# On Reprensentation Theory and the Cohomology Rings of Irreducible Compact Hyperkähler Manifolds of Complex Dimension Four

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In this paper, we continue the study of the possible cohomology rings of compact complex four dimensional irreducible hyperkähler manifolds. In particular, we prove that in the case $b_2 = 7$, $b_3 = 0$ or 8. The latter was achieved by the Beauville construction.

## 1 Introduction

The study of higher dimensional hyperkähler manifolds caught much attentions. It is evident that there are only few examples of these manifolds, especially in complex dimension four (see [11]). Can we classify them as in the case of K3 surfaces?

In [11], we combine the results of Riemann-Roch formula and the representation generated by the Kähler classes to give a picture of the Hodge diamonds of irreducible compact hyperkähler manifolds of complex dimension four and obtained

**Proposition 1.** *If $M$ is an irreducible compact hyperkähler manifold of complex dimension* 4*, then* $3 \leq b_2 \leq 23$*. Moreover,*

1) *if* $b_2 = 23$*, then* $b_3 = 0$*. The Hodge diamond of $M$ is the same as that of Fujiki's first example.*

2) *if* $b_2 \neq 23$*, then* $b_2 \leq 8$*, and when* $b_2 = 8$*, we have* $b_3 = 0$*.*

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3) *if* $3 \le b_2 \le 7$, *then* $b_3 \le \min(\frac{4(23-b_2)(8-b_2)}{b_2+1},\ \frac{1}{2}(b_2+4)(23-b_2))$.

4) *in the case of* $b_2 = 3, 4, 5, 6,\ b_3 = 4l$ *with* $l \le 17$ *if* $b_2 = 3$, $l \le 15$ *if* $b_2 = 4$, $l \le 9$ *if* $b_2 = 5$, *and* $l \le 4$ *if* $b_2 = 6$.

5) *in the case of* $b_2 = 7$, $b_3 = 0,\ 4$ *or* $8$.

6) $c_2 \in \wedge^2 H^2$ *if and only if* $(b_2,\ b_3) = (5, 36), (7, 8), (8, 0), (23, 0)$.

The last item is very important since all the known examples are in this category. Therefore, *those Hodge diamonds are the best candidates for being the Hodge diamonds of irreducible compact hyperkähler manifolds*. We also *doubt* the existence of the case of $b_2 = 3$. Therefore, much more work should be done in these directions.

We announced there that in the statement 5) the case of $b_3 = 4$ can be removed. The proof was left out there since we need a deeper knowledge on representations on the cohomology. One of our main result here is:

**Theorem A.** *If $M$ is a complex four dimensional irreducible compact hyperkähler manifold with $b_2 = 7$, then $b_3 = 0$ or $8$.*

This means that if $b_2 = 7$, then we have the Hodge diamond of the Kummer variety or $b_3 = 0$. Together with the result in the case of $b_2 = 23$, we see that our results are quite sharp.

To prove this we have to deal with the representation of a Lie algebra generated by the Kähler classes.

We obtain some results on the structure of the cohomology ring of compact irreducible hyperkähler manifolds. We first obtain in general:

**Theorem B.** *Fix a Kähler class $x$ on an irreducible compact hyperkähler manifold. The multiplication by $x$ can be regarded as the root vector of the root $e_1$ if we regard $g = so(4, b_2 - 2)$ as a special orthogonal group with a properly chosen metric*

$$\mathrm{diag}(1, -1, 1, 1, -1, \cdots, -1, 1).$$

We shall explain the choice of the metric in the proof.

Applying this result to the case of complex dimension four we obtain:

**Theorem C.** *If $M$ is a complex four dimensional irreducible compact hyperkähler manifold, then $b_3 = k2^{[\frac{b_2}{2}]}$ for some integer $k$. In particular, if $b_2 = 7$, $b_3 = 8k$.*

Combining Theorem C with the results from [11] we obtain the Theorem A.

In our argument we use some technique used in [7], [8]. It turns out that the theory of spinors makes the proofs more efficient.

We shall give the real structure of the cohomology rings in more detail and some examples for the possible Hodge algebras as Jordan-Lefschetz modules (see [14]) in Section 4.

## 2 The Multiplication of a Kähler Class on Irreducible Compact Hyperkähler Manifolds

By the results in [22], there is an $so(4, b_2-2)$ action on the cohomology ring which is induced by the Kähler classes. When $b_2$ is odd this Lie algebra is a simple Lie algebra of type B with rank $\frac{b_2+1}{2}$. When $b_2$ is even this Lie algebra is a simple Lie algebra of type D with rank $\frac{b_2+2}{2}$. Here we only deal with the case of odd $b_2$. The case of a even $b_2$ is similar.

In this section we only apply the complex representation theory. We shall study the real structure in the next section.

To apply the representation theory of the complex orthogonal groups, we eventually adopt the spinor theory in this paper.

Here we recall some results about Clifford algebras, see [6, Chapter II, III], [13, Chapter II, VI] and [17, Chapter 9] for the details. Let $V$ be a complex vector space of dimension $n$ with the standard norm, $S$ be the spinor representation of the even Clifford algebra if $n = 2r-1$, $S_1,\ S_2$ be the even and odd half spinor representations of the even Clifford algebra if $n = 2r$. Then $S_1,\ S_2$ are the irreducible representations of $so(2r,\ \mathbf{C})$ corresponding to the fundamental weights $\frac{1}{2}(e_1+\cdots+e_{r-1}\pm e_r)$ of dimension $2^{r-1}$. $\wedge^r V$ is a sum of two irreducible representations $V_1^r$ and $V_2^r$ of highest weights $e_1+\cdots+e_{r-1}\pm e_r$. All other $\wedge^k V = V^k,\ k<r$ are irreducible with $V^k,\ 0<k<r-1$ being the fundamental representations. We have (see [Cv p.96]):

$$\otimes^2 S_1 = \sum_{k=r(\mathrm{mod}2)\ 0\le k\le r-1} V^k + V_1^r,$$

$$\otimes^2 S_2 = \sum_{k=r(\mathrm{mod}2)\ 0\le k\le r-1} V^k + V_2^r,$$

$$S_1\otimes S_2 = \sum_{k\ne r(\mathrm{mod}2)\ 0\le k\le r-1} V^k.$$

And for the symmetric products and the skew-symmetric products we have:

$$S^2S_1 = \sum_{k=r(\mathrm{mod}4)\ \ 0\le k\le r-1} V^k + V_1^r,$$

$$\wedge^2 S_1 = \wedge^2 S_2 = \sum_{k=r+2(\mathrm{mod}4)\ \ 0\le k\le r-1} V^k,$$

$$S^2S_2 = \sum_{k=r(\mathrm{mod}4)\ \ 0\le k\le r-1} V^k + V_2^r.$$

The case of $n = 2r-1$ can be regarded as that of a codimension 1 subspace of a space with dimension $2r$, as in [6, p.107]. Then $S$ comes from $S_1$, From $V^k,\ 0 < k < r$ (resp. $V^0$ or $V_1^r$) in the even case we obtain $V^{k-1} + V^k$ (resp. $V^0$ or $V^{r-1}$). $V^k$ are irreducible with $V^k,\ 0 < k < r-1$ fundamental representations. And we have (see also [6, p.107]):

$$\otimes^2 S = \sum_{0\le k\le r-1} V^k,$$

$$S^2S = \sum_{k=r+3,\ r(\mathrm{mod}4)\ \ 0\le k\le r-1} V^k,$$

$$\wedge^2 S = \sum_{k=r+1,\ r+2(\mathrm{mod}4)\ \ 0\le k\le r-1} V^k.$$

This discussion could be thought as a partial extension of some arguments we used in [8].

Proof of Theorem B: By [22](see also [23] , [14]), the complex cohomology ring is a representation of the Lie algebra $g = so(4,\ b_2-2)$ which is generated by all Lefschetz $sl(2)$ for all the Kähler classes in $H^2$ for all deformations of the manifold $M$.

As in [22] , we have a decomposition of $g = g_{-2} \oplus g_0 \oplus g_2$ as a graded Lie algebra, where $g_{-2}$ (resp. $g_2$) is consist of the linear combinations of $\Lambda_\omega$ (resp. $L_\omega$) with all the possible Kähler classes $\omega$ in $H^2$. We have $g_0 = so(3, b_2-3) \oplus \mathbf{R}H$.

Fix a Kähler class $x_1$. We want to prove that the multiplication by $x_1$ ($L_3$ in [22]) can be regarded as a root vector with root $e_1$ if we regard $g = so(4,\ b_2-2)$ as a special orthogonal group with a properly chosen metric

$$\mathrm{diag}(1, -1, 1, 1, -1, \cdots, -1, 1)$$

(we call $so(1, 1, 2, b_2-3, 1)$).

We can first choose the metric such that the $so(3, b_2-3)$ component of $g_0$ is:

$$g^0 = \begin{pmatrix} 0_{2,2} & 0_{2,b_2} \\ 0_{b_2,2} & so(2,\ b_2-3,1) \end{pmatrix},$$

where $so(2, b_2-2, 1)$ is the special orthogonal Lie algebra with metric $\mathrm{diag}(1,1,-1,\cdots,-1,1)$ and $0_{k,l}$ is the $k\times l$ zero matrix. We let our Cartan subalgebra be the set of the elements $\mathrm{diag}(A_1,\cdots A_r, 0)$ with

$$A_i = \begin{pmatrix} 0 & a_i \\ s_i a_i & 0 \end{pmatrix}$$

$s_1 = s_r = 1,\ s_i = -1$ if $2 \le i \le r-1$ and $r = \frac{b_2+1}{2}$. Then, the root vector with root $e_1$ is

$$\begin{pmatrix} 0 & 0 & 0_{1,b_2-1} & 1 \\ 0 & 0 & 0_{1,b_2-1} & 1 \\ 0_{b_2-1,1} & 0_{b_2-1,1} & 0_{b_2-1,b_2-1} & 0_{b_2-1,1} \\ -1 & 1 & 0_{1,b_2-1} & 0 \end{pmatrix}$$

and is in $g = so(1,1,2,b_2-3,1)$. If $\{x_1, H, \Lambda_{x_1}\}$ is the standard Kähler triple, then $H$ is $\mathrm{diag}(2A, 0,\cdots,0)$ with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The subalgebra generated by all the $K_{ij}$ in [22] is a $K_1 = so(3)$ factor of a maximal compact Lie subalgebra in $so(2, b_2-3, 1)$. And we can second choose our metric such that $K_{12}$ generates $\mathrm{diag}(0,0,so(2),0,\cdots,0)$ and the other factor of this compact Lie subalgebra is $K_2 = \mathrm{diag}(0,0,0,0,so(b_2-3),0)$, which commutes with $L_3$. We obtain that $x_1$ can be found by the condition

$$[x_1, K_2] = 0,$$

$$[x_1, K_{12}] = 0,$$

$$[H, x_1] = 2x_1$$

and is in the form we expected.

Q. E. D.

Proof of Theorem C: Since the cohomology ring $H^*$ is a representation of the graded Lie algebra $g = so(4,\ b_2-2)$ with elements of even degrees, we see that $B = H^3 \oplus H^5$ is a representation of $g$. And for any element in $B$, the $sl(2)$ representation introduced by $x_1$ has dimension 2. This is only possible if all the irreducible representations are with highest weight $\frac{1}{2}(e_1 + \cdots + e_r)$ since all the possible highest weights are

$$\sum_{k=1}^{r-1} a_k(\sum_{i=1}^{k} e_i) + \frac{a_r}{2}\sum_{i=1}^{r} e_i,$$

where $a_k$ are positive integers, i.e. if any $a_k > 0$ with $k < r$, then the coefficient of $e_1$ is at least 1. Therefore, the irreducible representations are all the even spinor representations (see, e.g., [7, Appendix], [12, p.69], [13, p.378]). The dimension of $B$ must be $k2^{\frac{b_2+1}{2}}$ for some integer $k$. We obtain our desired conclusion.

Q. E. D.

# 3 Real Structures and Examples

To give a more precise picture of the possible cohomology ring we shall apply a delicate representation theory of the real semisimple Lie groups. By [20] and [13] we shall calculate the class of the Tits algebra with applying [13, p.378–379].

**Lemma 1** *The standard spinor representation of the real even Clifford algebra of* $so(3,\ 2k)$ *has dimension* $2^{k+2}$ *for* $k = 0,\ 3(\mathrm{mod}4)$ *and* $2^{k+1}$ *for* $k = 1,\ 2(\mathrm{mod}4)$.

Proof: We apply [17, p.333 Remark 9.2.12] to our case. We only need to calculate the Clifford invariant $c = c(3,\ 2k)$. If $c$ is the quaternion (i.e., $c = -1$), then the dimension is $2 \times 2^{k+1}$ where 2 is the degree of the quaternion and $2^{k+1}$ is the degree of the even Clifford algebra. If $c$ is the real field (i.e., $c = 1$), then the dimension is the degree of the even Clifford algebra, i.e., $2^{k+1}$.

We apply the formula in [17, p.81]. Notice that $(1, -1) = (1, 1) = 1$ and $(-1, -1) = -1$, we obtain:

$$c(3,\ 8l) = -s(3,\ 8l) = -(-1)^{C_{8l}^2} = -(-1)^{4l(8l-1)} = -1,$$

$$c(3,\ 8l+6) = s(3,\ 8l+6) = (-1)^{C_{8l+6}^2} = (-1)^{(4l+3)(8l+5)} = -1,$$

similarly for $k = 4l + 1$ and $4l + 2$.

Q. E. D.

**Lemma 2.** *The even Cifford algebra $C_0(3, 2k-3)$ is a simple algebra over $\mathbf{C}$ for even $k$ and is a sum of two simple algebras if $k = 2r+1$.*

Proof: The discriminant of $(3, 2k-3)$ is $D = (-1)^{\frac{2k(2k-1)}{2}}(-1) = (-1)^{-k+1}$ (see [13, xix]). Hence $D = -1$ if $k$ is even and $D = 1$ if $k$ is odd. We obtain our lemma by [13, p.88 Theorem 8.2].

Q. E. D.

**Lemma 3.** *The even Clifford algebra $C_0(3, 4r-1)$ is a sum of two simple algebras of class $\mathbf{R}$ if $r$ is odd and is a sum of two simple algebras over quaternions if $r$ is even.*

Proof: Here we first prove that $C_0(3, 2k+1) = C(3, 2k)$ (see [6, p.47]). Let $V = V_0 + \mathbf{R}x_{2k+4}$ such that $q(x_{2k+4}) = -1$ and $V_0$ is orthogonal to $x_{2k+4}$. We consider the map

$$f: \ C(3, 2k) \to C_0(3, 2k+1) = C_1(3, 2k)x_{2k+1} + C_0(3, 2k)$$

generated by $f(1) = 1$ and $f(y) = yx_{2k+4}$ for all $y \in V_0$. Then,

$$f(y^2) = f(q(y)) = q(y) = -q(y)q(x_{2k+4}) = q(yx_{2k+4}).$$

We obtain an isomorphism.

Apply Lemma 1. Then we have Lemma 3.

Q. E. D.

Now we are ready to give some examples for the possible cohomology rings:

1. $b_2 = 3$. $g = so(4,1) = sp(1,1)$, $g^0 = so(3) = su(2) = sp(1)$. $K_{12}$ is in a Cartan subalgebra of $su(2)$ and can have the form $\mathrm{diag}(i, -i)$ (see [21, Lemma 2.2]). The spinor representation of $so(3)$ is $su(2)$, which comes from the quaternion representation $sp(1)$. In the case $l = 1$, the real dimension of $H^3$ is the real dimension of the standard representation of $su(2)$, i.e., 4. If the complex basis of the representation of $su(2)$ is $e_1, e_2$, then $H^{2,1}$ is determined by $K_{12}$ and is generated by $e_1$, $\bar{e}_2$. If $(z_1, z_2)$ is the coordinates, then $H^6$ contains the tracefree skew-Hermitian forms on $(z_1, z_2)$. $H^{4,2} =< z_1 \wedge \bar{z}_2 >$, $H^{3,3} =< iz_1 \wedge \bar{z}_1 - iz_2 \wedge \bar{z}_2 >$, $H^{2,4} =< z_2 \wedge \bar{z}_1 >$.

2. $b_2 = 4$. $g = so(4,2) = su(2,2)$, $g^0 = so(3,1) = sl(2, \mathbf{C})$. $K_{12}$ is in a Cartan subalgebra of a compact Lie subalgebra $su(2)$ of $sl(2, \mathbf{C})$ and can have the form $\mathrm{diag}(i, -i)$. The spinor representation of $so(3,1)$ is $sl(2, \mathbf{C})$, which is complex, with real dimension 4. If $l = 1$ and the complex basis of the representation of $sl(2, \mathbf{C})$ is $(e_1.e_2)$, then $H^{2,1}$ is generated by $e_1, \bar{e}_2$. Let

$(z_1, z_2)$ be the coordinates, then $H^6$ contains all the skew-Hermitian forms on $(z_1, z_2)$. $H^{4,2} =< z_1 \wedge \bar{z}_2 >$, $H^{3,3} =< iz_1 \wedge \bar{z}_1, iz_2 \wedge \bar{z}_2 >$, $H^{2,4} =< z_2 \wedge \bar{z}_1 >$.

3. $b_2 = 5$. $g = so(4,3)$, $g^0 = so(3,2) = sp(2, \mathbf{R})$. A maximal compact Lie subalgebra of $so(3,2)$ is $so(3) \times so(2)$ with center $so(2)$. A maximal compact Lie subalgebra of $sp(2, \mathbf{R})$ is $sp(2, \mathbf{R}) \cap so(4)$ containing all the matrices

$$(A|B) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

with $A$ a $2 \times 2$ skew-symmetric matrix and $B$ a $2 \times 2$ symmetric matrix. This Lie algebra is isomorphic to $u(2)$ with the isomorphism:

$$\alpha : \ A + iB \to (A|B).$$

Therefore, the $so(3)$ in $so(3,2)$ is corresponding to the $su(2)$ in $u(2)$ and $K_{12}$ can have a form

$$\alpha(\mathrm{diag}(i, -i)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The spinor representation of $so(3,2)$ is $sp(2, \mathbf{R})$, which is real, with real dimension 4. If $l = 1$ and $(e_1, e_2, e_3, e_4)$ be the real basis of the spinor representation, then $K_{12}$ acts as the complex structure and $K_{12}e_1 = -e_3$, $K_{12}e_2 = e_4$. $H^{2,1} =< e_1 + ie_3, e_2 - ie_4 >$. $H^{4,2} =< (e_1 + ie_3) \wedge (e_2 - ie_4) >$, $H^{3,3} =< e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3, e_1 \wedge e_2 - e_3 \wedge e_4 >$, $H^{2,4} =< (e_1 - ie_3) \wedge (e_2 + ie_4) >$.

4. $b_2 = 6$. $g = so(4,4)$, $g^0 = so(3,3) = sl(4, \mathbf{R})$. The even half spinor representation of $so(3,3)$ is $sl(4, \mathbf{R})$ with real dimension 4. The odd half spinor representation is the adjoint of the even representation. A maximal compact Lie subalgebra of $so(3,3)$ is $so(3) \times so(3)$ and a maximal compact Lie subalgebra of $sl(4, \mathbf{R})$ is $so(4)$ with a root system generated by $\mathrm{diag}(J, -J), \mathrm{diag}(J, J)$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore, $K_{12}$ can have a form $\mathrm{diag}(J, J)$. If $l = 1$, we might assume that $H^3$ is the even half spinor representation and $(x_1, x_2, x_3, x_4)$ is a real basis, then $K_{12} = \mathrm{diag}(J, J)$ acts as a complex structure on $\mathbf{R}^4$. For our convenient, we let $e_k = x_{2k-1} + ix_{2k}$ for $k = 1, 2$. Then, $H^{2,1} =< e_1, e_2 >$. $H^6$ contains all the skew-symmetric forms on $\mathbf{R}^4$ and $H^{4,2} =< e_1 \wedge e_2 >$, $H^{3,3} =< ie_k \wedge \bar{e}_l, \ k, l = 1, 2 >$.

5. $b_2 = 7$. If $b_3 = 8$, this occurs as the Kummer manifold $K_2$ which was first appeared in [5]. $g = so(4,5)$, $g^0 = so(3,4)$. The spinor representation is real with dimension 8.

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