On the Automorphism Group of Compact Symplectic Manifold

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In this note we give a structure theorem for a finite dimension subgroup of the automorphism group of a compact symplectic manifold. An application of this result is a simpler and transparent proof of the classification of compact homogeneous spaces with invariant symplectic structures. We also give another proof of the classification from the general theory of compact homogeneous space which lead us to a splitting conjecture on compact homogeneous spaces with symplectic structures (which are not necessarily invariant under the group action) that makes the classification of this kind of manifolds possible.

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1 Introduction

A smooth $2n$-dimensional manifold M equiped with a smooth transitive action of a Lie group is called a *homogeneous space*. If in additional M is a symplectic manifold, we refer to it as a *homogeneous space with a symplectic structure* and, if the structure is invariant, a *homogeneous space with an invariant symplectic structure*.

Recently there has been much progress in the area of symplectic manifolds and group actions. I was interested in the *classical* problem of classifying compact homogeneous spaces with symplectic structure. The difficulty is that we do not know anything about the transitive group and the isotropy group (Cf. [DG1,2,Hk]). In the Kähler case we know that the isometric group is compact. In this note we prove following theorem:

Main Theorem: *Every finite dimension Lie subgroup of the automorphism group of a compact symplectic manifold is locally a product of a compact semisimple group and a 2-step solvable group R. Moreover, the adjoint representation of R on R' is a subgroup of a compact torus.*

An application of this result is a simpler and more prospective proof of the classification of the compact homogeneous manifolds with invariant symplectic structures.

I am also interested in the structure of compact homogeneous manifold with symplectic structure (which might not be invariant under the group action)

In [Gu1,2] we also proved the following theorem:

Proposition 1. *Every compact homogeneous complex manifold with a 2-cohomology class ω such that ω^n is not zero in the top cohomology is a* *product of a rational homogeneous space and a complex parallelizable solvmanifold with a right invariant symplectic structure on its universal covering.*

This generalized the result of [BR] for the Kähler case (one does not assume that the Kähler form is invariant).

These results suggest further study in two directions along the lines of Proposition 1. One is the classification of compact complex homogeneous space; the other is the classification of compact homogeneous space with a symplectic structure. The first problem can be solved by the method in [Gu3,7], where in the first paper we prove that every compact complex homogeneous space with an invariant volume is a torus bundle over a product of a rational homogeneous space and a complex parallelizable manifold and in the second paper we classify compact complex homogeneous spaces of 1-step. In the present paper we are working in the direction of the second problem. This is quite analogous to the results in [Gu1,2]:

Proposition 2. *Every compact homogeneous space with an invariant symplectic structure is a product of a rational homogeneous space and a torus with invariant symplectic structures.*

In this classification, the tori which occur are not necessarily standard. The following conjecture arises naturally from our arguments for the proof of the above proposition:

CONJECTURE. *If G/H is a compact homogeneous space with a symplectic structure, then G/H is diffeomorphic to a product of a rational homogeneous space and a finite quotient of a compact locally flat parallelizable manifold with a symplectic structure.*

Here we call a manifold N *locally flat parallelizable* if $N = G/H$ for a simply connected Lie group G which is diffeomorphic to $\mathbf{R}^k$ for some integer k and H is a uniform discrete subgroup.

In our future work we will attempt to prove this conjecture and to classify the compact locally flat parallelizable manifolds with a symplectic structure.

We prove our Proposition 2 at the end of 1997 in the line of the section 3—we prove the diffeomorphism in [Gu5], then the symplectic isomorphism in [Gu6]. Then we get the Main Theorem. But we just found the paper [ZB] which literally included all our results in [Gu5,6] with a different proof. So in this note we combine our three papers in one and keep the proof in the section 3 to show the reason of the Conjecture and the relation to the general compact homogeneous manifold theory.

Another consequence of our main theorem is that *we can start to consider the classification of compact symplectic manifold of codimension* 1.

2 The Proof of the Main Theorem

2.1 Preliminary

Let $(M,\ \omega)$ be a symplectic manifold and G a Lie group of symplectic diffeomorphisms of M, i.e., a smooth action $G\times M\to M$ such that $g^*\omega=\omega$ for all $g\in G$. Let $\mathrm{Ham}_{loc}(M)$ be the set of smooth vector fields X on M such that $L_X\omega=0$. In this situation we have the following sequence:

$$0\to\mathbf{R}\xrightarrow{i}C^\infty(M)\stackrel{\mathrm{sgrad}}{\to}\mathrm{Ham}_{loc}(M)\stackrel{\alpha}{\leftarrow}\mathcal{G}$$

where i realizes the real numbers as constant functions, the skew-gradient $\mathrm{sgrad}(f)$ is a vector field X_f such that $i_{X_f}\omega=\omega(X_f,\ )=df$, and α is the natural Lie algebra homomorphism arising from the G–action. The associated Lie algebra structure $\{\ ,\ \}$ on $C^\infty(M)$ is defined by

$$\{f,g\} = \omega(\mathrm{sgrad}(f), \mathrm{sgrad}(g)).$$

It follows that sgrad: $C^\infty(M) \to \mathrm{Ham}_{loc}(M)$ is a Lie algebra homomorphism, and we are confronted with a *lifting* question: Does there exist a Lie morphism $\lambda : \mathcal{G} \to C^\infty(M)$ such that $\mathrm{sgrad}\circ\lambda = \alpha$? If such a lifting exists, we refer to the G–action as a *Poisson action* (with regard to the lifting). In this case the G–equivariant dual map

$$\Phi :\ M \to \mathcal{G}^*, \quad \Phi(x)(\xi) = \lambda(\xi)(x)$$

is called the moment map. If every $\mathcal{G}$–field is the skew-gradient of some function, i.e., if for every $\xi \in \mathcal{G}$ the associated vector field is of the form $\xi_M = \mathrm{sgrad}(f_\xi)$, then the G–action is called a *Hamiltonian action* (our definition is different from the one in [GS]).

The following is a list of elementary observations in the above setting (see [GS]):

(1) Let G' be the commutator of G, then the G'–action is Hamiltonian, and if G is a semisimple group, then it induces a Poisson action (see also [GS p.185]).

(2) Suppose that $\xi \in \mathcal{G}$ can be lifted. Then

$$\{\ x \in M \mid df_\xi(x) = 0\ \} = \{\ x \in M \mid \xi_M(x) = 0\ \}.$$

(3) If the G–action is Poisson with moment map $\Phi :\ M \to \mathcal{G}^*$, then

$$\ker(d\Phi_x) = \{\ v \in T_xM \mid \omega_x(v,w) = 0 \text{ for all } w \in T_xG(x)\ \} = (T_xG(x))^\perp,$$

where $(T_xG(x))^\perp$ is the skew-orthogonal complement to the tangent space of the G–orbit $G(x)$.

(1), (2) and (3) come from the properties of the Poisson bracket. If M is compact, we might use $C_0^\infty(M) = \{f \in C^\infty(M)|_{\int_M f\omega^n=0}\}$ instead of $C^\infty(M)$ in (1) and we see that G' is actually Poisson by the argument of [GS p.186–187]. We have:

Lemma 1. *If M is compact, then the Lie algebra of the exact Hamiltonian group of M is C_0^∞. The group is "compact" in the sense that its Lie algebra has an invariant positive definite inner product $(f, g) = \int_M fg\omega^n$.*

Proof: It is known that the exact Hamiltonian group of M is generated by functions h with $\int_M h\omega^n = 0$. All these kind of functions consists a infinite dimensional Lie algebra since

$$\int_M \{f, g\}\omega^n = \int_M df \wedge dg \wedge \omega^{n-1} = 0.$$

This metric is invariant since

$$\begin{aligned}
&(\{h,\ f\},\ g) + (f,\ \{h,\ g\}) \\
&= \int_M (gdh \wedge df + fdh \wedge dg) \wedge \omega^{n-1} \\
&= \int_M d(fg) \wedge dh \wedge \omega^{n-1} = 0.
\end{aligned}$$

Q. E. D.

For a further application of this section, we also list the following property:

(4) If G is as in (3) and $G(x) = G/H$ is a generic orbit with moment fibering

$$\Phi|_{G(x)} :\ G/H \to G/J = G(\Phi(x)) \quad x \in M,$$

then $H^0 \triangleleft J^0$ and J^0/H^0 is abelian (see [GS p.190]).

2.2 The Proof

Now we are ready to prove the Main Theorem:

Since G' is a subgroup of the exact Hamiltonian group, there is an invariant inner product (f, g) on its Lie algebra. Moreover, G acts on the Lie algebra of G' and keeps the inner product invariant. We see that the adjoint group of G on G' is a subgroup of a compact group and it splits locally into a product of its compact semisimple part and the abelian part. So G is locally a product of a compact semisimple group and a solvable 2-step Lie group R. And the adjoint action of R on R' is a subgroup of a torus.

3 On the Classification of Compact Homogeneous Manifolds with Invariant Symplectic Structures

3.1 Preliminaries

1. A *rational homogeneous manifold* Q is a compact complex manifold which can be realized as a closed orbit of a linear algebraic group in some projective space. Equivalently, $Q = S/P$ where S is a complex semisimple Lie group and P a parabolic subgroup, i.e., a subgroup of S which contains a maximal connected solvable subgroup (Borel group). Every homogeneous rational manifold is simply-connected and is therefore an orbit of a compact group. In general, a quotient K/L with K compact and semisimple carries a K-invariant complex structure which is projective algebraic if and only if L is the centralizer $C(T)$ of a torus $T \subset K$.

A *parallelizable manifold* is the quotient of a Lie group by a discrete subgroup.

A *solv-manifold* is a homogeneous space with a solvable Lie group.

In our research we are interested in *locally flat parallelizable manifolds* in the sense that the Lie group is *locally flat*, i.e., its universal covering is diffeomorphism to $\mathbf{R}^k$ for some integer k. By the classification of the Lie group, we see that a Lie group G is locally flat if and only if its semisimple part is locally isomorphic to a product of $SL(2,\ \mathbf{R})$'s.

2. In this subsection we recall some basic results about simply connected Lie groups (see [On]). First, any simply connected Lie group has a Levi decomposition $G = SR$ with S a semisimple subgroup, R a simply connected maximal solvable normal subgroup and $S \cap R = \{e\}$.

A connected closed subgroup $U \subset G$ is said to be a *k-subgroup* if the manifold G/U is compact. Let $\mathcal{G}$ be the Lie algebra of G, $\mathrm{Int}\mathcal{G}$ the Lie group of its inner automorphisms. A subalgebra $\mathcal{U} \subset \mathcal{G}$ is said to be *compact* in $\mathcal{G}$ if the connected subgroup in $\mathrm{Int}\mathcal{G}$ corresponding to the subalgebra $\mathrm{Ad}_{\mathcal{G}}\mathcal{U} \subset \mathrm{Ad}\mathcal{G}$ is compact. A subalgebra $\mathcal{H} \subset \mathcal{G}$ is said to be a *k-subalgebra* if there exists a subalgebra $\mathcal{U}$ which is compact in $\mathcal{G}$, such that $\mathcal{G} = \mathcal{U} + \mathcal{H}$.

Let $\mathcal{G}$ be a semismple Lie algebra, $\mathcal{U}$ a maximal compact subalgebra in $\mathcal{G}$, and $\mathcal{G} = \mathcal{U} + \mathcal{P}$ the Cartan decomposition. Let θ denote an involutive automorphism of $\mathcal{G}$ which is given by the formula $\theta(x + y) = x - y$, where $x \in \mathcal{U},\ y \in \mathcal{P}$. Let $\mathcal{H}_-$ be a maximal abelian subalgebra in $\mathcal{P}$ and let $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-$ be a Cartan subalgebra in $\mathcal{G}$ which contains $\mathcal{H}_-$ and $\mathcal{H}_+ \subset \mathcal{U}$. We consider the complexification $\mathcal{G}^{\mathbf{C}}$ of $\mathcal{G}$ and let us denote by σ the corresponding conjugation in $\mathcal{G}^{\mathbf{C}}$. Then $\mathcal{H}^{\mathbf{C}}$ is a Cartan subalgebra in $\mathcal{G}^{\mathbf{C}}$. Let us denote by $\Sigma \subset (\mathcal{H}^{\mathbf{C}})^*$ the corresponding system of roots of $\mathcal{G}^{\mathbf{C}}$. If $\alpha \in \Sigma$, then by $\mathcal{G}_\alpha$ we denote its root subspace in $\mathcal{G}^{\mathbf{C}}$. The roots of Σ are real in the subspace $\mathcal{H}_* = i\mathcal{H}_+ + \mathcal{H}_-$. Let us determine on $(\mathcal{H}^{\mathbf{C}})^*$ an anti-involution σ^* by the formula

$$(\sigma^*\phi)(x) = \overline{\phi(\sigma x)} \quad (x \in \mathcal{H}^{\mathbf{C}}).$$

If $\alpha \in \Sigma$, then $\sigma^*\alpha$ is also a root, and $\sigma\mathcal{G}_\alpha = \mathcal{G}_{\sigma^*\alpha}$. Let $\Sigma_0 = \{\alpha \in \Sigma,\ \sigma^*\alpha = -\alpha\}$. Then $\Sigma_0 = \{\alpha \in \Sigma|_{\alpha|_{\mathcal{H}_-}=0}\}$. Let $\Sigma_1 = \Sigma\backslash\Sigma_0$. It is clear that Σ_0 and Σ_1 are invariant relative to σ^*.

If in $(\mathcal{H}^{\mathbf{C}})^*$ we introduce an ordering and if $\Delta \subset \Sigma$, let us denote by Δ^+ and Δ^- the sets of positive and negative roots respectively, which belong to Δ. It is known that this ordering may be introduced in such a manner that Σ_1^+ will be invariant relative to σ^*. Then Σ_1^- is also invariant, and $\sigma^*\Sigma_0^+ = \Sigma_0^-$.

The subsystem $\Delta \subset \Sigma$ is said to be *closed* if for any $\alpha,\ \beta \in \Delta$ such that $\alpha + \beta \in \Sigma$, we have $\alpha + \beta \in \Delta$. Clearly, Σ_0, Σ_1^+ and Σ_1^- are closed subsystems.

Let us now set $\mathcal{N}^{\mathbf{C}} = \Sigma_{\alpha\in\Sigma_1^+}\mathcal{G}_\alpha$. From the fact that system Σ_1^+ is closed, it is clear that $\mathcal{N}^{\mathbf{C}}$ is a nilpotent subalgebra in $\mathcal{G}^{\mathbf{C}}$. Moreover, $\sigma(\mathcal{N}^{\mathbf{C}}) = \mathcal{N}^{\mathbf{C}}$. Hence, if we let $\mathcal{N} = \mathcal{N}^{\mathbf{C}} \cap \mathcal{G}$, we have that $\mathcal{N}^{\mathbf{C}}$ is a complex envelope of $\mathcal{N}$. As it is well known, a so-called Iwasawa decomposition holds; namely:

$$\mathcal{G} = \mathcal{U} + \mathcal{H}_- + \mathcal{N}.$$

or $G = UH_-N$ with H_-N a maximal triangular subgroup in G. Let $\Pi \subset \Sigma^+$ be a system of simple roots, $\Pi_0 = \Pi \cap \Sigma_0$, $\Pi_1 = \Pi \cap \Sigma_1$. It appears that for every $\alpha \in \Pi_1$ there exist a $\beta \in \Pi_1$ such that $\sigma^*\alpha - \beta = \sum_{\gamma\in\Pi_0} k_\gamma\gamma$, where $k_\gamma \geq 0$. If we set $\beta = \tilde{\sigma}\alpha$, then we have an involution $\tilde{\sigma}$ of system Π_1.

An algebra $\mathcal{G}$ is said to be *normal* if $\mathcal{H}_- = \mathcal{H}$. In this case $\Sigma_1 = \Sigma$ and $\tilde{\sigma} = 1$. It is clear that in every complex semisimple Lie algebra there exists exactly one (up to conjugacy) normal real form.

We find it convenient to write a real semisimple Lie algebra $\mathcal{G}$ by means of its Satake scheme. This scheme is constructed as follows. We take the Dynkin diagram of $\mathcal{G}^{\mathbf{C}}$; that is, the diagram of the system Π. The circles which correspond to elements which belong to Π_0 are blackened; those belonging to Π_1 are left white. In addition, in Π_1 by means of arrows we show pairs of roots which are permuted by the involution $\tilde{\sigma}$.

We call a subgroup L of G a *t-subgroup* if $H_-NN_R \subset L$, where H_-N is a maximal triangular subgroup of the semisimple part of G and N_R is the maximal nilpotent normal subgroup in R. To describe a t-subgroup it is sufficient to describe its intersection with the semisimple part of G.

Using above terminology, let us describe the well-known method for the construction of t-subalgebras in a semisimple Lie algebra $\mathcal{G}$. Let $\Gamma \subset \Pi_1$ be some subset which is invariant relative to $\tilde{\sigma}$. Let us denote by Σ' the system of roots which can be linearly expressed by the system $\Pi_0 \cup \Gamma$ and by Σ'' a system of all roots which can not be expressed linearly by $\Pi_0 \cup \Gamma$. It is clear that Σ' and Σ'' are closed systems. In what follows, it is convenient to study an element $x_\Gamma \in \mathcal{H}$ which is defined by the formulas

$$\alpha(x_\Gamma) = \begin{cases} 0, & \text{if} \quad \alpha \in \Pi_0 \cup \Gamma, \\ 1, & \text{if} \quad \alpha \in \Pi_1 \backslash \Gamma. \end{cases}$$

Clearly, such an element exists and is uniquely determined. We will show that $x_\Gamma \in \mathcal{H}_-$. In order to do this, one must verify that $\sigma x_\Gamma = x_\Gamma$ or that $(\sigma^*\alpha)(x_\Gamma) = \overline{\alpha(x_\Gamma)} = \alpha(x_\Gamma)$ for all $\alpha \in \Pi$. For $\alpha \in \Pi_0$ this is true because $\sigma^*\alpha = -\alpha$. If $\alpha \in \Gamma$, then $\sigma^*\alpha = \tilde{\sigma}\alpha + \sum_{\gamma \in \Pi_0} k_\gamma \gamma$, where $\tilde{\sigma}\alpha \in \Gamma$. For this reason $(\sigma^*\alpha)(x_\Gamma) = (\tilde{\sigma}\alpha)(x_\Gamma) = 0 = \alpha(x_\Gamma)$. Similarly we verify it for $\alpha \in \Pi_1 \backslash \Gamma$.

We notice that $\Sigma' = \{\alpha \in \Sigma, \alpha(x_\Gamma) = 0\}$, $\Sigma''^+ = \{\alpha \in \Sigma,\ \alpha(x_\Gamma) > 0\}$. Because $x_\Gamma \in \mathcal{H}_-$, it follows at once from what we have proved that Σ' and

Σ''^{+} are invariant relative to σ^*.

Let us denote by $\mathcal{Z}_\Gamma$ the centralizer of the element x_Γ in $\mathcal{G}$. It is clear that $\mathcal{Z}_\Gamma^{\mathbf{C}}$ is the centralizer of x_Γ in $\mathcal{G}^{\mathbf{C}}$ and

$$\mathcal{Z}_\Gamma^{\mathbf{C}} = \mathcal{H}^{\mathbf{C}} + \sum_{\alpha\in\Sigma'} \mathcal{G}_\alpha.$$

Let us denote by $\mathcal{C}_\Gamma$ the subspace in $\mathcal{H}$ which is annihilated by all roots in $\Pi_0 \cup \Gamma$ or, what comes to the same thing, in Σ', it is the center of $\mathcal{Z}_\Gamma$. And $\mathcal{Z}_\Gamma = \mathcal{C}_\Gamma + \mathcal{S}_\Gamma$, where $\mathcal{S}_\Gamma$ is the semisimple part of $\mathcal{Z}_\Gamma$.

Let us set $\mathcal{N}_\Gamma^{\mathbf{C}} = \sum_{\gamma\in\Sigma''^{+}} \mathcal{G}_\gamma$. It is a nilpotent subalgebra of $\mathcal{G}^{\mathbf{C}}$ which is invariant under σ. Consequently $\mathcal{N}_\Gamma^{\mathbf{C}}$ is a complex envelope of $\mathcal{N}_\Gamma = \mathcal{N}_\Gamma^{\mathbf{C}} \cap \mathcal{G}$. In particular, $\mathcal{N}_\emptyset = \mathcal{N}$.

Finally, let us set $\mathcal{U}_\Gamma = \mathcal{Z}_\Gamma + \mathcal{N}_\Gamma$. Then $\mathcal{U}_\Gamma^{\mathbf{C}}$ is a parabolic subalgebra in $\mathcal{G}$ and $\mathcal{N}_\Gamma$ is the nilradical of $\mathcal{U}_\Gamma$. Let $\mathcal{C}_\Gamma^{+} = \mathcal{C}_\Gamma \cap \mathcal{U}$, $\mathcal{C}_\Gamma^{-} = \mathcal{P} \cap \mathcal{C}_\Gamma$ and $\mathcal{M}_\Gamma = \mathcal{C}_\Gamma^{+} + \mathcal{E}_\Gamma$ be the maximal compact ideal of $\mathcal{Z}_\Gamma$ with $\mathcal{E}_\Gamma$ semisimple, $\mathcal{Z}'_\Gamma$ be its complement in $\mathcal{Z}_\Gamma$. Then we call a subalgebra

$$\mathcal{T} = \mathcal{M} + \mathcal{Z}'_\Gamma + \mathcal{N}_\Gamma$$

with $\mathcal{M}$ a subalgebra of $\mathcal{M}_\Gamma$ a *standard t-subalgebra*. Then we have:

Proposition 3. *Every t-subalgebra of a semisimple Lie algebra $\mathcal{G}$ is a standard t-subalgebra. Moreover, the normalizer of $\mathcal{T}$ in $\mathcal{G}$ is $P(\mathcal{M}) + \mathcal{Z}'_\Gamma + \mathcal{N}_\Gamma$, where $P(\mathcal{M})$ is the normalizer of $\mathcal{M}$ in $\mathcal{M}_\Gamma$.*

3. In this subsection we recall some basic results on a generalization of the *Tits fibration*, introduced by V. V. Gorbatservich [Gb1] to compact homogeneous spaces. It coincides with a fibration considered by Tits [Ti] in the case of compact complex homogeneous spaces. We call it the *double normalizer fibration* as in [Gb1] or the *the Gorbatservich fibration*. Let

$M = G/H$, H^0 be the identity component of H and $P(L) = Norm_G(L^0)$ the normalizer of the identity component of a subgroup L in G, $P^k(L) = P(P^{k-1}(L))$ we have:

Proposition 4. *Let G be a connected real Lie group acting almost effectively and transitively on the manifold $M = G/H$ and let $G/H \to G/P^2(H)$ be the double normalizer fibration.*

Then

- (*a*) *$P^k(H) = P^2(H)$ for all $k \geq 2$. In particular, the double normalizer fibration of $G/P^2(H)$ is a trivial fibering.*
- (*b*) *$G/P(H)$ is a compact homogeneous space in $\mathbf{R}P^k$ for some integer k such that G acts as a linear group and $P(H)$ is a t-subgroup of G. In particular, the nilradical of G is in $P(H)$ and the intersection of a semisimple part of G with $P(H)$ is a standard t-subgroup.*
- (*c*) *Any normalizer bundle of $G/P^2(H)$ is itself.*
- (*d*) *The semisimple part S of G acts on $G/P^2(H)$ transitively. There is a maximal connected compact subgroup K of S acts on $G/P^2(H)$ transitively, i.e., $G/P^2(H) = K/K \cap P^2(H)$.*

If G is a complex Lie group and $H \subset G$ is a closed complex subgroup, then we have the *normalizer fibration* $G/H \to G/P(H)$ and $P^k(H) = P(H)$. Let $\mathcal{G}$ and $\mathcal{H}$ denote the Lie algebras of G and H, respectively. The base space G/N is realized as the $\mathrm{Ad}(G)$–orbit of the subspace $\mathcal{H}$ in the Grassmann manifold of subspaces of $\mathcal{G}$ that have the same dimension as that of $\mathcal{H}$. And $G/P(H)$ is a rational homogeneous manifold and $P(H)/H$ is a compact parallelizable homogeneous manifold.

4. In this subsection we will recall the foliation fibering induced by the action of the maximal compact subgroup of G. This was considered by V. V. Gorbatservich in [Gb2]. He proved following structure theorem:

Proposition 5. *Let $M = G/H$ be a compact homogeneous, K be a maximal compact Lie subgroup in G. Then all the K orbits have same dimension and $M//K$ is a good orbifold, i.e., has a compact manifold as a covering. Moreover, there is a subgroup $H' \subset H$ of finite index such that all the K orbits on $M' = G/H'$ are the same and $M'//K$ is smooth with $\mathbf{R}^k$ as its universal covering for some integer k.*

5. In the rest of this paper we will use frequently arguments on the Lie algebra level.

First we recall the following result due to Koszul [Kz]:

Proposition 6. *Let G be a real Lie group and H a closed subgroup. Then G/H admits a G-invariant symplectic structure if and only if there exist a 2 form ρ on $\mathcal{G}$ which satisfies following conditions for all $x,\ y,\ z \in \mathcal{G}$ and $h \in H$*

$$\rho([x,\ y],\ z) + \rho([y,\ z],\ x) + \rho([z,\ x],\ y) = 0,$$
$$\rho(\mathrm{Ad}h(x),\ \mathrm{Ad}h(y)) = \rho(x,\ y).$$

6. Here we collect some results we need from the splitting theory of the Lie group (see [Gb3]). Let $G = SR$ be a Levi decomposition of a semisimple Lie group. We call G a *splittable Lie group* if $R = TU$ with $T \cap U = \{e\}$ such that T acts semisimplely and U acts unipotently on the Lie algebra $\mathcal{G}$. We call a Lie group embedding $\alpha :\ G \to M(G)$ from G to a splittable simply connected Lie group $M(G) = T \cdot S \cdot U$ a *Mal'cev splitting* or *M-splitting* if $\alpha(G)$ is a normal subgroup of $M(G)$ and $M(G)$ is a semidirect product of T and $\alpha(G)$, and $\alpha(G) \cdot U = M(G)$.

Proposition 7. *For any simply connected Lie group G there is a unique Mal'cev splitting.*

The Mal'cev splitting can be constructed as following:

Let $G = S \cdot R$ be the Levi decomposition of a connected simply connected Lie group G. Consider the adjoint representation $\mathrm{Ad}_G : G \to GL(\mathcal{G})$; put $G^* = \mathrm{Ad}_G(G)$, and let $< G >$ be the algebraic closure of G^* in $GL(\mathcal{G})$. Since $< G >$ is algebraic, it has a Chevalley decomposition $< G >= T^* S^* U^*$, where U^* is the unipotent radical, S^* is semisimple, and T^* is abelian and consists of semisimple (i.e., completely reducible) elements. Put $W^* = S^* U^*$; then $< G >= T^* W^*$, with $T^* \cap W^*$ finite. Let $t^* : T^* W^* \to T^*/T^* \cap W^*$ be the natural epimorphism, with kernel W^*. Writing $\hat{T} = T^*/T^* \cap W^*$, we have clearly $t^*(\mathrm{Ad}G) \subset (\hat{T})^0$, since G is connected. If for the connected abelian Lie group $(T^*)^0$ we consider the universal covering for $\pi_T : \tilde{T} \to (T^*)^0$, it is obvious that $t^* \cdot \pi_T : \tilde{T} \to (\hat{T})^0$ is the universal covering for $(\hat{T})^0$. Since G is connected and simply connected, there exists a unique homomorphism $\tilde{t} : G \to \tilde{T}$ such that $t^* \cdot \pi_T \cdot \tilde{t} = t^* \cdot \mathrm{Ad}_G$. Put $T = \tilde{t}(G)$, $T^*_G = \pi_T \cdot \tilde{t}(G)$; then T is a connected simply connected abelian Lie group covering of T^*, while $T^*_G \subset < G >$. We see that T^*_G can be regarded as a subgroup of $\mathrm{Aut}G$. The imbedding $T^*_G \to \mathrm{Aut}G$ and the homomorphism π_T induce a homomorphism $\phi : T \to \mathrm{Aut}G$, with $\ker\phi = \ker\pi_T \cap T$ discrete. Then we can get the Mal'cev splitting $M(G) = T \times_\phi G$ and $M(G) = TSU$ for a unipotent group U such that $\dim U = \dim R$, $\dim U/N_R = \dim T$, where N_R is the nilpotent radical of G. Now we let $W_G = SU$, $W_{G,l} = S/l(S) \cdot U$, then $\mathrm{Aut}W_{G,l}$ and the semidirect product $\mathrm{Aut}W_{G,l} \propto W_{G,l}$ are prealgebraic groups. We can regard T_G^* as a subgroup of $\mathrm{Aut}W_{G,l}$. Let $a(T_G^*)$ be the prealgebraic hull of T_G^* in $\mathrm{Aut}W_{G,l}$, and $\mathcal{A}_l(G) = a(T_G^*) \propto W_{G,l}$. We see that $\mathcal{A}_l(G)$ is prealgebraic. Let $M_l(G) = T_G^* S_l U$ as a quotient of $M(G)$, then:

Proposition 8. *The group $\mathcal{A}_l(G)$ is prealgebraic, and there exists an imbedding $\beta:\ M_l(G) \to \mathcal{A}_l(G)$ such that the following properties hold:*

1) *$\mathcal{A}_l(G)$ is splittable, and if $\mathcal{A}_l(G) = T'S'U'$, where U' is unipotent, S' semisimple and T' a prealgebraic torus, then $\beta(M_l(G)) \supset S'U'$ and $S' = S_l$, where S is the semisimple part of G and $U' = U$.*

2) *The prealgebraic closure of each of the subgroup $\beta(G_l)$ and $\beta(M_l(G))$ in $\mathcal{A}_l(G)$ is $\mathcal{A}_l(G)$ itself.*

Here we like to give a very simple example: Let $G = G_1 \times G_2,\ G_1 = TN$ with T, N, G_2 abelian and T acts on N almost faithfully and as a compact torus without any eigenvector. Then $< G >= \mathrm{Ad}_G(T)N,\ W^* = N$,

$$t^*:\ \mathrm{Ad}_G(T)N \to \mathrm{Ad}_G(T) = \hat{T}$$

$$\pi_T:\ T \to \mathrm{Ad}_G(T)$$

$$\tilde{t}:\ TN \times G_2 \to T$$

$$T_G^* = \mathrm{Ad}_G(T),\ \phi:\ T \to \mathrm{Ad}_G(T)$$

$$M(G) = T \times_\phi G = TU,\ U = \{(t, t^{-1}, n, g)|_{t\in T\ n\in N\ g\in G_2}\}$$

$$W_G = W_{G,l} = U,\ \mathcal{A}_l(G) = M_l(G) = \mathrm{Ad}_G(T)U.$$

7. Here we recall the *Gorbatservich modification* for a compact homogeneous spaces. This is first used in [Gb4]. Similar construction can be found in the study of homogeneous Kähler manifolds, e.g., [Dm], [DN].

Let $M = G/H$ be a compact homogeneous space of a simply connected Lie group G. We set $G_* = G_l = G/l(S)$ be the image of $G(H_* = H/H \cap l(S)$ be the image of $H)$ in $\mathcal{A}_l(G)$. We also set $P_* = N_{\mathcal{A}_l(G)}(H_*^0)$, the normalizer of H_*^0. Since the subgroup H_*^0 is connected, then its normalizer is a prealgebraic group, i.e., the identity component of an algebraic group. Hence the group $\pi_0(P_*)$ is finite. Passing from H to the subgroup $H_1 = H \cap \pi^{-1}(P_*^0 \cap H_*)$ of finite index, where $\pi : M(G) \to M_l(G)$ is the natural epimorphism, we might assume that $H_* \subset P_*^0$ by considering a finite covering M' of M. This inclusion will be assumed to hold in what follows.

We consider the natural epimorphism $\gamma : \mathcal{A}_l(G) \to \mathcal{A}_l(G)/W_l$. We have $\mathcal{A}_l(G)/W_l = T_* \times \pi(W_G)/W_l$ with $W_G = SU$, $W_l = S_l N_R$ (our W_l is the same as in [Gb4] but different from the one in [Gb3], in [Gb3] $W_l = S/l(S) \cdot U$) and T_* is a prealgebraic torus; $\pi(W_G)/W_l = U/N_R$. So $\mathrm{Im}\gamma = T_* \times U/N_R$, we denote it by A. A is connected and Abelian. There is a natural embedding of the group $G_*/W_l = R/N_R$ in $M_l(G)/W_l$ which is contained in A.

We denote the image of R/N_R by B. By $B \cap T_* = \{e\}$ we see that the projection $\mu : T_* \times U/N_R \to U/N_R$ to the second factor is an isomorphism on B, i.e., B is closed in A. Now we consider the subgroup $H_*/H_* \cap W_l$ of A and its closure $\overline{H_*/H_* \cap W_l}$ (in the Euclidean topology) which we denote by A_1. Since $H_*/H_* \cap W_l \subset B$ we have $A_1 \subset B$. Since the group B is simply connected and Abelian, A_1 is a closed subgroup of it, A_1 is torsion free and isomorphic to $\mathbf{R}^p \times \mathbf{Z}^q$ for some $p, q \geq 0$.

Finally we consider the subgroup $\gamma(P_*) \subset A$. The subgroup $\mathrm{Ker}\gamma = W_l$ is closed in the "Zariski topology" on $\mathcal{A}_l(G)$, so does the Lie group P_*, therefore $\gamma(P_*)$ is a closed subgroup of A. But $H_* \subset P_*$, so $H_*/H_* \cap W_l \subset \gamma(P_*)$ and hence $A_1 \subset \gamma(P_*)$, i.e., $A_1 \subset \gamma(P_*^0)$ by our convention. The group $\gamma(P_*^0)$ is connected and Abelian and hence $\gamma(P_*^0) = K \times V$, where K is a maximal compact subgroup of $\gamma(P_*^0)$ (which is a torus), and V is simply connected. Since A_1 is closed in A and torsion free, $A_1 \cap K = \{e\}$. Hence the projection $K \times V \to Y$ onto the second direct factor on A_1 is a monomorphism. Now it follows from this that there exists a closed simply connected subgroup $C \subset \gamma(P_*^0)$, such that $A_1 \subset C$ and A_1 is uniform in C (we notice that C is not always in B). We set $\Phi_l = \gamma^{-1}(C)$. Then Φ_l is a closed connected subgroup of $\mathcal{A}_l(G)$. To it corresponds a closed connected subgroup Φ of $\mathcal{A}(G)$.

With this construction at hand, V. V. Gorbatservich proved in [Gb4] the following theorem:

Proposition 9. *Let $M = G/H$ be a compact homogeneous space of a simply connected Lie group G. Then there exists a subgroup H' of finite index in H and a subgroup Φ of $\mathcal{A}(G)$, such that:*

(*a*) *Φ is a connected, simply connected, closed subgroup of $\mathcal{A}(G)$, containing H',*

(*b*) *$W_\Phi = W_G$, in particular $S_\Phi = S_G, U_\Phi = U_G$ (although $M(\Phi)$ and $M(G)$ are not generally isomorphic),*

(*c*) *for the decomposition $\mathcal{A}(G) = TW_G$ with T an Abelian subgroup of $\mathcal{A}(G)$ we have $\Phi \subset TG$, $G \subset T\Phi$, where $\Phi \cap T = G \cap T = \{e\}$,*

(*d*) *there exists a diffeomorphism $\eta : \Phi \to G$ which is the identity on the subgroup H' and induces a diffeomorphism $\Phi/H' \to G/H'$,*

(e) $\Phi = N_{\Phi}((H')^0)S_{\Phi}$,

(f) *the Lie group* $(N_{\Phi}((H')^0))_l = N_{\Phi}((H')^0)/N_{\Phi}((H')^0) \cap l(S)$ *has a finite number of connected components.*

Here we also test this construction with a simple example that G is the same as the example in last subsection and H is in the kernel of $\mathrm{Ad}|_{\mathcal{N}}$ such that $\mathcal{H} \subset \mathcal{N}$ and does not contain any ideal of $\mathcal{G}_1$, $N_{\mathrm{Ad}(T)}(H_*^0)$ is discrete. Then:

$$G_* = G_l = \{(\mathrm{Ad}t, (t, t^{-1}), n, g)|_{t \in T \ n \in N \ g \in G_2}\}$$

$$H_* = H,\ H_* \subset \{(\mathrm{Ad}t, (t, t^{-1}), n, g)|_{\mathrm{Ad}t=1}\} \subset U$$

$$P_* = N_{\mathrm{Ad}(T)}(H_*^0)U,\ P_*^0 = U,\ H_1 = H,\ M' = M,\ W_G = U,\ W_l = N$$

$$\gamma : \mathrm{Ad}(T)U \to \mathrm{Ad}(T) \times U/N = A = \{(\mathrm{Ad}t_1, (t_2, t_2^{-1}), g)|_{t_1, t_2 \in T \ g \in G_2}\}$$

$$T_* = \mathrm{Ad}(T),\ G_*/N_{\mathbf{R}} = \{(\mathrm{Ad}t, (t, t^{-1}), g)|_{t \in T \ g \in G_2}\} = B$$

$$A_1 = \overline{H_*/H_* \cap W_l} = H_*/H_* \cap W_l \subset \{(\mathrm{Ad}t, (t, t^{-1}), g)|_{\mathrm{Ad}t=1}\}$$

$$p = 0,\ K = e,\ V = U/N,\ C = V.\ \Phi_l = U = \Phi.$$

We will see that this modification turns out to be very useful in our classification.

3.2 The Splitting Theorem

Theorem 1. *Let* $(G/H, \omega)$ *be a compact homogeneous space with an invariant symplectic structure. Then the double normalizer bundle is a product of a rational homogeneous space and a solv-manifold with an invariant symplectic structure.*

Proof: Assume that G is simply connected, then the maximal compact subgroup K is semisimple. So K is Poisson, by the Proposition 5 and (4) of 2.1 we get that every K orbit is, up to a finite covering, a same torus bundle over a rational homogneous space $Q = K/J$ which is induced by the moment map $K/K \cap H \to K/J$. In particular, if $\xi \in \mathcal{J}$ (the Lie algebra of J) and if T is the closure of the 1-parameter group $\exp t\xi$ in J, then

$$\mathrm{Fix}_M(T) = \{ \ x \mid \xi_M(x) = 0 \ \} = \{ \ x \mid df_\xi(x) = 0 \ \} \neq 0.$$

Now let T be a maximal torus in J. We see that one of the K orbit must be Q since not other homogeneous space of this type can be locally isomorphic to Q (the isotropy group J is connected and is the normalizer of itself). So all the K orbits are isomorphic to Q. Now for any K orbit there is only one point on it with isotropy group J. So we get a section s of the compact foliation, i.e., M is a product of Q and s. Now we see that s is a symplectic reduction of the moment map and therefore s is a symplectic manifold with the induced symplectic structure.

Now from the normalizer of J being itself, i.e., $J = K \cap H = K \cap P^2(H)$, we see that the double normalizer fibration (see (d) of Proposition 4) has Q as the base and s as a fiber and s itself is a compact homogeneous space with an invariant symplectic structure. We have $s = P^2(H)/H$ as a fiber of the double normal fibration. By $P(H)$ be a t-subgroup in G we see that all the semsimple factors of $P^2(H)$ is in $P(H)$. If a simple factor S_1 of $P(H)$ is not in H it acts almost freely on s, i.e., only elements in its center may have fixed points, then since S_1 is Poisson, every element in its Lie algebra have fixed points, a contradiction. We see that all the semisimple factors must be in H, i.e., the radical of $P^2(H)$ acts on s transitively. So we get that s is a solv-manifold with an invariant symplectic structure.

Q. E. D.

Remark 1. (1) In our proof above we have to use the structure of the double normalizer fibration and t-subalgbra in Propositon 3,4 to prove that the moment map is actually equivariant.

(2) An easy calculation can show that the Levi decomposition is actually a product. We first prove this in [Gu6], and found that this is a conclusion of our Main Theorem later. See also [ZB].

3.3 Compact Solv-manifolds with Invariant Symplectic Structures

Lemma 2. *Every compact solv-manifold with an invariant symplectic structure is a two step solv-manifold. Moreover, the orbits of the commutator are isotropic, and any element in the Lie algebra correspond to the commutator is conjugate to an element in the Lie algebra of the isotropy group.*

Proof: We consider the moment map introduced by $\mathcal{N} = [\mathcal{G},\ \mathcal{G}]$ (see the sentence after (3) of 2.1). The corresponding subgroup N is unipotent, in particular on $\mathcal{N}^*$ and each orbit in the image is compact hence must be a point. We see that $N/N \cap H$ is abelian, $[\mathcal{N},\ \mathcal{N}] \subset \mathcal{H}$ as an ideal must be zero. Hence G is a two step solvable group. Moreover, since the moment map is a constant on each commutator orbit, we see that $df(X) = \omega(X_f,\ X) = 0$ for all $X_f,\ X$ in the commutator, i.e., the commutator orbits are isotropic.

We also see that any element in the Lie algebra of the commutator corresponds to a function on the manifold, and hence has a zero point, i.e., this element is in the Lie algebra of the isotropy group at that point. So every element in the Lie algebra of the commutator is conjugate to an element in the Lie algebra of the isotropy group.

Q. E. D.

Now we assume that $\mathcal{G}$ is a direct sum $\mathcal{A} + \mathcal{N}$ as a vector space, where

$\mathcal{N} = [\mathcal{G}, \mathcal{G}]$ and such that $\mathcal{H} = \mathcal{H} \cap \mathcal{A} + \mathcal{H} \cap \mathcal{N}$. We let $\mathcal{H}_1 = \mathcal{H} \cap \mathcal{A}$, $\mathcal{H}_2 = \mathcal{H} \cap \mathcal{N}$ and

$$\mathcal{B} = \{a \in \mathcal{A}|_{\rho(a,\ \mathcal{N})=0}\},$$

$\mathcal{B} = \mathcal{H}_1 + \mathcal{A}_1$ as a direct sum of vector speces, then $\mathcal{A}$ is a direct sum $\mathcal{H}_1 + \mathcal{A}_1 + \mathcal{A}_2$, $\mathcal{N}$ is a direct sum $\mathcal{H}_2 + \mathcal{N}_1$ as vector spaces, where $\mathcal{A}_2$ is a complement of $\mathcal{H}_1$ in

$$\mathcal{C} = \{a \in \mathcal{A}|_{\rho(a, \mathcal{A}_1)=0}\}.$$

We have:

Lemma 3. $[\mathcal{B},\ \mathcal{B}] = 0,\ [\mathcal{B},\ \mathcal{N}] = 0.$

Proof: Since $\rho([\mathcal{B},\ \mathcal{B}],\ \mathcal{A}) = \rho(\mathcal{B},\ [\mathcal{B},\ \mathcal{A}]) \subset \rho(\mathcal{B},\ \mathcal{N}) = 0$, we see that $[\mathcal{B},\ \mathcal{B}] \subset \mathcal{H}$. In the same way we see that $[\mathcal{B},\ \mathcal{N}] \subset \mathcal{H}$. And $[[\mathcal{B},\ \mathcal{B}],\ \mathcal{G}] \subset [\mathcal{B},\ \mathcal{N}]$, $[[\mathcal{B},\ \mathcal{N}],\ \mathcal{G}] \subset [\mathcal{B},\ \mathcal{N}]$, we see that $[\mathcal{B},\ \mathcal{N}], [\mathcal{B},\ \mathcal{B}]$ generates an ideal in $\mathcal{H}$, that is, $[\mathcal{B},\ \mathcal{B}] = [\mathcal{B},\ \mathcal{N}] = 0$.

Q. E. D.

By this Lemma 3 we can see that $\mathcal{H}_1$ is a Lie subalgebra, by modification we can assume that $\mathcal{H}_1 = 0$. We also see that $\mathcal{A}_1 \subset P(\mathcal{H})$. Now by the last sentence of the Lemma 1 and counting the dimension we see that $P(\mathcal{H}) = \mathcal{N} + \mathcal{A}_1$. Now we consider the G action on the Grassmanian $G(\mathcal{G}, \mathcal{H})$ of $\mathcal{H}$ in $\mathcal{G}$. The orbit through $\mathcal{H}$ is exactly the base of the normalizer fibering, being compact and with an abelian transitive group it must be a torus with its action on itself. We can also regard it as an orbit in $G(\mathcal{N}, \mathcal{H})$ and the action of G on it is an almost faithful representation of adjoint action $\mathrm{Ad}G|_{\mathcal{N}}$ on $\mathcal{N}$ as the restriction of the adjoint action. In particular, we see that $\mathrm{Ad}G$ acts on $\mathcal{N}$ as a compact group. To see this we notice that $H(N + A_1)$ is an open and closed subgroup of $P(\mathcal{H})$, hence is closed in G, i.e., $G/H(N + A_1)$ is compact. We see that $H(N + A_1)$ is cofinite in $P(\mathcal{H})$. But H acts on $\mathcal{N}/\mathcal{H}$

trivially since $(\mathcal{A}_2, \mathcal{N}/\mathcal{H})$ is a perfect pair to ρ and any invariant subspace of $\mathcal{H}$ on which H acts nontrivially must be an ideal of $\mathcal{G}$, i.e., H is in the kernel of $\mathrm{Ad}|_{\mathcal{N}}$. Now for a generic element in G, it does not have any eigenvector in $\mathcal{N}$, otherwise this eigenvector will generate an ideal in $\mathcal{H}$. Now we consider a minimal invariant subspace V of the linear transformation of $A = \mathrm{ad}(a)$ on $\mathcal{G}$ for an element $a \in \mathcal{A}_2$, then either $\mathrm{ad}(a)|_V = 0$ or $V \cap \mathcal{N} \neq 0$. In the latter case we must have $V \subset \mathcal{N}$ since the eigenvalues of $\mathrm{ad}(a)|_{V^{\mathbf{C}}}$ are not zero, hence V has dimension 2 and $\mathrm{ad}(a)$ is semisimple. This implies that $\mathcal{A}_1$ can be chosen to be an abelian ideal, and $\mathcal{A}_2 + \mathcal{N}$ is another ideal. So we get:

Lemma 4. *If we let $\mathcal{G}_1 = \mathcal{A}_2 + \mathcal{N}$ and $\mathcal{G}_2 = \mathcal{A}_1$. Then $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ as direct sum of Lie algebras. Moreover, $\mathcal{G}_2$ is abelian and $\mathcal{A}_2$ acts on $\mathcal{G}_1$ as torus.*

Now we can apply Gorbatservich modification (Proposition 9) to our case with some modification from the 7. in the Preliminary. Instead of $P_* = N_{\mathcal{A}_l(G)}(H_*^0)$ we consider $P_* \cap D$ where

$$D = \{a \in \mathcal{A}_l(G)|_{\rho(\mathrm{Ad}(a)x,\ \mathrm{Ad}(a)y)=\rho(x,\ y) \text{ for all } x,y\in\mathcal{G}}\},$$

then all the construction go through. and basically we get same modification modulo a finite covering. Especially in the case of the example in the 7. of the Preliminaries we see that $P_*^0 = U$ acts trivially on N, we will see that this is what exactly happens here latter on.

Moreover if we defind on $M(G)$ that $\rho(t_1 + x,\ t_2 + y) = \rho(x, y)$, then on Φ we have $\rho([t_1+x,\ y],\ z)+\rho(y, [t_1+x,\ z]) = 0$ for all $t_1+x \in \mathrm{Lie}\Phi, y, z \in \mathcal{G}$ and hence

$$\rho([t_1 + x,\ t_2 + y],\ t_3 + z) + \rho([t_2 + y,\ t_3 + z],\ t_1 + x)$$

$$
\begin{aligned}
&+\rho([t_3+z,\ t_1+x],\ t_2+y)\\
&=\rho([t_1+x,\ t_2+y],\ z)+\rho([t_2+y,\ t_3+z],\ x)\\
&+\rho([t_3+z,\ t_1+x],\ y)\\
&=\rho([t_1+x,\ y],\ z)+\rho([x,\ t_2+y],\ z)-\rho([x,\ y],\ z)\\
&+\rho([t_2+y,\ z],\ x)+\rho([y,\ t_3+z],\ x)-\rho([y,\ z],\ x)\\
&+\rho([t_3+z,\ x],\ y)+\rho([z,\ t_1+x],\ y)-\rho([z,\ x],\ y)\\
&=0
\end{aligned}
$$

for all $t_1+x, t_2+y, t_3+z \in \mathrm{Lie}\Phi$.

Here we see that H is in the kernel of $\mathrm{Ad}|_{\mathcal{N}}$. So it is easy to see by the example in Preliminaries 7. that in our case Φ is abelian and $\rho(\mathrm{Ad}(h)(t_1+x),\ \mathrm{Ad}(h)(t_2+y))=\rho(t_1+x,\ t_2+y)$, i.e., by Proposition 7 we see that Φ/H is also a compact homogeneous space with an invariant symplectic structure. Now by Φ abelian we see that it is a torus. So we get:

Theorem 2. *Every solvable compact homogeneous space with an invariant symplectic structure is a torus.*

Remark 2. The actually symplectic torus structure was obtained in [Gu6] by completely integrable system. See also [ZB] for a very interesting version of the proof of Theorem 2.

Conbining Theorem 1 and 2 we obtain Proposition 2.

3.4 Examples of Nonstandard Symplectic Torus

In last section we get some nonstandard symplectic torus in the case $\mathcal{A}_2 \neq 0$ and $\mathcal{H}_2 \neq 0$. Although they have standard torus as their modification, themselves are not standard. The dimension of $\mathcal{H}_1$ can be as big as possible. So can be the dimension of G. In this section we will give some simple examples which show that this situation does occur.

Let $\mathbf{C}^n$ be a complex vector space generated by vectors $e_1, \cdots, e_n$ with $S_1 = \{e^{i\theta}|_{\theta\in[0,2\pi]}\}$ action $e^{i\theta}: \ e_k \to e^{ki\theta} e_k$.

Regarding S_1 as a subgroup of the automorphism group of the abelian group $\mathbf{C}^n$ we get a Lie group $G = S_1 \propto \mathbf{C}^n$ as a semiproduct.

Now we let $Z = (\mathbf{Z} + i\mathbf{Z})^n$ be the standard lattice in $\mathbf{C}^n$,

$$H^0 = \{(z_1, \cdots, z_n) \in \mathbf{C}^n|_{\mathrm{Re}(z_1+\cdots+z_n)=0}\},$$

and $H = Z + H^0$.

Then G/H is a nonstandard torus occured in last section with symplectic structure introduced by ρ in the Proposition 7 such that $\rho(s, x) = 1$, $\rho(s, h) = 0$, $\rho(x, h) = 0$ for all $h \in \mathcal{H}$, where s is a generator of the Lie algebra of S_1, $x \in \mathbf{C}^n$ is an element such that $\mathrm{Re}(x_1 + \cdots + x_n) = 1$.

See also [ZB] for similar examples.

It is very surprising fact to me that we can get a classification for compact homogeneous space with an invariant symplectic structure since the automorphism group of a compact symplectic manifold has infinite dimension. From any smooth function we can construct a one parameter group which keep the symplectic structure invariant. So we can see that the base manifold is always "homogeneous" under the symplectic automorphism group. But we see from our classification that a compact symplectc manifold is homogeneous under a finite dimensional Lie subgroup of the symplectic automorphism group is so different. Moreover, we are seemly be able to classify compact homogeneous space with a symplectic structure which is nonnecessary invariant under the group action.

Using the nonstandard torus we can construct new examples of simply connected compact symplectic manifolds as in [Gu4] and [Bo].

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References

[Bo] F. A. Bogomolov: On Guan's Examples of Simply connected Non-Kähler Compact Complex Manifolds, Amer. J. Math. 118(1996), 1037–1046.

[BR] A. Borel & R. Remmert: Über Kompakte Homogene Kählersche Mannigfaltigkeiten, Math. Ann. 145 (1962), 429–439.

[DG1] J. Dorfmeister & Z. Guan: Classifications of Compact Homogeneous Pseudo-Kähler Manifolds, Comm. Math. Helv. 67 (1992), 499–513.

[DG2] J. Dorfmeister & Z. Guan: Pseudo-Kählerian Homogeneous Spaces Admitting a Reductive Transitive Group of Automorphisms, Math. Zeischrift 209 (1992), 89–100.

[Dm] J. Dorfmeister: Homogeneous Kähler Manifolds Admitting a Transitive Solvable Group of Automorphisms, Ann. Scient. Ec. Norm. Sup., 4 Serie, vol 18 (1985), 143–180.

[DN] J. Dorfmeister & K. Nakajima: The Fundamental Conjecture for Homogeneous Kähler Manifolds, Acta Math. 161(1988), 23–70.

[Gb1] V. V. Gorbatservich: On the Double Normalizer of the Stationary Subalgebra of a Plesiocompact Homogeneous Spaces, Siberian Math. J. 34 (1993), 451–456.

[Gb2] V. V. Gorbatservish: On a Fibration of Compact Homogeneous Spaces, Trans. Moscow Math. Soc. vol 1 (1983), 129–157.

[Gb3] V. V. Gorbatservich: Splittings of Lie Groups and Their Application to the Study of Homogeneous Spaces, Math. USSR Izvestija. vol 15 (1980), 441–467.

[Gb4] V. V. Gorbatservich: Plesiocompact Homogeneous Spaces, Siber. Math. J. 30 (1989), 217–226.

[GS] V. Guillemin & S. Sternberg: *Symplectic Techniques in Physics*, Cambridge Univ. Press. 1984.

[Gu1] Z. Guan: Examples of compact holomorphic symplectic manifolds which admit no Kähler structure. In *Geometry and Analysis on Complex Manifolds—Festschrift for Professor S. Kobayashi's 60th Birthday*, World Scientific 1994 63–74.

[Gu2] D. Guan: A Splitting Theorem for Compact Complex Homogeneous Spaces with a Symplectic Structure. Geom. Dedi. 67(1996), 217–225.

[Gu3] D. Guan: Classification of Compact Complex Homogeneous Spaces with Invariant Volumes, to appear in Transactions of AMS.

[Gu4] D. Guan: Examples of Compact holomorphic Symplectic Manifolds which are not Kählerian II, Invent. Math. 121(1995), 135–145.

[Gu5] D. Guan: Classification of Compact Homogeneous Space with an Invariant Symplectic Structure, preprint 1997.

[Gu6] D. Guan: Fine Structure of Compact Homogeneous Space with an Invariant Symplectic Structure, preprint 1997.

[Gu7] D. Guan: Toward a Classification of Compact Complex Homogeneous Spaces, preprint 1998.

[Hk] A. T. Huckleberry: Homogeneous Pseudo-Kählerian Manifolds: A

Hamiltonian Viewpoint, Note di Matematica 10(1990) suppl. 2, 337–342.

[HO] A. T. Huckleberry & E. Oeljeklaus: *Classification Theorems for Almost Homogeneous Spaces*, Publ. de l'Inst. Elie Cartan, Nancy, Janvier 1984, 9. 178 pages.

[Kz] J. L. Koszul: Sur la Form Hermitienne Canonique des Spaces Homogenes Complexes, Canad. J. Math. 7(1968), 562–576.

[On] A. L. Onishchik: On Lie Groups Transitive on Compact Manifolds II, Math. USSR Sbornik. vol. 3 (1967), 373–388.

[Ti] J. Tits: Espaces Homogènes Complexes Compacts, Comm. Math. Helv. 37 (1962), 111–120.

[ZB] Zwart & Boothby: On Compact, Homogeneous Symplectic Manifolds, Ann. Inst. Fourier Grenoble 30, 1(1980), 129–157.

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