Type I Almost-Homogeneous Manifolds of Cohomogeneity One—I

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Abstract

In this paper, we generalize our results in [GC, Gu4, 5] on the existence of Kähler metrics with constant scalar curvatures to the general type I almost homogeneous manifolds of cohomogeneity one. We actually carry out all the results in [Gu5] to the type I cases. We prove that the existence of Kähler metrics with constant scalar curvatures is equivalent to the negativity of an integral, and is also equivalent to the geodesic stability. We also prove the existence of smooth geodesic connecting any two given metrics on the Mabuchi moduli space of Kähler metrics, which leads to the uniqueness of our Kähler metrics with constant scalar curvatures if they exist. The similar proofs of the results other than the existence of Kähler metrics with constant scalar curvatures for the type II cases are more complicated and will be done in [Gu6]. In particular, we also deal with the existence of Kähler-Einstein metrics on these manifolds and obtain a lot of new Kähler-Einstein manifolds as well as Fano manifolds without Kähler-Einstein metrics. With applying our results to the canonical circle bundles we also obtain Sasakian manifolds with or without Sasakian-Einstein metrics.

In memory of Professor Hong-You Wu

Key Words: Kähler manifolds, Einstein metrics, Ricci curvature, constant scalar curvature, fibration, almost-homogeneous, cohogeneity one, semisimple Lie group, semisimple elements, root vectors, duality, geodesic stability, fourth order equations, nonlinear second order equations, sasakian Einstein, Calabi-Yau metrics.

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1. Introduction

I met Hong You in the Institute of Mathematics, Academia Sinica, Beijing China around the middle of the 1980's as graduate students. Following Professor Zhong Jia-Qiing, I started to study the Kähler Einstein metrics. Hong You came to USA in the

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summer of 1986 as a graduate student of Professor Dorfmeister. In the summer 1987, I also came to USA and became another graduate student of Professor Dorfmeister. He met me at the airport and we were close friends. He was also interested in the Einstein geometry. In the summer of 1988, he moved to another university with Professor Dorfmeister. And to pursuit my Kähler-Einstein dream, I eventually moved to Berkeley and became one of Ph. D. students of Professor Kobayashi. While in Berkeley, then Princeton, I worked very hard but did not get any new example of Kähler-Einstein manifold (but see [Gu1,2,11] and our Lemma 5 in the last section, also that in [Gu9,12]) until we applied the Hilbert scheme constructions in [GC](see [Gu3]). I dedicate this paper to the memory of Hongyou Wu.

One major problem in differential geometry is to find compact Riemannian Einstein manifolds. A Riemannian manifold is Einstein if the Ricci curvature is proportional to the Riemann metrics.

In general, this is a very difficult problem. When the manifold is Kähler, it is much easier. In that case, after rescaling, the Ricci class is either the negative of the Kähler class, or the zero class, or the Kähler class. The case in which the Ricci class is the same as the Kähler class is still open. Although considerable progress has been made in the area of the existence of Kähler Einstein metrics, see [Ya], [Su], [Ti], [Gu10], [Do], [WxZu], [TZ] (also [Mt], [CO], [Kb1] etc. for obstructions) for example, the examples for the positive first Chern class case are still very isolated. There is no very clear picture what is the difference between the Kähler Einstein ones and the Kähler non-Einstein ones. Even after Perelman's stunning breakthrough on the Kähler Ricci flow, see [TZ], we still do not have many general and systematic methods to find Kähler Einstein metrics. In this paper, we shall finish all the type I cohomogeneity one cases, that lead to the finishing of all the cohomogeneity one cases. It provides us with many new examples. It also shows the clear relation between the existence and the geodesic stability.

The Kähler-Einsten metrics is a special case of Kähler metrics with constant scalar curvatures.

Main Theorem *For any simply connected type I compact Kähler complex almost homogeneous manifolds of cohomogeneity one with a hypersurface end, there is a Kähler metric with a constant scalar curvature in a given Kähler class if and only if the condition (7) holds.*

The condition (7) can be found as an integral inequality in the seventh section as a necessary condition. It will be appeared again as (16) in the eighth section. We should see in the second part that it is a checkable topological condition and a polynomial of some topological constants. For the higher condimensional end cases and the general cases, similar results are obtained in the second part of this paper.

As an application, considering the canonical circle bundle, we also obtained Sasakian manifolds with and without Sasakian-Einstein metrics (see [BG1 Theorem 2.4 (iv)], also [Kb2], [WmZi]). By using the Riemannian cone of the Sasakian-Einstein metrics we obtained many open Calabi-Yau manifolds (see [BG2 p.374 Corollary 11.1.8]).

Therefore, we finished all the possible cases in which the existence of the extremal

metrics could be reduced to an ordinary differential equation problem. We also give many examples for both the stable and the unstable cases in the second part of this paper, as we promised in [Gu5].

For the definition and the classification of compact almost homogeneous of cohomogeneity one, on emight look at the papers [Ak], [HS]. The more detail classification can be found in [Gu5, 12]. For the type I cases, one only need to consider three classes of manifolds, i.e., manifolds with the typical fibers of (1) the second and third cases, (2) the fourth case, (3) the eighth and ninth cases in [Ak p.67]. There are five different cases. We have to separate them into each cases in this paper, just like what people did for those four classical Hermitian symmetric domains.

For (1), the fiber is either $\mathbf{C}P^n$ the complex projective space or Q^n the hyperquadric with a $SO(n+1, \mathbf{C})$ almost homogeneous action. We denote the fiber by $F = F(OP_n)$ or $F = F(OQ_n)$. $F(OQ_n)$ is the double branched covering of $F(OP_n)$ along the exceptional divisor Q^{n-1}. For (2), the fiber is $Gr(2k, 2)$ with a $Sp(k, \mathbf{C})$ almost homogeneous action. We denote the fiber by $F = F(Gr_k)$. For (3), the fiber is either $\mathbf{C}P^7$ or Q^7 with a $Spin(7, \mathbf{C})$ almost homogeneous action. We denote the fiber by $F = F(Sp_7^p)$ or $F = F(Sp_7^q)$. And we denote them by $F(Sp_7)$ if there is no confusion.

2. The Complex Structures of the Type I Almost Homogeneous Manifolds

In this section, we shall deal with the complex structure of the type I almost homogeneous manifolds. Let us recall some basic notations of the Lie algebras.

Let G be the complex Lie group action and S be the connected complex Lie subgroup acting on a given fiber. According to [Gu5 p.283 Theorem 12.1(ii)], a compact complex almost homogeneous manifold of cohomogeneity one is type I if and only if the fiber F is one of (1) the second and third case with $n \geq 3$, (2) the fourth case, (3) the eight and ninth cases, (4) the fifth case in [Ak p.67].

The fiber F in (4) has $S = F_4$, so $G = F_4 = S$, that is, $M = F$ is homogeneous. Therefore, every Kähler class of M has a metric with constant scalar curvature. So, we do not need to do anything with (4).

To make the things simpler, we look at three special possible fiber cases [Ak p.67] first:

(1) $F = F(OP_n)$: The third case with $n \geq 3$. We shall treat $G = S = SO(n+1, \mathbf{C})$ and $X = F = \mathbf{C}P^n$ first. The corresponding compact rank one symmetric space is the real n dimensional real projective space. It has an equivariant branched double covering Q^n of the second case.

(2) $F = F(Gr_k)$: The fourth case with an standard $G = S = Sp(k, \mathbf{C})$ action on the manifold $X = F = Gr(2k, 2)$. The corresponding compact rank one symmetric space is the quarterion projective space.

(3) $F = F(Sp_7)$: The ninth case with a $G = S = Spin(7, \mathbf{C})$ action on $X = F = \mathbf{C}P^7$. This is the restriction of (1) with $n + 1 = 8$ to the complex Lie subgroup $Spin(7, \mathbf{C})$. It has an equivariant branched double covering Q^7 of the eighth case.

3

In the case (1), we consider the case $n = 3$ first. $G = S = SO(4)$ with roots $\pm(e_1 \pm e_2)$. The roots $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_1 + e_2$ constitute a fundamental root system of this Lie algebra.

$G = D_2$ has a Cartan subalgebra

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & -a_1 i & 0 & 0 \\ a_1 i & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 i \\ 0 & 0 & a_2 i & 0 \end{bmatrix} \bigg|_{a_1, a_2 \in \mathbf{C}} \right\}.$$

e_1 corresponds to $(a_1, a_2) = (1, 0)$ and e_2 corresponds to $(a_1, a_2) = (0, 1)$. The open orbit is generated by the action of D_2 on $[1, 0, 0, 0]^T$. $U = S(O(1, \mathbf{C}) \times O(3, \mathbf{C}))$.

We let

$$E_{e_1 \pm e_2} = \begin{bmatrix} 0_{2 \times 2} & A \\ -A^T & 0_{2 \times 2} \end{bmatrix} \quad \text{for} \quad A = \frac{1}{2} \begin{bmatrix} 1 & \pm i \\ i & \mp 1 \end{bmatrix}, \quad E_{-\alpha} = \bar{E}_\alpha^T.$$

$F_\alpha = E_\alpha - E_{-\alpha}$, $G_\alpha = i(E_\alpha + E_{-\alpha})$, then $[F_\alpha, G_\alpha] = 2H_\alpha$ and $[H_\alpha, F_\alpha] = i(H_\alpha, H_\alpha)_0 E_\alpha$ where $(\ ,\ )_0$ is the standard inner product such that $(e_i, e_i)_0 = 1$.

Similar to the cases of [Gu8,9,12], we consider the semisimple orbit generated by $-iH = e_1$. Now, $p_s = \exp(-isH)[1, 0, 0, 0]^T = [1, i \tanh s, 0, 0]^T$, $p_\infty = [1, i, 0, 0, 0]^T$. As before, we can check that

$$J(F_{\alpha_1} \pm F_{\alpha_2}) = -(\tanh s)^{\mp 1}(G_{\alpha_1} \pm G_{\alpha_2}),$$

$$F_{\alpha_1}(0) - F_{\alpha_2}(0) = G_{\alpha_1}(0) + G_{\alpha_2}(0) = 0.$$

Let T be the tangent vector of the curve p_s, then $JH = -T$.

Similarly, we consider the case $n = 4$. $G = S = B_2 = SO(5, \mathbf{C})$. The long roots of B_2 are $\alpha = \pm(e_1 \pm e_2)$ and the short roots of B_2 are $\beta = \pm e_i$. We have long simple root $\alpha_1 = e_1 - e_2$ and short simple root $\alpha_2 = e_2$. They constitute a fundamental root system of this Lie algebra. B_2 has other positive roots $\alpha_1 + \alpha_2 = e_1$, $\alpha_1 + 2\alpha_2 = e_1 + e_2$. B_2 has a Cartan subalgebra

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & -a_1 i & 0 & 0 & 0 \\ a_1 i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 i & 0 \\ 0 & 0 & a_2 i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bigg|_{a_1, a_2 \in \mathbf{C}} \right\}.$$

e_1 corresponds to $(a_1, a_2) = (1, 0)$ and e_2 corresponds to $(a_1, a_2) = (0, 1)$. The open orbit is generated by the action of B_2 on $[1, 0, 0, 0, 0]^T$.

We let

$$E_{\pm e_1} = \begin{bmatrix} 0_{2 \times 2} & 0_{2 \times 2} & B^T \\ 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 1} \\ -B & 0_{1 \times 2} & 0 \end{bmatrix} \quad \text{with} \quad B = \frac{1}{\sqrt{2}}[\pm 1, i],$$

4

$$E_{e_1 \pm e_2} = \begin{bmatrix} 0_{2\times2} & A & 0_{2\times1} \\ -A^T & 0_{2\times2} & 0_{2\times1} \\ 0_{1\times2} & 0_{1\times2} & 0 \end{bmatrix} \quad \text{for} \quad A = \frac{1}{2}\begin{bmatrix} 1 & \pm i \\ i & \mp 1 \end{bmatrix},$$

$E_{-\alpha} = \bar{E}_\alpha^T$. $F_\alpha = E_\alpha - E_{-\alpha}, G_\alpha = i(E_\alpha + E_{-\alpha})$, then $[F_\alpha, G_\alpha] = 2H_\alpha$ and $[H_\alpha, F_\alpha] = i(H_\alpha, H_\alpha)_0 E_\alpha$ where $(\ ,\ )_0$ is the standard inner product such that $(e_i, e_i)_0 = 1$.

We also have that $[E_{\pm e_i}, E_{\pm(e_j - e_i)}] = \mp E_{\pm e_j}$, $[E_{e_i}, E_{\pm e_j}] = \mp E_{e_i \pm e_j}$ and $[E_{-e_i}, E_{\pm e_j}] = \mp E_{-e_i \pm e_j}$, $[E_{\pm e_i}, E_{\pm(e_i + e_j)}] = \pm E_{\mp e_j}$.

As above, we consider the semisimple orbit generated by $-iH = e_1$.

Now,
$$p_s = \exp(-isH)[1, 0, 0, 0, 0]^T = [1, i\tanh s, 0, 0, 0]^T,$$

$p_\infty = [1, i, 0, 0, 0]^T$. As before, we have that

$$J(F_{e_1 + e_2} \pm F_{e_1 - e_2}) = -(\tanh s)^{\mp 1}(G_{e_1 + e_2} \pm G_{e_1 - e_2}),$$

$$F_{e_1 + e_2}(0) - F_{e_1 - e_2}(0) = G_{e_1 + e_2}(0) + G_{e_1 - e_2}(0) = 0.$$

Let T be the tangent vector of the curve p_s, then $JH = -T$.

Similarly, $JF_{e_1} = -(\tanh s)G_{e_1}, F_{e_1}(0) = 0$ and $F_{e_2} = G_{e_2} = 0$. In particular, at p_∞ we have that $JF_\alpha = -G_\alpha$.

Similarly, we consider the case (1) with $G = S = SO(n+1, \mathbf{C})$, which is either D_k with $n = 2k - 1$ or B_k with $n = 2k$. The open orbit is a $SO(n+1, \mathbf{C})$ action on $[1, 0, \cdots, 0]^T$.

For D_k (or B_k), we have $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < k$ and $\alpha_k = e_{k-1} + e_k$ (or $\alpha_k = e_k$). In particular, as above we have that:

Proposition 1. For $F(OP_n)$ and $F(OQ_n)$, we have:

$$J(F_{e_1 + e_i} \pm F_{e_1 - e_i}) = -(\tanh s)^{\mp 1}(G_{e_1 + e_i} \pm G_{e_1 - e_i})$$

(and $JF_{e_1} = -(\tanh s)G_{e_1}$). We also have that

$$F_{e_i \pm e_k} = G_{e_i \pm e_k} = 0$$

(and $F_{e_i} = G_{e_i} = 0$) for $i > 1$. In particular, at p_∞, $JF_\alpha = -G_\alpha$ for $\alpha \neq e_i \pm e_k$ (and e_i) $1 < i < k$.

In the case of (2), we first consider the case in which $G = S = Sp(2, \mathbf{C}) = C_2$ acting on $Gr(4, 2)$. The short simple root $\alpha_1 = e_1 - e_2$ and the long simple root $\alpha_2 = 2e_2$ constitute a fundamental root system of the Lie algebra. C_2 has other positive roots $\alpha_1 + \alpha_2 = e_1 + e_2$ and $2\alpha_1 + \alpha_2 = 2e_1$.

C_3 has Cartan subalgebra $\mathcal{H} = \{\mathrm{diag}(a_1, a_2, -a_1, -a_2)|_{a_1, a_2 \in \mathbf{C}}\}$. e_1 corresponds to $(a_1, a_2) = (1, 0)$, e_2 to $(0, 1)$. The open orbit is generated by the C_2 action on

$$A = \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \end{bmatrix}^T$$

which represents the complex 2 dimensional column space π of A in $\mathbf{C}^4$.

5

We let

$$E_\alpha = \begin{bmatrix} A_\alpha & 0 \\ 0 & -A_\alpha^T \end{bmatrix} \quad \text{with} \quad A_{e_1-e_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{-e_1+e_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We let

$$E_\alpha = \begin{bmatrix} 0 & B_\beta \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad B_{2e_1} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_{e_1+e_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We also let $E_{-\beta} = E_\beta^T$.

We have $[F_\alpha, G_\alpha] = 2H_\alpha$ and $[H_\alpha, E_\alpha] = i(H_\alpha, H_\alpha)_0 E_\alpha$ where $(\ ,\ )_0$ is the standard inner product such that $(e_1 - e_2, e_1 - e_2)_0 = 2$,

$$[E_{\pm 2e_i}, E_{\mp(e_i+e_j)}] = \pm\sqrt{2}E_{\pm(e_i-e_j)}, [E_{\pm(e_i-e_j)}, E_{\pm 2e_j}] = \pm\sqrt{2}E_{\pm(e_i+e_j)},$$

$$[E_{\pm(e_i-e_j)}, E_{\pm(e_i+e_j)}] = \pm\sqrt{2}E_{\pm 2e_i}, [E_{e_i-e_j}, E_{e_j-e_k}] = E_{e_i-e_k},$$

$$[E_{\pm(e_i-e_j)}, E_{\mp(e_j+e_k)}] = \pm E_{\pm(e_i+e_k)}.$$

As above, we consider the semisimple orbit generated by $-iH_{\alpha_1}$.

Now,

$$p_s = \exp(-isH_{\alpha_1}) \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \end{bmatrix}^T = \begin{bmatrix} 1 & ie^{-2s} & 0 & 0 \\ 0 & 0 & 1 & -ie^{2s} \end{bmatrix}^T,$$

$$p_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T.$$

The complex projective space $\mathbf{C}P^n$ has canonical line bundle $K = -(n+1)D_H$ with D_H being the hyperplane divisor line bundle. Our exceptional divisor D is a hyperquadric and hence $D = 2D_H$ for the corresponding line bundles.

When $Q = Q^n$ is a hyperquadric in $\mathbf{C}P^{n+1}$, D_H be the restriction of the hyperplane line bundle to Q^n, then $K_Q = -nD_H$. And D_H can be represented as a hyperquadric Q^{n-1} in Q.

Consider the equivariant double branched covering map $f : Q^n \to \mathbf{C}P^n$, we see that the branched locus is just the exceptional divisor $D = Q^{n-1}$. So we have that $D = D_H$ in this case.

Let $F_\alpha = E_\alpha - E_{-\alpha}, G_\alpha = i(E_\alpha + E_{-\alpha})$, then as above we have $JF_{e_1-e_2} = -(\tanh 2s)G_{e_1-e_2}$. Let T be the tangent vector of the curve p_t, then $JH = -T$. Similarly,

$$J(F_{2e_1} \pm F_{2e_2}) = -(\tanh 2s)^{\pm 1}(G_{2e_1} \mp G_{2e_2}), F_{e_1+e_2} = G_{e_1+e_2} = 0.$$

Similarly, we consider the case (2) with $G = S = Sp(3, \mathbf{C}) = C_3$. The two short simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$ and the long simple root $\alpha_3 = 2e_3$ constitute a fundamental root system of the Lie algebra. C_3 has other positive roots

$$\alpha_1 + \alpha_2 = e_1 - e_3, \ \alpha_1 + \alpha_2 + \alpha_3 = e_1 + e_3, \ \alpha_1 + 2\alpha_2 + \alpha_3 = e_1 + e_2,$$

6

$$\alpha_2 + \alpha_3 = e_2 + e_3, \ 2\alpha_2 + \alpha_3 = 2e_2, \ 2\alpha_1 + 2\alpha_2 + \alpha_3 = 2e_1.$$

C_3 has a Cartan subalgebra

$$\mathcal{H} = \{\mathrm{diag}(a_1, a_2, a_3, -a_1, -a_2, -a_3)|_{a_1, a_2, a_3 \in \mathbf{C}}\}.$$

e_1 corresponds to $(a_1, a_2, a_3) = (1, 0, 0)$, e_2 to $(0, 1, 0)$, e_3 to $(0, 0, 1)$. The open orbit is generated by the C_3 action on

$$A = \begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 \end{bmatrix}^T$$

which represents the complex 2 dimensional column space π of A.

We let

$$E_\alpha = \begin{bmatrix} A_\alpha & 0 \\ 0 & -A_\alpha^T \end{bmatrix} \ \text{ with } \ A_{e_1-e_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ \text{ and } \ A_{-e_1+e_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We let

$$E_\beta = \begin{bmatrix} 0 & B_\beta \\ 0 & 0 \end{bmatrix} \ \text{ with } \ B_{2e_1} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ \text{ and } \ B_{e_1+e_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We also let $E_{-\beta} = E_\beta^T$.

We have that $[F_\alpha, G_\alpha] = 2H_\alpha$ and $[H_\alpha, E_\alpha] = i(H_\alpha, H_\alpha)_0 E_\alpha$ where $(\ ,\ )_0$ is the standard inner product such that $(e_1 - e_2, e_1 - e_2)_0 = 2$,

$$[E_{\pm 2e_i}, E_{\mp(e_i+e_j)}] = \pm\sqrt{2}E_{\pm(e_i-e_j)}, [E_{\pm(e_i-e_j)}, E_{\pm 2e_j}] = \pm\sqrt{2}E_{\pm(e_i+e_j)},$$

$$[E_{\pm(e_i-e_j)}, E_{\pm(e_i+e_j)}] = \pm\sqrt{2}E_{\pm 2e_i}, [E_{e_i-e_j}, E_{e_j-e_k}] = E_{e_i-e_k},$$

$$[E_{\pm(e_i-e_j)}, E_{\pm(e_j+e_k)}] = \pm E_{\pm(e_i+e_k)}.$$

As above, we consider the semisimple orbit generated by $-iH_{\alpha_1}$.

Now,

$$p_s = \exp(-isH_{\alpha_1}) \begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & ie^{-2s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -ie^{2s} & 0 \end{bmatrix}^T$$

$$p_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

Let $F_\alpha = E_\alpha - E_{-\alpha}, G_\alpha = i(E_\alpha + E_{-\alpha})$, then as above we have that $JF_{e_1-e_2} = -(\tanh 2s)G_{e_1-e_2}$. Let T be the tangent vector of the curve p_s, then $JH = -T$. Similarly,

$$J(F_{2e_1} \pm F_{2e_2}) = -(\tanh 2s)^{\mp 1}(G_{2e_1} \mp G_{2e_2}), F_{2e_3} = G_{2e_3} = 0,$$

$$J(F_{e_1-e_3} \pm G_{e_2-e_3}) = -(\tanh s)^{\mp 1}(G_{e_1-e_3} \pm F_{e_2-e_3}),$$

7

$$J(F_{e_1+e_3} \pm G_{e_2+e_3}) = -(\tanh s)^{\mp 1}(G_{e_1+e_3} \pm F_{e_2+e_3}).$$

At p_∞, $F_{2e_3} = G_{2e_3} = 0 = F_{e_1+e_2} = G_{e_1+e_2}$, and $JF_{2e_2} = G_{2e_2}$, otherwise $JF_\alpha = -G_\alpha$.

Similarly, we consider the case (2) with $G = S = C_n$, then the roots of U are $\pm(e_i \pm e_j)$ with $1 < i < j \le n$ and $\pm 2e_i, 2e_1$. The open orbit is a combination of the C_n action on

$$\begin{bmatrix} 1 & i & 0 & \cdots & 0; & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0; & 1 & -i & 0 & \cdots & 0 \end{bmatrix}^T.$$

For C_n, we have that $\alpha_i = e_i - e_{i+1}$ for $1 \le i < n$ and $\alpha_n = 2e_n$, Therefore,

$$e_i - e_k = \sum_{j=i}^{k-1} \alpha_j, e_i + e_k = \sum_{j=i}^{k-1} \alpha_j + 2\sum_{j=k}^{n-1} \alpha_j + \alpha_n, 2e_i = 2\sum_{j=i}^{n-1} \alpha_j + \alpha_n.$$

Therefore, similarly we have:

Proposition 2. *For $F(Gr_k)$, we have*

$$JF_{\alpha_1} = -(\tanh 2s)G_{\alpha_1},$$

$$J(F_{2e_1} \pm F_{2e_2}) = -(\tanh 2s)^{\mp 1}(G_{2e_1} \mp G_{2e_2}),$$

$$J(F_{e_1-e_k} \pm G_{e_2-e_k}) = -(\tanh s)^{\mp 1}(G_{e_1-e_k} \pm F_{e_2-e_k}),$$

$$J(F_{e_1+e_k} \pm G_{e_2+e_k}) = -(\tanh s)^{\mp 1}(G_{e_1+e_k} \pm F_{e_2+e_k}).$$

$$F_\alpha = G_\alpha = 0$$

for

$$\alpha = e_1 + e_2, e_i - e_k, 2e_i, e_i + e_k$$

with $i > 2$.

At p_∞, we have $F_\alpha = G_\alpha = 0$ if $\alpha = e_1 + e_2, 2e_i, e_i \pm e_k, \ i > 2$; and $JF_\alpha = G_\alpha$ if $\alpha = 2e_2, \ e_2 \pm e_k$. Otherwise $JF_\alpha = -G_\alpha$.

Before we consider the isolated case (3), we can look at the general cases in which $G \ne S = \pi(G_F) \subset Aut(F)$, where G_F is the subgroup that acts on the fiber F and $\pi \colon G_F \to Aut(F)$ is the induced map from G_F to $Aut(F)$. As in [Ak], G is semisimple, U_G is the 1-subgroup. There is a parabolic subgroup $P = SS_1R$ with S, S_1 semisimple and R solvable such that $U_G = US_1R$ where U is a 1-subgroup of S. The manifold is a fibration over G/P with the completion of $P/U_G = S/U$ as the isotropic open orbit of the almost homogeneous fiber. In this case, the root system of S is a subsystem of the root system of G. In the Lie algebra of G, we also have some other F_α, G_α outside $\mathcal{S}$. Let K be a maximal connected compact Lie subgroup of G and L be the isotropic subgroup of K at a generoc orbit. Let $\mathcal{K}, \mathcal{L}$ be the corresponding Lie algebras. The tangent space of G/U_G along p_t is decomposed into irreducible $\mathcal{L}$ representations. These F_α, G_α are in the complement representation of the Lie algebra $\mathcal{S}$ of S. $JF_\alpha = -G_\alpha \pmod{\mathcal{S}}$ as it is in the tangent space of G/P. Therefore, we have $JF_\alpha = -G_\alpha$ for any α which is not in the root system of

S. This discussion is corresponding to the discussion in the last paragraph of the second section of [Gu8] and similar discussions in [Gu9, 12]..

If S is B_2, G can be B_n, C_n, F_4. If S is B_3, G can be B_n, F_4. If S is C_3, G can be C_n, F_4. If S is B_n with $n > 3$, G can only be B_{m+n}. If S is C_n with $n > 3$, then G can be C_{n+m}. The case of B_2 action which has an isotropic group of $SO(4, \mathbf{C})$ generated by roots $\pm e_1 \pm e_2$ is exactly the same as the case of $Sp(2, \mathbf{C})$ action, which has an isotropic subgroup of $Sp(1, \mathbf{C}) \times Sp(1, \mathbf{C})$ generated by $\pm 2e_1, \pm 2e_2$. All these are similar to the discussions in [Gu9, 12]. Here we have a few more possibilities. If $S = D_k$, $k > 3$, G can only be D_n, $n > 3$ or E_n $n > k$. If $S = D_3$, that is an A_3, G can be A_n $n > 2$, B_n, $n > 3$, C_n $n > 3$, D_n $n > 2$ and E_n. If $S = D_2$, G can be any simple group or product of simple groups other than G_2.

We now treat the isolated case (3) of the $Spin(7, \mathbf{C})$ action on $\mathbf{C}P^7$. This case is the restriction of the case (1) with an $G = S = SO(8, \mathbf{C})$ action to the $Spin(7, \mathbf{C})$ action induced by the spinor representation.

Let $\pm e_i$ $1 \leq i \leq 4$ be the weights of the $SO(8, \mathbf{C})$ action, they generate a Cartan subalgebra. Let h_1, h_2, h_3 be the short positive roots which generated a Cartan subalgebra of $Spin(7, \mathbf{C})$, then the spinor representation has weights $\frac{1}{2}(\pm h_1 \pm h_2 \pm h_3)$ and the Cartan subalgebra of G_2 is generated by $h_i - h_j$. We can identify $Spin(7, \mathbf{C})$ as a subgroup of $SO(8, \mathbf{C})$ by letting

$$h_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4), h_1 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4),$$

$$h_2 = \frac{1}{2}(e_1 - e_2 + e_3 - e_4), h_3 = \frac{1}{2}(e_1 - e_2 - e_3 + e_4).$$

One can easily check that the Cartan subalgebra of $Spin(7)$ is just the complement of h_0. We have that $h_1 - h_2 = e_2 - e_3$, $h_2 - h_3 = e_3 - e_4$, therefore, $E_{h_1 - h_2} = E_{e_2 - e_3}$ and similarly, $E_{h_i - h_j} = E_{e_{i+1} - e_{j+1}}$. All of them fix p_s. Therefore, G_2 fixes p_s.

We also have that

$$h_1 + h_2 = e_1 - e_4, h_2 + h_3 = e_1 - e_2, h_1 + h_3 = e_1 - e_3.$$

Therefore,
$$E_{h_i + h_j} = E_{e_1 - e_{k+1}} \ \{i, j, k\} = \{1, 2, 3\}.$$

And
$$h_0 + h_i = e_1 + e_{i+1}, h_0 - h_i = e_{j+1} + e_{k+1},$$

therefore,
$$E_{h_i} = aE_{e_1 + e_{i+1}} + bE_{-e_{j+1} - e_{k+1}}.$$

It is not difficult to check that $a = b = \frac{1}{\sqrt{2}}$ will make them generating a $so(7)$ subalgebra. Therefore,

$$E_{h_i} = \frac{1}{\sqrt{2}}(E_{e_1 + e_{i+1}} + E_{-e_{j+1} - e_{k+1}}).$$

9

That is, the semisimple element which we should use is a nonzero multiple of h_0. We use
$$-iH = \frac{1}{\sqrt{3}}(h_1 + h_2 + h_3) = \frac{\sqrt{3}}{6}(3e_1 - e_2 - e_3 - e_4).$$

Therefore, the open orbit is generated by $p_0 = [1,0,0,0,0,0,0,0]$ with the $Spin(7, \mathbf{C})$ action. And we have $p_s = \exp(-isH)p_0 = [1, i\tanh\frac{\sqrt{3}}{2}s, 0, 0, 0, 0, 0, 0]$. Therefore, $p_\infty = [1, i, 0, 0, 0, 0, 0, 0]$. As before, we have:

Proposition 3. *For $F(Sp_7)$, we have*

$$J(\sqrt{2}F_{h_i} \pm F_{h_j+h_k}) = -(\tanh\frac{\sqrt{3}}{2}s)^{\mp 1}(\sqrt{2}G_{h_i} \pm G_{h_j+h_k})$$

and

$$JH = -T,$$

$$F_{e_i-e_j} = G_{e_i-e_j} = 0 \ 0 < i < j < 4.$$

At p_∞, $JF_{h_i} = -G_{h_i}$, $JF_{h_j+h_k} = -G_{h_j+h_k}$, $F_{h_i-h_k} = G_{h_i-h_k} = 0$.
However, in this case $S = B_3$, G can only be B_n or F_4.

3.   The Kähler Structures I

In this section, we first examine the Kähler structure for the $S = SO(n, \mathbf{C})$ actions and shall deal with other actions in the next section. We shall summarize our conclusion of the volume calculation in our Theorem 1 in the next section, which is needed in calculating the Ricci and the scalar curvatures.

If $G = S = D_2$ or B_2, by regarding the open D_2 (or B_2) orbit as a homogeneous space, the vector fields which corresponding to the Lie algebra are the pushdown of the right invariant vector fields on the Lie group D_2 (or B_2). As we did in [Gu8], we study the corresponding left invariant vector fields on the Lie group. To make the things simpler, we still use our original notation for the left invariant vector fields. Since the Kähler form is (left)invariant under the action of the maximal compact Lie algebra $\mathcal{K}$, the pullback of this Kähler form is left $\mathcal{K}$ invariant form on S. Therefore, $T(\omega(X, Y)) = -\omega(T, [X, Y])$ for any $X, Y \in \mathcal{K}$.

Let $(\ ,\ )$ be an invariant metric on $\mathcal{K}$ such that $(H, H) = 1$. Then

$$[X, Y] = ([X, Y], H)H + [X, Y]_{\mathcal{L}} + [X, Y]_{(\mathcal{A}+\mathcal{L})^\perp}.$$

Therefore, $\omega(T, [X, Y]) = ([X, Y], H)\omega(T, H) + \omega(T, [X, Y]_{(\mathcal{A}+\mathcal{L})^\perp})$. But we also have $\omega(T, [X, Y]_{(\mathcal{A}+\mathcal{L})^\perp}) = \omega(sH, J([X, Y]_{(\mathcal{A}+\mathcal{L})^\perp})) = 0$ since $JX \in (\mathcal{A} + \mathcal{L})^\perp$ if $X \in (\mathcal{A} + \mathcal{L})^\perp$. We also have that $\omega(X, Y) = (aH + I, [X, Y])$ with I in the center of $\mathcal{L}$. Therefore,

$$T(\omega(X, Y)) = (a'H + I', [X, Y]) = -\omega(T, [X, Y]) = -([X, Y], \omega(T, H)H),$$

i.e., $I' = 0$ and $a' = -\omega(T, H)$. The first equality means that I does not depend on s, i.e., $I = Bie_2$ (or $I = 0$) for some constant B. Therefore, the Kähler form is

$$\omega(X, Y) = (a(s)H + Bie_2, [X, Y]) = (H(s), [X, Y])$$

10

where $H(s) = aH + I$. Here we have $B = 0$ for the B_2 action. We also notice that when $S = D_2$ we have $\omega(F_{e_1+e_2} - F_{e_1-e_2}, X) = 0$ at $s = 0$. Therefore, by letting $X = G_{e_1+e_2}$ and $X = G_{e_1-e_2}$ we have $a(0) = B = 0$. $a(-s)\coth(-s) = a(s)\coth s$ implies that a is an odd function and $a(s) < 0$ for $s > 0$.

Therefore, as observed in [Si] and [PS] that the tangent space has following orthogonal basis:

$$\{T, H\}, \{F_{e_1-e_2} + F_{e_1+e_2}, G_{e_1-e_2} + G_{e_1+e_2}\}, \{F_{e_1-e_2} - F_{e_1+e_2}, G_{e_1-e_2} - G_{e_1+e_2}\}$$

(and $\{F_{e_1}, G_{e_1}\}$).

We have that $\omega(T, JT) = \omega(T, H) = -a'$. We see that a is decreasing. We also have that

$$\omega(F_{\alpha_1} + F_{\alpha_2}, J(F_{\alpha_1} + F_{\alpha_2})) = -4a\coth s,$$
$$\omega(F_{\alpha_1} - F_{\alpha_2}, J(F_{\alpha_1} - F_{\alpha_2})) = -4a\tanh s$$

(and $\omega(F_{e_1}, JF_{e_1}) = -2a\tanh s$).

Therefore, the volume is equal to $V = -16a^2a'$ (or $32a^3(\tanh s)a'$).

For the case of $G = S = D_n$ (or B_n), we can do the same and almost everything are the same and $I_n = 0$ since $L = D_{n-1}$ (or B_{n-1}) is semisimple. In that case, we have one basis element $\{T, H\}$ with the metric value $-a'$, $n - 1$ basis elements

$$\{F^+_{i-1} = F_{e_1+e_i} + F_{e_1-e_i}, G^+_{i-1} = G_{e_1+e_i} + G_{e_1-e_i}\}$$

with the metric values $-4a\coth s$, $n - 1$ basis elements

$$\{F^-_{i-1} = F_{e_1+e_i} - F_{e_1-e_i}, G^-_{i-1} = G_{e_1+e_i} - G_{e_1-e_i}\}$$

with metric values $-4a\tanh s$ (and one basis element $\{F_n = F_{e_1}, G_n = G_{e_1}\}$ with metric value $-2a\tanh s$). Therefore, the volume is

$$V = -4^{2(4n-1)}a'a^{2(n-1)} \quad (\text{or } V = 2^{4n-3}a'a^{2n-1}\tanh s).$$

In the case $S = D_n$ (or B_n), $G = D_{m+n}$ (or B_{m+n}) and the $\mathbf{C}^n$ is generated by $e_{m+1}, \cdots, e_{m+n}$. The metric is $\omega(X, Y) = (aH + i\sum_{i=1}^m B_i e_i, [X, Y])$. Other roots related to $-iH = e_{m+1}$ are $e_i \pm e_{m+1}$. The restricted metric values are $-2(B_i \pm a)$. Other elements of the orthogonal basis only produce positive constants.

Therefore, the volume is

$$V = -Ma'a^{2(n-1)}\prod_{i=1}^m(B_i^2 - a^2) \quad (\text{or } Ma'a^{2n-1}(\tanh s)\prod_{i=1}^m(B_i^2 - a^2))$$

with a constant $M > 0$.

Now, it is not difficult to see that for any possible G and $S = SO(n, \mathbf{C})$ we always have Kähler metric: $\omega([X, Y]) = (aH + I, [X, Y])$ with the I in the C center of l and we always have $I_S = 0$. Therefore, we have that

$$V = -Ma'a^{2(n-1)}\prod_1^r(a_i - a)\prod_1^s(b_j + a)$$

11

$$\text{(or } V = Ma'a^{2n-1}(\tanh s)\prod_1^r(a_i - a)\prod_1^s(b_j + a))$$

with positive a_i and b_j.

One more observation: actually a_i, b_j come in pairs and $b_{j(i)} = a_i$. This comes from an involution symmetry of positive roots. This symmetry is induced by the element H. If S is B_k, then $H = ie_1$ is from a root vector e_1. The representation of the Lie subalgebra $sl(2)$ corresponding to e_1 decomposes the Lie algebra of G into irreducible representations, which we call *strings* as in [Gu9]. The involution symmetry is induced by reversing the signs of the eignvalues. One might check this using the case by case checking. However, it will be tedious for us. Here, we only need to prove the pairing up a_i and b_j. To do this we can use $H = ie_k$, then H corresponds to a simple root. Therefore, the reversing of the signs of the eignvalues induces the pairing of positive roots except the cases in which either the irreducible representation is the $sl(2)$ itself, where it reverses e_k with $-e_k$, or we have a 0 eigenvalue string. For these extreme cases, we just say that the involution symmetry of e_k or the positive roots corresponding to the 0 eigenvalue strings are themselves. The effect of the involution symmetry of the positive roots induces the pairing of the a_i and b_j we needed. In the case in which $S = D_k$, we can also consider the choice of $H = ie_k$. Then, it is proportional to the sum of two simple roots $e_{k-1} - e_k$ and $e_{k-1} + e_k$, which are orthogonal to each other. This time we have a Lie subalgebra $sl(2) \times sl(2)$ and we can decompose the Lie algebra of G into irreducible representations, which we call **double strings**. Then the same method above leads to an involution symmetry of the positive roots related to the two simple roots and pairs up the coefficients a_i and b_j in our volume formula.

4. The Kähler structures II

In this section, we shall deal with the Kähler metrics with $Sp(k, \mathbf{C})$ and $Spin(7, \mathbf{C})$ actions.

As above, we always have that

$$T(\omega(X,Y)) = -\omega(T, [X,Y]), \omega(X,Y) = (aH + I, [X,Y])$$

with $I_S = 0$ since $\mathcal{L} \cap \mathcal{S} = sp(k-2) \times sp(1)$ or $su(3)$ (see [Gu5 p.284]) are both semisimple, $I' = 0$.

If $S = Sp(n, \mathbf{C})$, the tangent space has an orthogonal basis

$$\{T, H\}, \{F_{2(k-1)} = F_{e_1-e_2}, G_{2(k-1)} = G_{e_1-e_2}\},$$

$$\{F_1^{\pm} = F_{2e_1} \pm F_{2e_2}, G_1^{\pm} = G_{2e_1} \mp G_{2e_2}\},$$

$$\{F_{i-1}^{\pm} = F_{e_1-e_i} \pm G_{e_2-e_i}, G_{i-1}^{\pm} = G_{e_1-e_i} \pm F_{e_2-e_i}\},$$

$$\{F_{k+i-3}^{\pm} = F_{e_1+e_i} \pm G_{e_2+e_i}, G_{k+i-3}^{\pm} = G_{e_1+e_i} \pm F_{e_2+e_i}\}$$

and $\{F_\alpha, G_\alpha\}$ with $\alpha \notin S$. The corresponding metric values are

$$-a', -2a\tanh 2s, \ -4a(\tanh 2s)^{\mp 1}, -4a(\tanh s)^{\mp 1}, -4a(\tanh s)^{\mp}$$

12

and $k_\alpha(a_\alpha - a)$ or $k_\alpha(b_\alpha + a)$ or k_α with positive numbers k_α. Therefore, the volumes are

$$V = Ma'a^{4k-5}(\tanh 2s) \prod_1^r (a_i - a) \prod_1^s (b_j + a).$$

As above, we also observe that a_i and b_j come in pairs, and $b_{j(i)} = a_i$. Since we have that H is proportional to the simple root $e_1 - e_2$, it induces an involution symmetry of the positive roots. That leads to our observation.

If $S = Spin(7, \mathbf{C})$, the tangent space has an orthogonal basis $\{T, H\}$, $\{F_i^\pm = \sqrt{2}F_{h_i} \pm F_{h_j + h_k}$, $G_i^\pm = \sqrt{2}G_{h_i} \pm G_{h_j + h_k}\}$ and $\{F_\alpha, G_\alpha\}$ with $\alpha \notin S$. The corresponding metric values are $-a'$, $-\frac{8}{\sqrt{3}}a(\tanh \frac{\sqrt{3}}{2}s)^{\mp 1}$ and $k_\alpha(a_i - a)$ or $k_\alpha(b_j + a)$ or k_α with positive numbers k_α. Therefore, the volumes are

$$V = -Ma'a^6 \prod_{i=1}^r (a_i - a) \prod_{j=1}^s ((b_j + a).$$

We also observe that a_i and b_j come in pairs, and $b_{j(i)} = a_i$. This can be seen in the last part of our last section. As for the $S = D_k$ case, we notice that $e_1 + e_2 + e_3 = (e_1 + e_2) + e_3$ is a sum of two roots. We can actually use $(e_1 - e_2) + e_3$ as our H, then it is proportional to a sum of two simple roots which are orthogonal to each other. Arguing as in the case of $S = D_k$ we can use double strings to get an involution symmetry of the positive roots. That leads to the pairing of the coefficients in the formula.

Altogether, we have:

Theorem 1. *For the type I case the volume is*

$$V = -Ma'a^{2m} \prod (a_i^2 - a^2) \tag{1}$$

for the cases $S = D_k$ or $Spin(7, \mathbf{C})$ and

$$V = Ma'a^{2m+1}(\tanh bs) \prod (a_i^2 - a^2)$$

for the cases $S = B_k$ (or C_k) with $b = 1$ (or 2), where M and a_i are positive numbers, m are nonnegative integers. We also have that $2m + 1$ (or $2m + 2$) are the dimensions of the fiber.

5. Calculating the Ricci Curvature

We now calculate the Ricci curvature. We have an orthogonal basis related to $T, F_i^\pm$ $1 \leq i < m$, F_α if the dimension of the fiber is $2m + 1$. If the dimension of the fiber is $2m + 2$, we have one extra element F_{m+1}. We also denote the corresponding restriction of Kähler metric by $\sigma, \sigma_i^\pm, \sigma_\alpha$ and σ_{m+1}. For any nonzero 2-form δ on $\mathbf{C}^2$, we let

$$A_{X,Y}(\delta) = 2^{-1}(\delta(X_1, X_2))^{-1}.$$

$$[\delta([J[X, JY], X_1] - J[[X, JY], X_1], X_2) + \delta(X_1, [J[X, JY], X_2] - J[[X, JY], X_2])]$$

for a given independent pair of vectors X_1, X_2. Let $h = \log V$. Following Koszul [Ks p.567], we have that

$$\rho(X, JY) = \frac{L_{J[X_r, JY_r]}(\omega^n)(T, JT, F, JF, F_\alpha, JF_\alpha)}{2\omega^n(T, JT, F, JF, F_\alpha, JF_\alpha)},$$

where X_r, Y_r are the corresponding right invariant vector fields and here we use F, JF to represent $F_1^+, JF_1^+, F_1^-, JF_1^-, \cdots, F_m^+, JF_m^+, F_m^-, JF_m^-$ (and F_{m+1}, JF_{m+1}), F_α, JF_α to represent $F_{\alpha_1}, JF_{\alpha_1}, \cdots, F_{\alpha_l}, JF_{\alpha_l}$ the array of $F_i^\pm$ (and F_{m+1} if it exists), F_α with its conjugate for positive roots $\alpha \notin S$ which have nonzero F_α and G_α. When $G = S$, all the notations related to α can be omitted.

To calculate the Ricci curvature for the case $S = G = B_k$ or C_k, we only need to consider $X = Y = F_{m+1} = F_{-iH}$.

$$[F_{m+1}, JF_{m+1}] = [F_{m+1}, -(\tanh bs)G_{m+1}] = -2(\tanh bs)H,$$

$$J[F_{m+1,r}, JF_{m+1,r}] = 2(\tanh bs)JH = -2(\tanh bs)T.$$

Again as what happened in [Ks p.567–570], usually it is not clear how to find JX for a right invariant vector field X along p_s and to deal with the left invariant form with right invariant vector fields. Therefore, the argument in [Si] does not work as we can see for our situation. We need something similar to the Koszul's trick in [Ks p.567–570](see also [Gu7]). It turns out that all the argument there still go through for our situation.

Therefore, we have that:

$$\rho(F_{m+1}, JF_{m+1}) = -(\tanh bs)h' + \frac{1}{2\omega^{2m+2}(T, JT, F, JF)}.$$
$$\begin{aligned}[\quad &\omega^{2m+2}([2(\tanh bs)T, T] - J[-2(\tanh bs)H, T], JT, F, JF)\\
+\quad &\omega^{2m+2}(T, [2(\tanh bs)T, JT] - J[-2(\tanh bs)H, JT], F, JF)\\
+\quad &\omega^{2m+2}(T, JT, [2(\tanh bs)T, F] - J[-2(\tanh bs)H - 2H, F], JF)\\
+\quad &\omega^{2m+2}(T, JT, F, [2(\tanh bs)T, JF] - J[-2(\tanh bt)H, JF])\\
=\quad &-(\tanh bs)h' + A_{F_{m+1}, F_{m+1}}(\tau)\\
+\quad &\sum_{i\pm} A_{F_{m+1}, F_{m+1}}(\tau_i^\pm) + A_{F_{m+1}, F_{m+1}}(\tau_{m+1}),\end{aligned}$$

here we use $\omega^n(\cdots, [A, F] - J[B, F], JF)$ to represent

$$\begin{aligned}\omega^n(&\cdots, [A, F_1^+] - J[B, F_1^+], JF_1^+, \cdots, F_{m+1}, JF_{m+1}) + \cdots\\
+\quad &\omega^n(\cdots, F_1^+, JF_1^+, \cdots, [A, F_{m+1}] - J[B, F_{m+1}], JF_{m+1})\end{aligned}$$

the sum of

$$\omega^n(\cdots, F_1^+, JF_1^+, \cdots, [A, F_i^\pm] - J[B, F_i^\pm], JF_i^\pm, \cdots, F_{m+1}, JF_{m+1})$$

for all the F elements in the orthogonal basis, similarly for $\omega^n(\cdots, F, [A, JF] - J[B, JF])$.

14

In general, we have a formula

$$\rho(X, JY) = \frac{1}{2} J[X_r, Y_r](h) + \sum A_{X,Y}(\sigma)$$

for σ runs through all $\sigma, \sigma_i^{\pm}, \sigma_{m+1}, \sigma_\alpha$.

To apply this formula, we have that $A_{F_{m+1}, F_{m+1}}(\sigma) = 0$ and

$$A_{F_{m+1}, F_{m+1}}(\sigma_i^{\pm}) = N_i \tanh bs \left(\tanh^{\pm 1} N_i s \mp \frac{1}{\sinh N_i s \cosh N_i s} + \tanh^{\mp 1} Ns \right)$$

with $N_i = 1$ except for the $F_1^{\pm}$ in the case $S = Sp(k, \mathbf{C})$, in which case $N_i = 2$. Similarly, we have that

$$A_{F_{m+1}, F_{m+1}}(\sigma_{n+1}) = b \tanh bs \left(\coth bs + \tanh bs + \frac{1}{\sinh bs \coth bs} \right).$$

Therefore, we have that

$$\rho(F_{m+1}, JF_{m+1})$$
$$= \tanh bs \left(-h' + \sum 2N_i \coth 2N_i s + b \coth 2bs + \frac{b}{\sinh bs \coth bs} \right) \quad (2)$$

which is

$$= \tanh bs(-(\log(a' a^{2m+1}))' + 4(m-1) \coth 2s + 6b \coth 2bs).$$

But, we also have that

$$\rho(F_{m+1}, JF_{m+1}) = (a_\rho H, [F_{m+1}, JF_{m+1}])$$
$$= -2 \tanh bs(a_\rho H, H) = -2a_\rho \tanh bs.$$

Therefore,

$$a_\rho = \frac{1}{2}((\log(a' a^{2m+1}))' - 4(m-1) \coth 2s - 6b \coth 2bs).$$

For the case in which G is strictly bigger than $S = B_k$ or C_k, we can easily check that $A_{F_{m+!}, F_{m+1}}(\sigma_\alpha) = 0$. Therefore, we have that

$$a_\rho = \frac{1}{2}((\log(a' a^{2m+1} \prod_1^r (a_i^2 - a^2)))' - 4(m-1) \coth 2s - 6b \coth 2bs)).$$

Now, we take care of the case $S = D_k$ or $Spin(7, \mathbf{C})$. In this case, we have the orthogonal basis determined by $T, F_i^{\pm}, F_\alpha$.

For the cases in which $G = S = D_k$, we only need to calculate $\rho(F_1^+, JF_1^+)$. When $S = D_k$, as before we have that

$$[F_1^+, JF_1^+] = -4(\coth s)H, J[F_{1,r}^+, JF_{1,r}^+] = 4(\coth s)JH = -4(\coth s)T.$$

15

This is proportional to what we have for $S = B_k$, therefore we have a similar formula

$$a_\rho = \frac{1}{2}((\log(a'a^{2k-2} \prod_1^r (a_i^2 - a^2)))' - 4(k-1) \coth 2s).$$

When $G = S = Spin(7, \mathbf{C})$, we have that

$$[F_1^+, JF_1^+] = -2i \coth(\frac{\sqrt{3}}{2}s)(2h_1 + h_2 + h_3)$$

$$= -2 \coth \frac{\sqrt{3}}{2}s \left(\frac{4}{\sqrt{3}}H + \frac{i}{3}(2h_1 - h_2 - h_3) \right),$$

$$J[F_{1,r}^+, JF_{1,r}^+] = -2 \coth \left(\frac{\sqrt{3}}{2}s \right) \cdot \frac{4}{\sqrt{3}}T.$$

Applying above formula, it is proportional to

$$(A, B) = (\frac{4}{\sqrt{3}}T, \frac{4}{\sqrt{3}}H + \frac{1}{3}(2h_1 - h_2 - h_3)) = \frac{4}{\sqrt{3}}(T, H) + \frac{1}{3}(0, 2h_1 - h_2 - h_3).$$

The second factor induces a zero since $[2h_1 - h_2 - h_3, F_1^+] = 2F_1^-$ and etc.. The first factor is proportional to what we had above and

$$[H, F_1^+] = \frac{\sqrt{3}}{2}G_1^+ - \frac{\sqrt{3}}{6}G_1^-$$

and etc., that is, the coefficient of G_1^+ is exactly N_1. Therefore, we have that

$$a_\rho = \frac{1}{2}((\log(a'a^6))' - 12N_1 \coth 2N_1 s.$$

Similarly, as above in general we have that

$$a_\rho = \frac{1}{2}((\log(a'a^6 \prod_1^r (a_i^2 - a^2)))' - 12 \cdot \frac{\sqrt{3}}{2}(\coth \sqrt{3}s).$$

Combining above results with those from the last section and as in [Gu9,12] we have:

Theorem 2. *If the fiber with the S action is of type I of complex dimension n, then*

$$a_\rho = \frac{1}{2}((\log(a'a^{n-1} \prod_1^r (a_i^2 - a^2)))' - 2\sum_1^{n-1} N_i \coth 2N_i s.$$

Moreover, (1) N_i are 1 for $S = SO(n+1, \mathbf{C})$ and (2) 1 except three of them being 2 for S of type C_k, (3) $\frac{\sqrt{3}}{2}$ for the case $S = Spin(7, \mathbf{C})$. Other coefficients come from the Ricci curvature of G/P which is

$$-(q_{G/P}, [X, Y])$$

with

$$q_{G/P} = \sum_{\alpha \in \Delta^+ - \Delta_P} H_\alpha$$

with the standard inner product.

We see that our result is basically the same as in [PS p.19 (4.18)] except the zero term $4B(Z^K, Z_D)$ there. But the middle steps are different. For example, we have a different JF_{m+1} if n is even.

6. Calculating the Scalar Curvature

To calculate the scalar curvature we again use the orthogonal basis. The Ricci curvature for T is $\frac{a'_\rho}{a'}$. The Ricci curvature of the other elements in F is $\frac{a_\rho}{a}$. The Ricci curvature for F_α which corresponding to factor $a_i \pm a$ is $\frac{a_{\rho,i} \pm a_\rho}{a_i \pm a}$. The Ricci curvature for constant fact k_j is $\frac{k_{\rho,j}}{k_j}$.

Now, the eigenvalues for the Ricci curvature must be a contiinuous functions. Therefore, in general we have that

$$\lim_{t \to 0} a_\rho = 0, \lim_{t \to 0} a'_\rho, \quad \lim_{t \to +\infty} a_\rho, \quad \lim_{t \to +\infty} \frac{a'_\rho}{a'}$$

exist and are finite (see Theorem 5 in the next section). We also see that if we can contract the hypersurface orbit and consider the manifold with a higher codimensional orbit, we have that $-l_\rho = \lim_{t \to +\infty} a_\rho = a_{i,\rho}$. We shall see more details in the following sections.

Then, by Theorem 1 we have $V = -Ma'\tilde{Q}(a) = -Ma'(-a)^{n-1}Q_1(a)g(s)$ where $g(s) = 1$ if $S = D_k$ or $Spin(7, \mathbf{C})$ and $g(s) = \tanh bt$ if $S = B_k$ or C_k. Let N be the complex dimension of the manifold, and $Q(a) = (-a)^{n-1}Q_1(a)$, we have that

$$\rho \wedge \omega^{N-1} = M(n-1)!\Big(-a'_\rho Q + (n-1)a'(-a)^{n-2}a_\rho Q_1$$

$$+ \quad a_\rho(-a)^{n-1}Q'_1 + (-a)^{n-1}\sum_i a_{\rho,i}\left(\frac{Q_1}{a_i + a} + \frac{Q_1}{a_i - a}\right)\Big)g(s)$$

$$+ \quad (n-1)!(\sum_j \frac{k_{\rho,j}}{k_j})V.$$

Therefore, $\rho \wedge \omega^{N-1} = M((-a_\rho Q(a))' + p_0 a')$.

Theorem 3. *The scalar curvature is*

$$\frac{2(-a_\rho Q)' + pa'}{-a'Q}.$$

Moreover, $p(a) = (-a)^{n-1}p_1(a)$ with $p_1(a)$ a polynomial of a and is a positive linear sum of Q_1 and product of $\deg Q_1 - 1$ factors of Q_1. The contribution of each factor k_j is $\frac{2k_{\rho,j}}{k_j}$ for the Q_1 factor. The contribution of each $a_i \pm a$ is

$$2\frac{a_{\rho,i}Q_1}{q_i}.$$

17

We shall see later on that the contribution of the k_j will not affect the equation although it has an effect in the scalar curvature.

7. Setting up the Equations

Now, we set up the equations for the metrics with constant scalar curvature. Before we do that, we need to understand more about the metrics. We have:

Theorem 4. ω *is a metric on the open orbit if and only if* a *is an odd function with* $a' < 0$ *and* $0 < a_i$, $b_j + a > 0$.

To understand the metrics near the hypersurface orbit, we let $\theta = \tanh ct$ with $c = 1, 1, \frac{\sqrt{3}}{2}$ for $S = SO(n + 1, \mathbf{C}), C_k, Spin(7, \mathbf{C})$ and we see that $\theta' = c(1 - \tanh^2 2s) = c(1 - \theta^2)$.

In the case of $S = SO(n + 1, \mathbf{C})$ or $Spin(7, \mathbf{C})$, we can use $[1, z_1, z_2, \cdots, z_n]$ as the coordinate near $[1, i, 0, \cdots, 0]$, then the metric is proportional to

$$dz_1 d\bar{z}_1 = d(i \tanh s) d(\overline{i \tanh s}) = (1 - \theta^2)^2 (ds)^2$$

at p_∞. Therefore, $\lim_{\theta \to 1} a_\theta = \lim_{\theta \to 1} \frac{a'}{2c(1 - \theta^2)} = 0$ and actually we have that $a_\theta = (1 - \theta)h(\theta)$ with $h(1) < 0$. If the fiber is a Q^n which is a branched double covering of $\mathbf{C}P^n$ near p_∞ with a local map equivalent to the double map $w_1 \to z_1 = w_1^2$, then the metric is proportional to

$$\begin{aligned} dw_1 d\bar{w}_1 &= d\sqrt{i \tanh s} d\overline{\sqrt{i \tanh s}} \\ &= 4^{-1}(1 - \tanh s)^{-1}(1 - \theta^2)^2(ds)^2 = 4^{-1}(1 - \theta^2)(1 + \theta)(ds)^2. \end{aligned}$$

Therefore, $a_\theta(1) < 0$.

Similarly, for the case of $S = C_k$ we can use

$$\begin{bmatrix} 1 & z_1 & z_2 & \cdots & z_l; & z_{l+1} & 0 & z_{l+2} & \cdots & z_{2l} \\ 0 & w_1 & w_2 & \cdots & w_l; & w_{l+1} & 1 & w_{l+2} & \cdots & w_{2l} \end{bmatrix}$$

as a coordinate near p_∞, then the metric along the curve is proportional to

$$dz_1 d\bar{z}_1 + dw_1 d\bar{w}_1 = 2d(ie^{-2s})d(\overline{ie^{-2s}}) = 8e^{-4s}(ds)^2,$$

which is proportional to $(\cosh s)^{-4}(ds)^2 = (1 - \theta^2)^2(ds)^2$. Therefore, $a_\theta(1) = 0$ and $a_\theta = (1 - \theta)h(\theta)$ with $h(1) < 0$.

In particular, a is bounded. Let $l = \lim_{s \to +\infty} a$. We also notice that l actually determine the homogeneous Kähler metric on the exceptional hypersurface divisor. Therefore, l is a topological invariant.

Theorem 5. ω *in Theorem 4 extends to a Kähler metric over the exceptional divisor if and only if*

$$\lim_{s \to +\infty} a = l > -a_i$$

18

and $a_\theta(1) = 0, a_{\theta\theta}(1) < 0$ *for the cases of the fiber*[1] $F = \mathbf{C}P^n$ *or* $Gr(2k, 2)$ *(or* $a_\theta(1) < 0$ *for the cases of the fiber* $F = Q^n$).

Now for any given l with $0 > l > -a_i$, we can check that $a(\theta) = l\theta$ (or $a = 2l\theta(1 + \theta^2)^{-1}$) satisfies Theorems 4 and 5. So we have:

Theorem 6. *The Kähler classes are in one to one correspondence with the elements in the set*

$$\Delta = \{l|_{0 > l > -a_i}\}.$$

To calculate the total volume, we notice that

$$T \wedge JT \bigwedge_i (F_i \wedge JF_i) \bigwedge_{\alpha \notin S} (F_\alpha \wedge JF_\alpha) = Mg(s)T \wedge H \bigwedge (F \wedge G) \bigwedge_{\alpha \notin S} (F_\alpha \wedge G_\alpha)$$

with a positive number M. $a(0) = 0$, $a(+\infty) = l$. Therefore, let $u = -a$, the total volume is $V_T = \int_0^{-l} Q(u)du$.

We also see that

$$a_\rho = \frac{1}{2}\left(\frac{a''}{a'} + \frac{Q'(a)a'}{Q(a)} - 4(k-1)N_1 \coth 2N_1 s - 2(n - 2k + 1)b \coth 2bs\right).$$

One can easily check that

$$a'(0) < 0, \left(\frac{a'}{a} - 2N_i \coth 2N_i s\right)(0) = a''(0) = 0$$

by a being odd and therefore $a_\rho(0) = 0$.

To make the things clearer, we replace s by $\theta = \tanh cs$. We have that

$$2a_\rho = c\left[[\log[a_\theta Q(a)(1 - \theta^2)]]_\theta (1 - \theta^2) - m_1 \frac{1 + \theta^2}{\theta} - 2m_2\left(\frac{1 + \theta^2}{2\theta} + \frac{2\theta}{1 + \theta^2}\right)\right]$$

$$= c\left[(1 - \theta^2)\left[\frac{a_{\theta\theta}}{a_\theta} + \frac{Q'(a)a_\theta}{Q(a)}\right] - 2\theta - (n - 1)\frac{1 + \theta^2}{\theta} - 4m_2\frac{\theta}{1 + \theta^2}\right]$$

which has a limit $-2(n + 1 + m_2)c$ if $F = \mathbf{C}P^n$ or $Gr(2k, 2)$; $-2cn$ if $F = Q^n$, here $m_1 = n - 1, m_2 = 0$ for $S = SO(n + 1, \mathbf{C})$ or $Spin(7, \mathbf{C})$ and $m_1 = n - 4, m_2 = 3$ for $S = C_k$. So $l_\rho = -c(n + 1) = -n - 1, -n - 4, -cn = -3\sqrt{3}$.

We can also check that $\lim_{\theta \to 1} \frac{a_{\rho,\theta}}{a_\theta}$ exists. If $F = Q^n$, $a_\theta(1) < 0$ and it is fine. If $F = \mathbf{C}P^n$ or $Gr(2k, 2)$, $a_\theta(1) = 0$, we need a little more work. We need to prove that $a_{\rho,\theta}(1) = 0$ also. Then

$$\lim_{\theta \to 1}\frac{a_{\rho,\theta}}{a_\theta} = \lim_{\theta \to 1}\frac{a_{\rho,\theta\theta}}{-h(1)}.$$

We actually notice that the curve p_t we considered is in a $\mathbf{C}P^1$. Therefore, we can consider the $SO(2)$ invariant metrics in $\mathbf{C}P^1$. Actually, we can see that the

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[1] In this case, if we require the metric to have some smoothness we need more, see the discussions two paragraph before (3). We already take this into our account in the proof of the next Theorem.

case $a = -\frac{\theta}{1+\theta^2}$ is the standard Fubini-Study metric. $a_\theta = (1 - \theta^2)(1 + \theta^2)^{-2}$ and $h = (1 + \theta)(1 + \theta^2)^{-2}$. In general we have

$$h = F(\theta)(1 + \theta)(1 + \theta^2)^{-2},$$

where F comes from an invariant function

$$F(\frac{1 + z^2}{1 + |z|^2}, \frac{1 + \bar{z}^2}{1 + |z|^2})$$

which must be a positive real function for all the possible z. Therefore, F comes from a function

$$G\left(\frac{(1 + z^2)(1 + \bar{z}^2)}{(1 + |z|^2)^2}\right).$$

Therefore

$$F(\theta) = G\left(\left(\frac{1 - \theta^2}{1 + \theta^2}\right)^2\right).$$

In particular, $F'(1) = 0$. Therefore, $h'(1) = F(1)2^{-2}(1 - 2 \times 2) = -\frac{3}{4}F(1)$ and $\frac{h'(1)}{h(1)} = -\frac{3}{2}$. Applying the latter to our formula of a_ρ, we see that $a_{\rho,\theta}(1) = 0$.

Now, we have the Kähler Einstein equation

$$c\left[\theta(1 - \theta^2)\left[\frac{a_{\theta\theta}}{a_\theta} + \frac{Q'(a)a_\theta}{Q(a)}\right] - 2\theta^2 - (n - 1)(1 + \theta^2) - 4m_2\frac{\theta^2}{1 + \theta^2}\right] = 2\theta a.$$

Here, we can also notice that if this equality holds then we have following inequality:

$$c[(1 - \theta^2)(\log(a'Q(a)))' - 2\theta - 2(n - 1 - m_2) - 4m_2]$$
$$> \quad c\left[(1 - \theta^2)(\log(a'Q(a)))' - 2\theta - (n - 1 - m_2)(\theta + \theta^{-1})\right] \tag{3}$$
$$- \quad 2m_2\left(\frac{1 + \theta^2}{2\theta} + \frac{2\theta}{1 + \theta^2}\right)\right] = 2a.$$

The total scalar curvature is

$$R_T = \int_0^1 [p(u)u' + 2(u_\rho Q(u))']d\theta = \int_0^l p du + 2l_\rho Q(l)$$

where we let $u = a$. And, from this we have the average scalar curvature

$$R_0 = \frac{R_T}{V_T} = \frac{\int_0^l p(u)du + 2l_\rho Q(l)}{\int_0^l Q(u)du}.$$

The equation of constant scalar curvature is $\frac{R}{V} = R_0$. Therefore, we have that

$$2u_\rho Q(u) + \int_0^u p(u)du = R_0 \int_0^u Q(u)du + A_0 \tag{4}$$

20

with A_0 a constant.

Let $\theta = 0$, we have that $0 = A_0$. If we put $\theta = 1$, we get the same A_0.

We have that

$$u_\rho = \frac{R_0 \int_0^u Q du - \int_0^u p\, du}{2Q(u)} = \frac{R(u)}{2Q_1(u)}$$

where $Q(u) = u^{n-1}Q_1(u)$ and $R(u)$ is a polynomial of u.

Actually, if we let

$$p_1 = 2u^{n-1}\sum_i a_{\rho,i}Q_1\left(\frac{1}{a_i+u} + \frac{1}{a_i-u}\right) = 4u^{n-1}\sum_i \frac{a_{\rho,i}a_iQ_1}{a_i^2-u^2} = 2u^{n-1}p_2$$

with

$$p_2 = 2\sum_i a_{\rho,i}a_i\frac{Q_1}{a_i^2-u^2} \quad \text{and} \quad R_1 = \frac{2l_\rho Q(l) + \int_0^l p_1 du}{\int_0^l Q du},$$

then

$$u_\rho = \frac{R_1 \int_0^u Q du - \int_0^u p_1 du}{2Q} = \frac{\int_0^u u^{n-1}(R_1Q_1 - 2p_2)du}{2Q}$$

$$= \frac{u^n m(u)}{2Q(u)} = \frac{um(u)}{2Q_1} = \frac{R(u)}{2Q_1(u)}$$

with $m(u)$ a polynomial. We can also check that

$$\frac{lm(l)}{2Q_1(l)} = \frac{R_0 \int_0^l Q du - \int_0^l p\, du}{2Q(l)} = \frac{2l_\rho Q(l)}{2Q(l)} = l_\rho.$$

Therefore, $u_\rho = \frac{um(u)}{2Q_1(u)}$. Obviously, this is a generalization of the Kähler Einstein equation, in which $m(u) = 2Q_1(u)$.

Therefore,

$$c[\theta(1-\theta^2)\left[\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}\right] - 2\theta^2 - (n-1)(1+\theta^2) - 4m_2\frac{\theta^2}{1+\theta^2}] = -\theta u\frac{m(u)}{Q_1(u)} \quad (5)$$

We also notice that

$$c[(1-\theta^2)(\log(u'Q(u)))' - 2\theta - 2(n-1-m_2) - 4m_2] \geq -u\frac{m(u)}{Q_1(u)} \quad (6)$$

by

$$-(n-1-m_2)\frac{1+\theta^2}{2\theta} \leq -(n-1-m_2) \quad \text{and} \quad -2m_2\left(\frac{2\theta}{1+\theta^2} + \frac{1+\theta^2}{2\theta}\right) \leq -4m_2.$$

The equality holds if and only if $\theta = 1$. That is,

$$c[(\log(u_sQ(u)))_s - 2(n-1+m_2)] \geq -u\frac{m(u)}{Q_1(u)},$$

and the equality holds if and only if $\theta = 1$.

By integration, we have that

$$0 = 2^{-1}(u_t Q(u))|_0^{+\infty} > \int_0^{-l} \left(n - 1 + m_2 - \frac{um(u)}{2cQ_1(u)} \right) Q(u)du.$$

Therefore, we have a necessary condition

$$0 > \int_0^{-l} (n - 1 + m_2 - \alpha)Q(u)du \tag{7}$$

with

$$\alpha = \frac{um(u)}{2cQ_1(u)} = \frac{R_0 \int_0^u Q du - \int_0^u p du}{2cQ(u)}$$

for existing an extremal metric. We shall see later on that this is also a sufficient condition.

The above equation might be a good equation. But, we could not obtain the estimates we had in [GC, Gu4,5,8,9,12]. Therefore, we use a square transformation $(u, \theta) \to (U = u^2, \theta_1 = \theta^2)$. By abusing the notation, we still use θ for the new free variable θ_1. We replace u' by $(\frac{U}{\theta})^{-\frac{1}{2}}U'$, and $'$ by $' \cdot 2\theta^{\frac{1}{2}}$. We also denote $Q(u)$, $m(u)$, $Q_1(u)$ by $Q(U)$, $m(U)$, $Q_1(U)$ for simplicity.

Therefore, we have that

$$c\left[2\theta(1-\theta)(\log(\theta^{\frac{1}{2}}U'U^{\frac{n-2}{2}} \prod_1^r (a_i^2 - U)))' - 2\theta - (n-1)(1+\theta) - 4m_2\frac{\theta}{1+\theta} \right]$$

$$= c\left[2\theta(1-\theta)(\log(U'Q(U)))' + (1-\theta) - 2\theta - (n-1)(1+\theta) - 4m_2\frac{\theta}{1+\theta} \right]$$

$$= c\left[2\theta(1-\theta)(\log(U'Q(U)))' - (n+2)\theta - (n-2) - 4m_2\frac{\theta}{1+\theta} \right]$$

$$= -(\theta U)^{\frac{1}{2}}\frac{m(U)}{Q_1(U)}. \tag{8}$$

From this, we have that

$$[\log[U'Q(U)]]' = \frac{P}{\theta(1-\theta)}.$$

Then, as in [Gu9,12] we have that

$$\frac{n+2}{2}\theta + \frac{n-2}{2} + 2m_2\frac{\theta}{1+\theta} - A_l\theta^{\frac{1}{2}} \le P \le \frac{n+2}{2}\theta + \frac{n-2}{2} + 2m_2\frac{\theta}{1+\theta} + A_l\theta^{\frac{1}{2}}.$$

for some positive constant A_l, C_l which only depend only on l. Since

$$P(1) = \frac{n+2}{2} + \frac{n-2}{2} + m_2 + c^{-1}l_\rho = n + m_2 - (n+1+m_2) = -1$$

for the case $F = \mathbf{C}P^n$ or $Gr(2k, 2)$ (or $P(1) = n + c^{-1}l_\rho = 0$ for the cases $F = Q^n$), we have $A_l \ge n + m_2 + 1$ (or n).

22

To integrate, we have that

$$\int \frac{d\theta}{1-\theta} = -\log(1-\theta) + C, \quad \int \frac{d\theta}{(1+\theta)(1-\theta)} = \frac{1}{2}\log\frac{1+\theta}{1-\theta} + C,$$

$$\int \frac{d\theta}{\theta(1-\theta)} = \log\frac{\theta}{1-\theta} + C, \quad \int \frac{\theta^{\frac{1}{2}}d\theta}{\theta(1-\theta)} = \log\frac{1+\theta^{\frac{1}{2}}}{1-\theta^{\frac{1}{2}}} + C.$$

By integration, we have that

$$\frac{a^{\frac{n-2}{2}}(1-a^{\frac{1}{2}})^{A_l-n-m_2}(1+a)^{m_2}(1+\theta^{\frac{1}{2}})^{A_l+n+m_2}}{\theta^{\frac{n-2}{2}}(1-\theta^{\frac{1}{2}})^{A_l-n-m_2}(1+\theta)^{m_2}(1+a^{\frac{1}{2}})^{A_l+n+m_2}} \leq \frac{U'(a)U^{\frac{n-2}{2}}(a)Q_1(U(a))}{U'(\theta)U^{\frac{n-2}{2}}(\theta)Q_1(U(\theta))} \quad (9)$$

$$\leq \frac{a^{\frac{n-1}{2}}(1-\theta^{\frac{1}{2}})^{n+m_2+A_l}(1+a)^{m_2}(1+\theta^{\frac{1}{2}})^{n+m_2-A_l}}{\theta^{\frac{n-2}{2}}(1-a^{\frac{1}{2}})^{n+m_2+A_l}(1+\theta)^{m_2}(1+a^{\frac{1}{2}})^{n+m_2-A_l}}$$

for $0 < \theta \leq a < 1$. We let $V = u^n$ and $x = \theta^{\frac{n}{2}}$, we obtain the following Harnack inequality:

$$\frac{(1-a^{\frac{1}{2}})^{A_l-n-m_2}(1+a)^{m_2}(1+\theta^{\frac{1}{2}})^{A_l+n+m_2}}{(1-\theta^{\frac{1}{2}})^{A_l-n-m_2}(1+\theta)^{m_2}(1+a^{\frac{1}{2}})^{A_l+n+m_2}} \leq \frac{V_x(a)Q_1(U(a))}{V_x(\theta)Q_1(U(\theta))} \quad (10)$$

$$\leq \frac{(1-\theta^{\frac{1}{2}})^{n+m_2+A_l}(1+a)^{m_2}(1+\theta^{\frac{1}{2}})^{n+m_2-A_l}}{(1-a^{\frac{1}{2}})^{n+m_2+A_l}(1+\theta)^{m_2}(1+a^{\frac{1}{2}})^{n+m_2-A_l}}.$$

Arguing as in [Gu4,5,9,12], we have that

Theorem 7. *If there is a solution $0 \leq U \leq l^2$ of above equation (8) with $U(0) = 0$ and $U(1) = l^2$, then there is a Kähler metric with constant scalar curvature in the considered Kähler class.*

Theorem 8. *For any small positive number β, we have a solution of (8) with*

$$U(0) = 0, U(1-\beta) = l^2.$$

This corresponds to a Kähler metric with a constant scalar curvature on the manifold with boundary $0 \leq \theta \leq 1 - \beta$.

8.   Global Solutions

In this section, we shall extend our solutions to the hypersurface orbit. We will let $\beta \to 0$. As we did in [Gu4], we let $\tau = -\log(1-\theta)$ and have

$$[\log[U_\tau Q(U)]]_\tau = \frac{P-\theta}{\theta}.$$

23

Therefore, we have that

$$\left[\log\left[\frac{\frac{n}{2}U^{\frac{n-2}{2}}U_\tau}{\theta^{\frac{n-2}{2}}}Q_1(U)\right]\right]_\tau = \frac{P-\theta}{\theta} - \frac{(n-2)\theta_\tau}{2\theta}$$

$$= \frac{P-\theta}{\theta} - \frac{n-2}{2}\left(\frac{1}{\theta}-1\right) = \frac{2P-n+2+(n-4)\theta}{2\theta} \qquad (11)$$

$$= n-1+\frac{2m_2}{1+\theta} - \frac{1}{2c}\left(\frac{U}{\theta}\right)^{\frac{1}{2}}\frac{m(U)}{Q_1(U)} = T(U,\theta)$$

$$\rightarrow \quad n-1+m_2 - \frac{U^{\frac{1}{2}}m(U)}{2cQ_1(U)} = n-1+m_2-\alpha$$

when θ tends to 1 and it converges unformly.

If ω is in the Ricci class, then $m(U) = 2Q_1(U)$ and $\alpha = (c)^{-1}\sqrt{U}$.

Let U_i be a series of solutions corresponding to $\beta_i \to 0$. By $P(1) = -1$ (or 0), for any $e_0 \in (n+m_2, n+m_2+1)$ (or $(n-1+m_2, n+m_2)$) there are two numbers $A(e_0) < l^2$ and $B(e_0) > 0$ such that if $U > A(e_0)$ and $\tau > B(e_0)$ then $\alpha > e_0 > n+m_2$ (or $n-1+m_2$) and $T(u,\theta(\tau)) < n-1+m_2-e_0$. Let τ_i be a point of τ such that $U_i(\tau_i) = A(e_0)$, and if we also have $\tau_i > B(e_0)$ then

$$\left[\log\left[\frac{\frac{n}{2}U_i^{\frac{n-2}{2}}U_{i,\tau}}{\theta^{\frac{n-2}{2}}}Q_1(U_i)\right]\right]_\tau = \frac{2P-n+2+(n-4)\theta}{2\theta} = T(U,\theta)$$

$$< \quad n-1+m_2-e_0$$

for $\tau \geq \tau_i$.

Let $w = \frac{\frac{n}{2}u^{\frac{n-2}{2}}u'}{\theta^{\frac{n-2}{2}}}Q_1(u)$, then $w_i \leq e^{(n-1+m_2-e_0)(\tau-\tau_i)}w_i(\tau_i)$.

If no subsequence of τ_i tends to $+\infty$, then a subsequence of τ_i tends to a finite number τ_0. By the left side of the Harnack inequality (10), we see that $V_{i,x}(\theta(\tau_0))$ must be bounded from above, otherwise $V_{i,x}$ will be bounded from below by a very large number such that V_i will be bigger than $(-l)^n$ before x reaching the point 1. That is, there is a subsequence of U_i converging to a solution U of our equation with $U(1) > A(e_0)$.

We shall observe that no subsequence of τ_i tends to $+\infty$ under our earlier necessary condition (7), which is just

$$\int_0^{l^2} (n-1+m_2-\alpha)Q(U)dU < 0. \qquad (12)$$

If there is a subsequence of τ_i tends to $+\infty$, we might assume that $\lim_{i\to+\infty}\tau_i = +\infty$, and $\tau_i > B(e_0)$. To make things simpler, we try to avoid the homogeneous cases, i.e., the cases in which $G = S$. In those cases, the second Betti number is 1, therefore, all the equations are basically the Kähler Einstein equations with different Einstein constants. There are unique solutions for our equations. Actually, one can easily see that $u = m\tanh 2s$ should solve all the equations for $F = \mathbf{C}P^n$

24

or $Gr(2k, 2)$ and $u = m \tanh s$ should solve all the equations for $F = Q^n$. Let u_0 be the solution, then

$$(\log \frac{u_i'}{u_0'})' + (n - 1)(\log \frac{u_i}{u_0})' = m_0(u_i - u_0)$$

with a positive constant m_0. We claim that $u_i \geq u_0$ always, otherwise $u_i(s) = u_0(s)$ at some point $s \neq 0$ and $u_i < u_0$ for some $s_1 < s$ with $u_i'(s_1) = u_0'(s_1)$, both $\frac{u_i'}{u_0'}$, $\frac{u_i}{u_0}$ increase near s_1 by $u_i(1 - \beta_i) = -l > u_0(1 - \beta_i)$ and $u_i(0) = u_0(0) = 0$. Then, the two sides of the above equality have a different sign, a contradiction. Now, by this inequality, τ_i has a finite upper bound τ_0 such that $U_0(\tau_0) = A(e_0)$. Actually, one can see later on that any convergent subsequence of u_i converges to the unique solution u_0, therefore u_i converges to u_0 itself. Therefore, we shall always assume that G is bigger than S and in that case we can see that there is at least one a_i. Now, from the equation we observe that if

$$U_{i,\tau}(\tau_i)U_i^{\frac{n-2}{2}}(\tau_i) > 2(-l)^{\frac{n-2}{2}}a_1^2 A_l > 2U^{\frac{n-2}{2}}(a_1^2 - U)A_l,$$

then $U_{i,\tau}(\tau_i) > 2(a_1^2 - U_i(\tau_i))A_l$, and we have that $V_\tau \prod_1^r = U_i^{\frac{n-2}{2}}U_{i,\tau}$ is increasing for $\tau \geq \tau_i$. This can not happen. Therefore, $U_{i,\tau}(\tau_i)$ is bounded from above.

We shall see that in that case a subsequence of

$$\tilde{U}_i(\tau) = U_i(\tau + \tau_i) \tag{13}$$

converges in the C^1 norm to a nonconstant function $\tilde{U}$. We see that for each $\tau \geq 0$, w_i is decreasing and $\tilde{U}_{i,\tau}$ are uniformly bounded. For each $\tau < 0$, $-A_l < [\log w_i]_\tau < n - 1 + m_2 + A_l$ when i big enough, that is, $\tilde{V}_{i,\tau}$ are also bounded uniformly on i over closed intervals. Therefore, a subsequence of $\tilde{V}_i$ converges in the C^1 norm to a function $\tilde{V}$. So does $\tilde{U}_i$.

To see that $\tilde{U}$ is not a constant, we can also notice that

$$\frac{\frac{n}{2}U_i^{\frac{n-2}{2}}U_{i,\tau}}{\theta^{\frac{n-2}{2}}} \leq C_i \frac{\frac{n}{2}U_i^{\frac{n-2}{2}}(\tau_i)U_{i,\tau}(\tau_i)}{\theta^{\frac{n-2}{2}}(\tau_i)}e^{(n-1+m_2-e_0)(\tau-\tau_i)}$$

for $\tau \geq \tau_i$, where C_i actually can be chosen that they do not depend on i. That is,

$$\frac{n}{2}U_i^{\frac{n-2}{2}}U_{i,\tau} \leq CU_{i,\tau}(\tau_i)e^{(n-1+m_2-e_0)(\tau-\tau_i)}.$$

Integrating both sides, we have $(-l)^{\frac{n}{2}} - A(e_0)^{\frac{n}{2}} \leq -\frac{C}{n-1+m_2-e_0}U_{i,\tau}(\tau_i)$, i.e., $U_{i,\tau}(\tau_i)$ is bounded from below. Therefore, $\tilde{U}_{i,\tau}(0)$ are bounded from below. We have $\tilde{U}_\tau(0) > 0$. This implies that $\tilde{U}$ is not a constant.

Then $\tilde{U}$ satisfies the equation $[\log[x^{\frac{n-2}{2}}x'Q_1(x)]]' = -\alpha + n - 1 + m_2$ on $(-\infty, +\infty)$. Therefore,

$$[x^{\frac{n-2}{2}}x'Q_1(x)]' = (-\alpha + n - 1 + m_2)x^{\frac{n-2}{2}}Q_1(x)x'.$$

25

Integrating as in [Gu4], we have that

$$\int_{x(-\infty)}^{x(+\infty)} f_l dx = 0, \quad \text{where } f_l = (-\alpha + n - 1 + m_2)x^{\frac{n-2}{2}}Q_1(x).$$

As in [Gu4], we see that $x(+\infty) = l^2$.

As in [Gu4], we shall prove:

Lemma 5. $n - 1 + m_2 - \alpha$ *has only one zero.*

Proof: As in [Gu4], we may expect that x is related to a Kähler metric of constant scalar curvature on the blow-downed $\mathbf{C}P^1$ bundle over the hypersurface orbit which is topologically equivalent to our manifold. Hence, we may apply the method of counting zeros in [Gu2,4] to this manifold. $x^{\frac{n-2}{2}}x'Q_1(x)$ is proportional to "φQ" in [Gu2]. Therefore, the counting of zeros of $n - 1 + m_2 - \alpha$ should be the same as counting the zeros of the derivative of "φQ" to "U" there.

However, it is obvious that f_l is, with square roots, not a polynomial at all. To make our argument work, we need to get rid of the square roots. Naturally, we have $u = \sqrt{U}$. We observe that $g_l = 2uf_l$ is a polynomial in u, which already appeared in the necessary condition (7), and should be proportional to the derivative of "φQ" in [Gu2]. Therefore, we may expect that $y = \frac{2}{l}u - 1$ corresponds to the "U" in [Gu2]. We let $q = 2uQ(u^2)$, which is actually our original $2Q(u)$ before we applying the square transformation before (8) and q is proportional to the "Q" in [Gu2].

We see that

$$2g_l = (n - 1 + m_2)q - c^{-1}m(u)u^n = (n - 1 + m_2)q - c^{-1}R(u)u^{n-1}$$
$$= (n - 1 + m_2)q - c^{-1}R_0\int Qdu + c^{-1}\int pdu. \tag{14}$$

Let g_l' be the derivative of g_l to u, we have that

$$2g_l' = (n - 1 + m_2)q' - (2c)^{-1}R_0q + c^{-1}p \tag{15}$$
$$= (n - 1 + m_2)q' + c^{-1}P_2 - (2c)^{-1}R_0q + c^{-1}P_3 = \Delta - mq,$$

where $P_3 = CQ$ is the Q term in p generated by the elements of the orthogonal basis with constant metric values and $P_2 = p - P_3$ is the positive linear combination of $\frac{Q}{a_i \pm u}$ corresponding to a_i, $\Delta = (n - 1 + m_2)q' + c^{-1}P_2$, $m = R_0 - C = R_1$. Therefore, $2g_l = \int_0^u(\Delta - mq)du$.

Lemma 6. *The coefficients of Δ are always positive.*

Proof of Lemma 6: From Theorem 3, we only need to check that $c^{-1}a_{\rho,i} - (n - 1 + m_2)$ is positive.

So we need to check that the last coefficient is also positive. There are two ways to prove this. First we notice that this actually is the same to check the coefficients $n - 1 + m_2$, $c^{-1}a_{\rho,i} + n - 1 + m_2$, $c^{-1}a_{\rho,i} - (n - 1 + m_2)$ are all positive. We claim that these are the components of the Ricci curvature of the exceptional divisor, then the positivity comes from the positivity of the Ricci curvature of the compact rational homogeneous spaces. The point is that u is corresponding to an H in the calculation

26

of the metric and the volume form, and we should prove that the contribution of H to the Ricci curvature is exactly $n - 1 + m_2$, i.e.,

$$(q_{G/P_\infty}, H)_0 = (q_{S/(S \cap P_\infty)}, H)_0 = c(n - 1 + m_2)$$

where P_∞ is the isotropic group of the exceptional divisor at p_∞. Notice that P_∞ is parabolic.

If $S = D_k$ (or B_k), then the semisimple part of $P_{\infty,1} = S \cap P_\infty$ is generated by $e_i \pm e_j$ $1 < i < j$ (and e_i) with the same orientation. $n = 2k - 1$ (or $2k$). Therefore, $(q_{S/P_{\infty,1}}, H)_0 = 2(k - 1)$ (or $2(k - 1) + 1 = 2k - 1$). But we also have that $c(n - 1 + m_2) = n - 1 = 2k - 2 = 2(k - 1)$ (or $n - 1 = 2k - 1$).

If $S = C_k$, then the semisimple part of $P_{\infty,1}$ is generated by $e_1 + e_2, \alpha_3, \cdots, \alpha_n$ with an orientation in which $e'_i = e_i$ $i \neq 2, e'_2 = -e_2$. $n = 3(2k - 2) = 4(k - 1)$. Therefore,

$$(q_{S/P_{\infty,1}}, H)_0 = 2 + 2(k - 2) + 1 + 2 + 2(k - 2) + 1 = 2(2k - 1).$$

But we also have that $c(n - 1 + m_2) = 4(k - 1) - 1 + 3 = 2(2k - 1)$.

If $S = Spin(7, \mathbf{C})$, then the semisimple part of $P_{\infty,1}$ is generated by α_1, α_2 with the same oriention. Therefore,

$$(q_{S/P_{\infty,1}}, H)_0 = \sqrt{3} + \sqrt{3} + \sqrt{3} = 3\sqrt{3}.$$

But we also have that $c(n - 1 + m_2) = \frac{\sqrt{3}}{2} \cdot 6 = 3\sqrt{3}$.

Secondly, we could check the positivity of the last coefficient with a case by case checking. That will also give all the $a_{\rho,i}$ in concrete calculations. But in practice the calculation is doable as in [Gu9] and it is tedious although they are needed to check the Fano property and apply our integral criterion for the existence of Kähler Einstein or extremal metrics. Therefore, we omit them here (but also see the second part of this paper).

<div align="center">Q. E. D.</div>

Therefore, as we argued in [Gu4 p.73], if $n - 1 + m_2 - \alpha$ has two zeros then $\Delta - mq$ has $\deg q - 3 + 4 = \deg q + 1$ zeros. That will be a contradiction to the degree $2 \deg Q + 1$. Thus, we obtain our Lemma 5.

<div align="center">Q. E. D.</div>

Now, we have that f_l has a unique zero. Therefore, if the necessary condition (7) which is also

$$\int_0^{l^2} f_l dx < 0 \tag{16}$$

holds we can not have $0 = \int_{x(-\infty)}^{l^2} f_l dx \leq \int_0^{l^2} f_l dx$. A contradiction.

By choosing $A(e_0)$ close to l^2 we have $u(1) = l^2$. Arguing as in [Gu4], we have that $u'(1) = 0$ (or exists and is finite). So are $u''(0)$ and $u''(1)$. Therefore, we have:

Theorem 9. *There is a Kähler metric of constant scalar curvature in a given Kähler class if and only if the condition* (7) *is satisfied.*

Fortunately, this result is much simpler than those in [Gu8,9,12] and those in [Gu4] (see the conjecture there), for which we need to check the converse in [Gu6]. For the type I case, the necessary part of (7) was observed in 2000 (see [Gu5 section 9]).

Assuming the converse in our Theorem 9, which we proved for the type I case in (7) and the type II case in [Gu6], our argument in the proof of our Theorem 9 actually shows that there is a convergent subsequence of U_i which converges to a metric with constant scalar curvature on the open orbit with $U(1) = a^2, 0 \le a \le -l$ and also converges at the exceptional divisor to a blowup metrics with a constant scalar curvature on the projective normal line bundle. If $a = -l$ we get what we need. If $a < -l$, then a is the only possible number such that $0 = \int_{a^2}^{l^2} f_l(x)dx$ by our lemma 5. Therefore, $U(1) = a^2$. If $a = 0$, the manifold piece collapses.

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