_e\delta_e - \alpha_e\delta_{1e})$ is in the level of $\frac{s^3}{-k^3}$. While $(\alpha_e + \beta_e)Q(-1)$ is in the level of $\frac{s}{-k}$.

Let

$$Q(x) = (1-x)^{m-1}Q_1(1) + a(1-x)^m + b(1-x)^{m+1} \ (\mathrm{mod}(1-x)^{m+2}),$$

then the denominator $s(\beta_e^2 - \alpha_e\gamma_e)$ is equivalent to $-s\frac{m(Q_1(1))^2}{k^2}L_{m-1}^2$. In particaular, when $m = 1$, i.e., $Q(1) \neq 0$, we have $-s^3\frac{(Q(1))^2}{k^4}$. Therefore, b is equivalent to k^2 when $-k$ turns to $+\infty$ as desired.

Now if $Q(1) = 0$, then $s(\beta_e\delta_e - \alpha_e\delta_{1e})$ dominate the numerator. Let

$$Q(x) = (1-x)^{m-1}Q_1(1) + a(1-x)^m \ (\mathrm{mod}(1-x)^{m+1})$$

and

$$\Delta(x) = m(m-1)(1-x)^{m-2}Q_1(1) + c(1-x)^{m-1} \ (\mathrm{mod}(1-x)^m),$$

then numerator is equivalent to $sm(m-1)(Q_1(1))^2L_{m-1}L_{m-2}/k$. Therefore, b is equivalent to $-(m-1)kL_{m-2}/L_{m-1}$ or k^2, as desired.

Q. E. D.

Similarly, when $-k$ turns to $-\infty$, b is dominated by $-k^2$. Therefore, $b = Ak$ with any given constant A will always have a solution. We have:

Corollary 3. *In the case as in Lemma 10, there is always a generalized Kähler-Ricci soliton of [Na] in any given Kähler class.*

9 Characteristic properties of complex homogeneous spaces as complex manifolds

For a Riemann surface, i. e., a complex manifold of complex dimension one, its universal covering is one of the three complex one dimensional homogeneous spaces: the Riemann sphere $\mathbf{C}P^1 = \mathbf{C} \cup \{\infty\} = S^2$; $\mathbf{C}$ and the unit ball $B = \{z \in \mathbf{C}|_{|z|<1}\}$.

In the higher dimension case, we also have three special homogeneous spaces: $\mathbf{C}P^n$; $\mathbf{C}^n$ and the unit ball $B^n = \{z \in \mathbf{C}^n|_{|z|<1}\}$.

There are also Hermitian symmetric spaces [Hel]; projective rational homogeneous spaces [Bo], [Ti]; the Wolf compact homogeneous quaternion spaces and complex homogeneous contact spaces [Wo1]; the quotient of the complex ball.

Question 2. How do we characterize these homogeneous spaces or quotient of these spaces among the complex Hermitian manifolds or their related Riemannian manifolds?

It was proved by Siu-Yau [SYau] that if a Kähler manifold M has a positive bisectional curvature, then M is bi-holomorphically to a complex projective space. On the other hand, Professor Mori [Mor] proved that

Theorem 38. *If a compact projective manifold has an ample holomorphic tangent bundle, then M is a projective space.*

When M is Kähler-Einstein and has nonnegative bisectional curvature, Professor N. Mok and J. Q. Zhong [MZ] proved that:

Theorem 39. *If a compact complex manifold M is Kähler-Einstein with nonnegative bisectional curvature, then M is isometric to a Hermitian symmetric space.*

With a help of Mori's method, Professor Mok [Mok] even went further and proved:

Theorem 40. *If a compact complex manifold M has nonnegative bisectional curvature with the Ricci curvature being positive at one point, then M is isometric to a product of several Hermitian symmetric spaces such that except for the factors of projective spaces, the metrics on each factor is just the Hermitian symmetric metric.*

For the positive curved case, one have the Campana-Peternell conjecture [CP]:

Conjecture 9. A compact projective manifold with a nef tangent bundle is a projective rational homogeneous space.

This conjecture was proven by Professor A. Kanemitsu [Ka] up to the complex five dimension.

Question 3. Does any co-homogeneity one compact Kähler manifold have a nef tangent bundle?

A related question is about the classification of compact quaternion manifolds with positive Ricci curvature [PS]:

Conjecture 10. A compact quaternion manifold with a positive Ricci curvature is a Wolf compact homogeneous quaternion space.

Another version of this conjecture is that

Conjecture 11. A compact complex contact Fano manifold is homogeneous.

These two conjectures were proven to be true up to real dimension 8. Notice that a quaternion Kähler manifold has a real dimension $4k$ with k a positive integer. Therefore, the situation for these two conjectures is not very positive at all, and is up to any imagination as far as now. Some serious efforts were done in [HH] and [He]. However, there was a gap in [HH]. See [HH1], for example. We notice that a compact complex Fano contact manifold is Kähler-Einstein. One needs more efforts to understand these manifolds.

There is also some characteristic conjecture for $\mathbf{C}^n$ (cf. [CZ]).

Conjecture 12. Any complete but noncompact Kähler manifold with positive bisectional curvature is bi-holomorphic to $\mathbf{C}^n$.

Considerably works have been done, for example, see [Liu]. The $U(n)$ co-homogeneity one examples of $\mathbf{C}^n$ have been classified in [WZ]. See also, for example, [Yang], [YZ].

Question 4. Does this Yau's conjecture (conjecture 12) holds for cohomogeneity one Kähler metrics?

There is also some characteristic conjecture for B^n.

Here we consider the possibly compact quotient of B^n.

A similar condition is that the holomorphic bisectional curvature is negative. Even for the compact surface case, there are examples of negative holomorphic bisectional curvature which is NOT a quotient of B^2. Therefore, one could only consider the Kähler-Einstein version. The first effort was done by Professors Siu and Yang in [SY]. Let K_{max}, K_{min}, K_{av} be the maximal, minimal, average of the holomorphic sectional curvature. There they proved that

Theorem 41. *Let M be a compact Kähler-Einstein surface of non-positive Ricci curvature. If M has non-positive holomorphic bisectional curvature and $K_{av} - K_{min} \leq a[K_{max} - K_{min}]$ with $a < \frac{2}{3(1+\sqrt{6/11})}$. then M is a quotient of either the complex unit ball or a complex plane.*

In [HGY] we improved that this is true for $a \leq \frac{2}{3(1+\sqrt{1/6})}$ and conjecture that this is true for $a = 1/2$. Eventually, in [Gu19], we were able to prove the conjecture and actually we do not need the non-positivity of the bisectional curvature. We actually proved:

Theorem 42. *Let M be a compact Kähler-Einstein surface of non-positive Ricci curvature. If we have a condition as in Theorem 41 with $a = 1/2$, then M is a quotient of either a complex unit ball or a complex plane.*

We do not expect the manifold to have B^2 as the universal covering if we do not have the condition $a = 1/2$ and expect that $a = 1/2$ is sharp even for manifolds with negative bisectional curvatures. However, as Professor Bun Wong pointed out, it should be true if we (only) have negative sectional curvature. More precisely, one have:

Conjecture 13. If a compact Kähler-Einstein surface has negative sectional curvatures, then M is a quotient of a complex two dimensional unit ball in $\mathbf{C}^2$.

10 Compact locally conformal Kähler manifolds

Definition 2. Let M be a complex manifold, h be an Hermitian metric. If h is locally conformal to a Kähler metric, i.e., for any point $m \in M$ there is an open neighborhood O such that on O, $g = e^f h$ is a Kähler metric with a function f, we say that (M, h) is a locally conformal Kähler. That is, let $\omega(\ ,\ ) = h(\ , J\ )$, then $d(e^f \omega) = 0$.

In 2012, Professor Hasegawa et al proved in [HKa] (see also [GMO]):

Theorem 43. *A compact homogeneous locally conformal Kähler manifold $M = G/H$ is a complex* 1*-dimensional torus bundle over a rational projective homogeneous space.*

It is striking that Theorem 1 and Theorem 43 are so close, yet the proofs of Theorem 43 in [HKa] and [GMO] are so complicated (to us). Therefore, at the end of 2012, in [Gu20] we took a simple approach from Theorem 1 to Theorem 43. As people could see, our machinery in the compact complex homogeneous spaces theory is powerful enough to solve this problem.

We also filled in the details of the argument in [HKa] (our argument was earlier than [GMO]) from a more complex homogeneous space and higher dimensional aspect.

We also took Vaisman's earlier approach in [V1, 2] into our account.

We proved the following result of a classification of the compact homogeneous locally conformal Kähler metrics:

Theorem 44. *The manifold is co-homogeneity one under the action of the semi-simple part S of the Lie group, i.e., S has hyper-surface orbits. $M = N \times S^1$ as a homogeneous space (but not necessary as a Riemannian manifold) with N the S orbits. Both the original locally conformal Kähler metrics and the related Kähler metrics are co-homogeneity one under the S action. Moreover, M is a complex one dimensional torus bundle, over a rational homogeneous projective space, which is a quotient of a $\mathbf{C}^*$ bundle by some action e^a with $\mathrm{Re} a \neq 0$. The metrics on M, as a submersion, is completely determined by the Kähler class of the base manifold and the Kähler class, as the restriction, of the fiber.*

At the beginning of Sept. 2012, Professor Hasegawa visited us and showed us their work on the classification of compact homogeneous locally conformal Kähler manifolds. They seemly had a difficulty to publish. Although not being a referee, as this was an important result, we came up these proofs.

Let us give a quick description for the case when $\dim_{\mathbf{C}} M = 2$. According to Theorem 1, M is either a complex torus of complex dimension 2 or a complex one dimensional torus bundle over $\mathbf{C}P^1$. The simply connected case was excluded by the locally conformal Kähler (but nonkähler) condition. For a complex torus, all the homogeneous hermitian metrics are actually Kähler. Therefore, we obtain the Theorem 37. We also notice that the manifold M might not be a Hopf surface itself but a finite covering of it. However, any homogeneous hermitian metrics comes from the Hopf surface covering and therefore, most the results about the Hopf surfaces apply to M.

Also, in this case, the semi-simple part of the group comes down to be an effective isometric subgroup on $\mathbf{C}P^1$ and therefore is $SU(2)$. This is because that the isometric group on the torus fiber is abelian. Now, the dimension of $SU(2)$ is 3 and the action is co-homogeneity one, i.e., it has hypersurfaces as orbits. To be homogeneous for $M = G/H$, the center of G is at least of a real dimension 1 and at most of real dimension 2. G is locally either $SU(2) \times \mathbf{R}$ or $SU(2) \times \mathbf{C}$. Moreover, the $SU(2)$ orbits are S^1 bundles over $\mathbf{C}P^1$. That leads to a line bundle and therefore a $\mathbf{C}^*$ bundle M^* over $\mathbf{C}P^1$. M^* is a covering of M. Let $M_a = (\mathbf{C}^2 - \{0\})/(a)$, $(a) = \{a^i|_{i\in\mathbf{Z}}\}$ with $a \in \mathbf{C}^*$ and $|a| \neq 1$, be a Hopf surface, then $M_a^* = \mathbf{C}^2 - \{0\}$. Then M_a^* is a finite covering of M^*. We denote M_a^* simply by $\mathbf{C}^{2,*}$. The covering map $\mathbf{C}^{2,*}$ to M^* introduces a Hopf surface covering over M (introduced by the corresponding S^1 bundles, which introduces a finite torus covering for each fiber).

This is our Theorem 44 for the complex dimension 2 case.

In a following paper, we were able to classify compact co-homogeneity one locally conformal Kähler manifolds.

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