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A Classification of Compact Cohomogeneity One Locally Conformal Kähler Manifolds

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Abstract: In this paper, we apply a result of the classification of a compact cohomogeneity one Riemannian manifold with a compact Lie group G to obtain a classification of compact cohomogeneity one locally conformal Kähler manifolds. In particular, we prove that the compact complex manifold is a complex one-dimensional torus bundle over a projective rational homogeneous, or cohomogeneity one manifold except of a class of manifolds with a generalized Hopf surface bundle over a projective rational homogeneous space. Additionally, it is a homogeneous compact complex manifold under the complexification $G^{\mathbf{C}}$ of the given compact Lie group G under an extra condition that the related closed one form is cohomologous to zero on the generic G orbit. Moreover, the semi-simple part S of the Lie group action has hypersurface orbits, i.e., it is of cohomogeneity one with respect to the semi-simple Lie group S in that special case.

Keywords: cohomology; invariant structure; homogeneous space; cohomogeneity one; complex torus bundles; Hermitian manifolds; reductive Lie group; compact manifolds; locally conformal Kähler manifolds

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1. Introduction

Let M be a complex manifold and h be a Hermitian metric. If h is locally conformal to a Kähler metric, i.e., for any point $m \in M$ there is an open neighborhood O such that on O we have that $g = e^f h$ is a Kähler metric with a function f, we say that (M, h) is a locally conformal Kähler. That is, let $\Omega(\ ,\ ) = h(\ , J\ )$, then $d(e^f\Omega) = 0$.

A (compact) Riemannian cohomogeneity one manifold M with a compact Lie group G has a real hypersurface orbit, which was classified as the following (see [1] page 198, for example):

A. The generic real hypersurface orbits are the same as a compact homogeneous space G/H, although they might have different (induced) Riemannian metrics. The generic orbits form an open set U of M.
B. $M - U$ has at most two components. Each of them is a compact homogeneous space G/H_i such that H is a subgroup of H_i for each i and H_i/H are spheres.
C. The quotient space $I = M//G$ has four possibilities:
    1. An open interval (a, b). In this case, all the orbits are generic and M is open with two ends.
    2. A half-open interval $[a, b)$. In this case, there is only one special orbit corresponding to the point a in the interval and M is open with one end.
    3. A closed interval $[a, b]$. In this case, there are two special orbits corresponding to the points a and b. M is closed.
    4. S^1. In this case, all the orbits are generic and M is closed.

In the compact case, only cases 3 and 4 could happen.

In the case in which M is complex and compact, the complexification $G^{\mathbf{C}}$ acts on M also and therefore, $G^{\mathbf{C}}$ has complex orbits. One of them contains a real hypersurface orbit and hence is an open orbit. We denote the open $G^{\mathbf{C}}$ orbit by $U^{\mathbf{C}}$. We notice that all the generic hypersurfaces are in some $G^{\mathbf{C}}$ open orbits. The generic hypersurfaces in the orbit space I consist of a connected interval. Therefore, all of them are in $U^{\mathbf{C}}$. Then, there are three possibilities:

3a. $U = U^{\mathbf{C}}$, $I = [a, b]$ and the special orbits are complex sub-manifolds. We say that M is a $G^{\mathbf{C}}$ almost homogeneous manifold with two ends. See [2] for examples.

3b. $I = [a, b]$ but $U^{\mathbf{C}} - U$ has only one component. We say that M is a $G^{\mathbf{C}}$ almost homogeneous space with one end. See [3,4] for examples.

c. M is $G^{\mathbf{C}}$ homogeneous. This comes from the case 4 as above.

We should see later on that only the case c would happen if we assume that the Lee form $\theta = -df$ (therefore closed) is in a trivial cohomology class on the generic G orbits. They are basically the quotient of the open orbits as the $\mathbf{C}^*$ bundle we studied earlier, e.g., in [2] or those mentioned in [5].

Remark 1. *By the local nature of the defining function f, θ is in general in a nonzero cohomology class even restricted to the generic orbits. We shall see some examples in Theorems 2 and 3. In those cases, we do see that all the cases of 3a, 3b and c can actually happen. One also can notice that θ being in a zero cohomology class to the generic orbits does not imply that θ is in a zero cohomology class on the manifold. See our Theorem 1 for example.*

Recently, Professor Hasegawa et al. proved in [6] (see also [7,8]):

Proposition 1. *A compact homogeneous locally conformal Kähler manifold $M = G/H$ is a complex 1-dimensional torus bundle over a rational projective homogeneous space.*

Remark 2. *Here, by locally conformal Kähler, we always mean that the manifold is not Kähler. Otherwise, the statement of this result is obviously not true.*

In [7], we actually prove the following in the G homogeneous case:

Proposition 2. *A compact homogeneous locally conformal Kähler manifold is cohomogeneity one under the action of the semi-simple part S of the Lie group, i.e., S has hypersurface orbits. $M = N \times S^1$ as a homogeneous space (but not necessary as a Riemannian manifold) with N the S orbits. Both the original locally conformal Kähler metrics and the related Kähler metrics are cohomogeneity one under the S action.*

Moreover,

Proposition 3. *A compact homogeneous locally conformal Kähler manifold M is a complex one-dimensional torus bundle, over a rational homogeneous projective space, which is a finite quotient of a quotient of a $\mathbf{C}^*$ bundle by some action e^a with $\mathrm{Re}a \neq 0$. The metrics on M, as a submersion, are completely determined by the Kähler class of the base manifold and the Kähler class, as the restriction, of the fiber. On the other hand, any positive homogeneous $\mathbf{C}^*$ bundle over a rational projective homogeneous space has a compact homogeneous locally conformal Kähler manifold as a compact quotient.*

In particular, the homogeneous case is a special case of the cohomogeneity one case, in which $G = S$.

Now, we come back to the general cohomogeneity one case.

We notice that on O one has $d(e^f\Omega) = 0$. That is

$$e^f(df \wedge \Omega + d\Omega) = 0.$$

As before, one defines $\theta = -df$. Then $d\Omega = \theta \wedge \Omega$. Since Ω is nondegenerate, θ is uniquely determined if $n = \dim_{\mathbb{C}} M > 1$. In the following, we always assume that $n \neq 1$ since when $n = 1$, M is always Kähler.

Lemma 1. *θ is closed and represents a nonzero cohomology class if h is not conformally Kähler itself. In the compact cohomogeneity one case, θ is invariant.*

Proof. $d\theta = d(-df) = 0$ implies that θ is closed. If θ represents a zero class in the cohomology, then $\theta = dF$ for a global function F. That is, $e^{-F}h$ is Kähler. Since G is reductive, the Lie algebra of G is $s + c$ with s compact semi-simple and c abelian. Since Ω and $d\Omega$ are invariant under G, so is θ. □

Next result is one of the major parts of this paper. This result basically says that the manifold is of type III but not of type I or II (see the similar definition in the Kähler case in [5,9] or the references in [9]) *if* θ is in a zero cohomology class on the generic G orbit. That would greatly simplify our proof. This would also generalize Proposition 2 to the cohomogeneity one case. We need prove the following:

Theorem 1. *Let M be a compact cohomogeneity one locally conformal Kähler manifold. If θ is in a zero cohomology class on the generic orbits, then the compact manifold M is $G^{\mathbb{C}}$ homogeneous and $I = S^1$.*

Proof. Since θ is invariant, we let $\pi : U \to U/G$ be the quotient map. Then, the bundle is a product if $I \neq S^1$. On U, θ is locally $\theta_0 + da$ with a a function of t and θ_0 is a closed 1 form on $\pi^{-1}(t)$. Moreover, by θ closed, we see that θ_0 is closed on each fiber and is constant on t. Therefore, *if* $\theta_0 = 0$, we can regard $f = a$ as a function on I up to adding a constant. If I is an interval, f can be globally defined and therefore, θ is a zero cohomology class. We obtain a contradiction whenever $\theta_0 = 0$. □

Now, if $\theta_0 \neq 0$ and I is an interval, locally, after modifying Ω by $\Omega^a = e^{-a}\Omega$ we can assume that $a = 0$. That is, $\theta = \theta_0$.

If $\theta_0 \neq 0$, we have as in [7], we have that

$$\Omega = k(t)\theta \wedge J\theta + \Omega_1.$$

Restricting to each fiber so we have that:

$$\Omega_t = k(t)\theta \wedge J\theta + \Omega_{1,t},$$

$$d_t\Omega_t = \theta \wedge \Omega_t = \theta \wedge \Omega_{1,t}.$$

We notice that actually on the orbit $\theta = \theta_1 + \theta_2$ with $\theta_1 \in \ker J$ and $\theta_2 \in J\mathcal{G}^*$. Here, $\mathcal{G}$ is the Lie algebra of G. It is classical that we identify $\mathcal{G}$ with the vector fields generated by the actions of the one parameter subgroups generated by the corresponding elements of the Lie algebra.

On the other hand,

$$d_t\Omega_t = -k_t\theta \wedge d_t(J\theta) + d_t\Omega_{1,t},$$

where $k_t = k(t)$.

This, as in [7], implies that $\Omega_{1,t} = -k_t d(J(\theta))$, which has a complex rank of $n-2$. We notice here that $J\theta$ is not zero on the orbits. If $J\theta = 0$, then $\Omega_t = \Omega_{1,t}$, $\theta \wedge \Omega_{1,t} = d\Omega_{1,t}$, which cannot happen since θ is in the center of the Lie algebra $\mathcal{G}$. This means that θ_2 will never be zero. We shall see actually that $\theta_1 = 0$ in the next two sections. That is,

$\Omega_t = k_t(\theta \wedge J\theta - d(J\theta))$. By assuming $J\theta$ and $J(dt)$ to be zero on the Lie algebra of H, we see that H is in the centralizer of $J\theta_2$. Also, we have, $\Omega = l(t)dt \wedge J(dt) + k(t)(\theta \wedge J\theta - d_t(J\theta))$ since $\Omega_1 = l(t)dt \wedge J(dt) + \Omega_{1,t}$.

Moreover, by the rank of $d(J\theta)$ being $n-2$, the centralizer $C(J\theta_2)$ of $J\theta_2$ introduces a fibration $p: G/H \to G/C(J\theta_2)$ of the generic orbits on rational homogeneous spaces of complex dimension $n-2$. This means the *generalized* Tits normalization fibration (Cf. [10] for the case with $U^{\mathbf{C}}$ compact) of $G^{\mathbf{C}}$, *on the open orbit*, has a complex two-dimensional parallelizable fiber with a rational G homogeneous space as the base if the base is homogeneous. If the base of the generalized Tits fibration is not homogeneous, we would again obtain a one-dimensional complex torus bundle. *See a proof of this in the next section*. That is, in the first case the fiber is a quotient of a complex two-dimensional reductive Lie group. However, a complex two-dimensional Lie group can only be abelian. That is, the fiber of p is a real three-dimensional torus. Since θ is invariant under G, θ is in the center and the center of G has at least one dimension. The complex center of $G^{\mathbf{C}}$ has at most two dimensions. Therefore, in the extreme case of a complex two-dimensional fiber, we should obtain some kind of Hopf surface bundle. This will be dealt with in the next two sections. See Theorems 2 and 3.

We will have some examples in the next section.

Therefore, we end up in a mild similar situation in the compact homogeneous case, except that the metric h itself is not necessarily homogeneous under any compact Lie group even in the setting of our Theorem 1.

Even so, we still obtain a similar structure result to the Proposition 1, or Theorem 1, in general in this paper:

Main Theorem: *A compact cohomogeneity one locally conformal Kähler manifold is a homogeneous holomorphic fiber bundle with fiber F and base B, such that either F is a complex one-dimensional torus and B a projective rational homogeneous space or a projective cohomogeneity one manifold, or F is a generalized Hopf surface and B is a projective rational homogeneous space.*

Remark 3. *By a (classical) Hopf surface, we mean that it is a complex surface defined as in the Example 1 in the next section with the base manifold to be* $\mathbf{C}P^1$. *A generalized Hopf surface is a finite quotient of a classical Hopf surface.*

2. The Normalization Fibration

Let M be a compact complex manifold with a cohomogeneity one G Hermitian structure. Then G is compact, and hence is reductive. Let $G^{\mathbf{C}}$ be the complexified Lie group of G. Then $G^{\mathbf{C}}$ also acts on M such that M is $U^{\mathbf{C}} \cup D$. We call D the complex end. D might be a divisor or a complex lower-dimensional submanifold. We have $U^{\mathbf{C}} = G^{\mathbf{C}}/H^{\mathbf{C}}$. We notice that, in general, $H^{\mathbf{C}}$ is not the complexification of H.

There is a natural fibration of $U^{\mathbf{C}} = G^{\mathbf{C}}/H^{\mathbf{C}} \to N = G^{\mathbf{C}}/Norm_{G^{\mathbf{C}}}(H_0^{\mathbf{C}})$, where $H_0^{\mathbf{C}}$ is the identity component of $H^{\mathbf{C}}$ and $Norm_{G^{\mathbf{C}}}(H_0^{\mathbf{C}}) = \{g \in G^{\mathbf{C}} \|_{gh_0^{\mathbf{C}}g^{-1} \subset H_0^{\mathbf{C}}}\}$ is the normalization of $H_0^{\mathbf{C}}$ in $G^{\mathbf{C}}$. Let $\mathcal{G}^{\mathbf{C}}$ be the Lie algebra of $G^{\mathbf{C}}$ and $\mathcal{H}^{\mathbf{C}}$ be the Lie algebra of $H^{\mathbf{C}}$. Then, N is just the $G^{\mathbf{C}}$ orbit of $\mathcal{H}^{\mathbf{C}}$ in $\mathcal{G}^{\mathbf{C}}$. Since M is G cohomogeneity one, N is either G homogeneous or G cohomogeneity one.

If N is G homogeneous, then the fiber is cohomogeneity one under $K = (G \cap Norm_{G^{\mathbf{C}}}(H_0^{\mathbf{C}}))/H_0 \subset Norm_G(H_0)/H^0$. Moreover, if K is a codimension one compact Lie subgroup of a complex Lie group, it must be abelian.

If N is G cohomogeneity one, then the fiber is K homogeneous and therefore is a compact quotient of K, which can only be a torus since K is both compact and complex. Therefore, again K is abelian. In the second case, there is a classification of the possible N in [3,4]. See also [5,9], as well as the references therein.

In both cases, the Lie algebra of $C(J\theta_2)$ is a direct product of H and an abelian subgroup $\mathcal{T}$.

This make the compact cohomogeneity one Hermitian structure being handleable in general.

In our case, we see that the normalization fibration has a fiber with a complex dimension of no more than two.

If $\theta_0 \neq 0$, then the fiber is not trivial. In this case, if N is cohomogeneity one, by the dimension bound, the fiber is of complex dimension one since dt is not in the fiber.

Lemma 2. *If $\theta_0 \neq 0$ and the base of the normalization fibration of the $G^{\mathbf{C}}$ open orbit is of cohomogeneity one, then the fibration is a homogeneous complex one-dimensional torus bundle over a cohomogeneity one quasi-projective complex $G^{\mathbf{C}}$ homogeneous manifold. Moreover, the complex one-dimensional torus bundle extends to the whole manifold. That is, the manifold itself is a complex one-dimensional torus bundle over a compact projective rational cohomogeneity one manifold.*

Proof. The compactification of the base manifold N in the Grassmannian is a cohomogeneity one projective variety N^c. The components of $N^c - N$ are G homogeneous compact complex manifolds. The ends in $M - U^{\mathbf{C}}$ are also complex and compact. They are also G homogeneous and locally conformal Kähler. Indeed, let $E_i = G/H_i$ be one of the complex ends in condition B in the Introduction, then near any point $e \in E_i$, there is a local neighborhood on which M is locally a product as a complex vector bundle over E_i and along a geodesic perpendicular to E_i, H_i has a sphere as an orbit and the torus of the normalization fibration projects to E_i as an embedding. Therefore, we find that the limit of the complex torus does not contract and is actually locally free. That is, the torus bundle structure extends over to the ends. □

Example 1. *Let N^c be any compact projective cohomogeneity one manifold. We also assume L is a positive homogeneous line bundle over N^c and L^* represents the nonzero points in the line bundle. There is a natural $\mathbf{C}^*$ action on L^*. Then $L^*/(a)$ with a nonzero complex number a with $|a| \neq 1$ is a compact cohomogeneity one locally conformal Kähler manifold. Similarly, this occurs if we replace N^c with a projective rational homogeneous manifold.*

Remark 4. *The projective rational homogeneous spaces are Kähler–Einstein with positive Ricci curvature. Therefore, they are simply connected. The classes of the line bundles are discrete. Thus, the identity components of the automorphism group acts trivially on these classes. This implies that all the line bundles are homogeneous. Similarly, if N^c is any compact simply connected projective cohomogeneity one manifold, then all its line bundles are homogeneous.*

In the case of Lemma 2, the complexified Lie group has a center of complex dimension one.

And the $G/H \to G/C(J\theta)$ must have a three-dimensional fiber and the three-dimensional fiber is included in the center of $C(J\theta)$. Therefore, M is exactly one of the manifolds from Example 1.

Therefore, we have:

Theorem 2. *Let M be a compact cohomogeneity one locally conformal Kähler manifold. If $\theta_0 \neq 0$ and the normalization fibration of the $G^{\mathbf{C}}$ open orbit is a G homogeneous torus bundle over cohomogeneity one base, then M itself is a quotient torus bundle of a positive $\mathbf{C}^*$ bundle over a projective rational cohomogeneity one manifold.*

We notice that in this case $\theta_1 = 0$.

When $\theta_0 \neq 0$ and N is G homogeneous, then the fibration has a complex dimension two fibration.

Lemma 3. *If θ is not in the zero cohomology class of the generic G orbit and the normalization fibration of the $G^{\mathbf{C}}$ open orbit has a G homogeneous base, then the fiber is an abelian complex two-dimensional Lie group that is cohomogeneity one. That is, M is a compact complex two-dimensional locally conformal Kähler manifold F bundle over a projective rational space. The group action of the fiber is a three-dimensional compact Lie group.*

In this case, $N = S^{\mathbf{C}}/P$ with $G^{\mathbf{C}} = S^{\mathbf{C}} \times A$, $S^{\mathbf{C}}$ being semi-simple and P a parabolic subgroup of $S^{\mathbf{C}}$. Here, A is a complex abelian Lie group of complex dimension one or two. Let π_2 be the projection map from $G^{\mathbf{C}}$ to $S^{\mathbf{C}}$, then $\pi_2(H^{\mathbf{C}}) \subset P$ is a complex codimension one normal subgroup if $\dim_{\mathbf{C}} A = 2$, or $\pi_2(H^{\mathbf{C}}) = P$ if $\dim_{\mathbf{C}} A = 1$.

Example 2. *Let L_i, $i = 1, 2$ be two of the line bundles in the examples 1 over N. We can obtain the Kähler metrics on L_i using some Hermitian metrics on them. Then, the product $V = L_1 \times L_2$ is a vector bundle over $N \times N$ with a product Kähler metric. Let V^* represent the nonzero points over the diagonal submanifold, which is itself an N. Now, we consider the group action $G^{\mathbf{C}} = S^{\mathbf{C}} \times (\mathbf{C}^*)^2$ with the second factor action:*

$$(c_1, c_2) : (l_1, l_2) \to (c_1(c_2)^b l_1, c_1(c_2)^a l_2).$$

This action comes down to $V^*/(a)$ with an a such that $|a| \neq 1$.

This gives some possible manifolds in Lemma 3. We shall deal with the structure of the the complex dimension two fiber in the next section.

3. Compact Complex Dimension Two Locally Conformal Kähler Manifold of Cohomogeneity One

In this section, we shall deal with the complex dimension two case. We notice that if $\dim_{\mathbf{C}} M = 2$, then G is a compact torus.

In this case, we have the standard case: $\Omega = \frac{|dz|^2}{|z|^2}$ and the group action is $G = [\mathbf{C}^*/(2)] \times S^1$ with the S^1 action (z_1, z_2) to $(e^{at}z_1, e^{bt}z_2)$ for an element $e^t \in S^1$. The group action is

$$(z, w) : (z_1, z_2) \to (e^{z+bw}z_1, e^{z+aw}z_2).$$

Here, we need to assume that both a and b are integers.

Now, in general, we let $z = x + iy, w = t + iv$ and we assume that the compact three-dimensional real Lie group is generated by x, y and v. Locally, after replacing Ω by $e^a\Omega$ we can assume that $\theta = \theta_0 = adx + bdy + cdv$. As we discussed earlier, we see that $\theta_2 = adx + bdy$ is not zero. We could assume simply that $a = 1$ and $b = 0$.

Before we go further, we want to prove that $c = 0$. If c is not zero, we let $c = 1$. Then we have $\theta = dx + dv$ and let

$$\Omega = Adz \wedge d\bar{z} + Bdz \wedge d\bar{w} + \bar{B}dw \wedge d\bar{z} + Cdw \wedge \bar{w}.$$

$$\theta = 1/2(dz + d\bar{z} - idw + id\bar{w})$$

$$d\Omega = A'dt \wedge dz \wedge d\bar{z} + B'dt \wedge dz \wedge d\bar{w} + \bar{B}'dt \wedge dw \wedge \bar{z}.$$

By $d\Omega = \theta \wedge \Omega$, we determine

$$A' = -B - iA.$$

Now, by A positive, we determine that $A' = -\mathrm{Re}B$ and $0 = \mathrm{Im}B + A$. Therefore, $B = -A' - iA$. We also obtain

$$B' = -C - iB.$$

That is, $-A'' - iA' = -C + iA' - A$. That is, $A'' = C + A$ and $A' = -A'$. The second identity implies that A is a constant and the first identity implies that $0 = C + A$. But both A and C are positive, which is a contradiction. This implies that $\theta = dx$.

Therefore, in the case we also obtain $\theta_1 = 0$.

Let $\Omega = f dx \wedge dy + gdx \wedge dv + hdy \wedge dv + dt \wedge (Adx + Bdy + Cdv)$.

$$d\Omega = dt \wedge (f'dx \wedge dy + g'dx \wedge dv + h'dy \wedge dv).$$

Now, by $dx \wedge \Omega = d\Omega$ we obtained that h is a constant c. However, we also have:

$$dx \wedge \Omega = cdx \wedge dy \wedge dv + dx \wedge dt \wedge (Bdy + Cdv).$$

We have $h = c = 0$, $B = -f'$, $C = -g'$. This is true if Ω is locally conformal symplectic.

However, we also have that Ω is from a Hermitian metric. That is, $J_*\Omega(\ , \ ) = \Omega(J \ , J \ ) = \Omega$. We have $Jdx = dy, Jdy = -dx, Jdt = dv, Jdv = -dt$. Then

$$\begin{aligned} J_*\Omega &= -fdy \wedge dx - gdy \wedge dt + Adv \wedge dy - dv \wedge (-f'dx - g'dt) \\ &= fdx \wedge dy - f'dx \wedge dv + Adv \wedge dy - dt \wedge (-gdy + g'dv). \end{aligned}$$

Then, $A = 0$, $g = -f'$. That is, we have

$$\begin{aligned} \Omega &= fdx \wedge dy - f'(dx \wedge dv + dt \wedge dy) + f''dt \wedge dv \\ &= -2(fdz \wedge d\bar{z} - f'(dz \wedge d\bar{w} + d\bar{z} \wedge dw) + f''dw \wedge d\bar{w}). \end{aligned}$$

For Ω to be a metric, we need $f > 0, \ f'' > 0$ and

$$ff'' - (f')^2 > 0.$$

that is, $f > 0$ and if $g = f'/f$ then $g' > 0$. Then both $A = \lim_{t\to+\infty} g$ and $B = \lim_{t\to-\infty} g$ exist, although one of them might be infinite.

Now, let us look at the metric restricted to the compact G orbits. We have that h is conformal to

$$dzd\bar{z} - g(dzd\bar{w} + dwd\bar{z}) + f''/fdwd\bar{w}.$$

Restricting to G, we obtain

$$dx^2 + dy^2 + f''/fdv^2 - 2gdydv.$$

We cannot have f''/f tending to infinity. Otherwise, at infinity the metrics are dv^2. That is, both x and y contract. This cannot happen by the condition B in the Introduction for the ends. So $\lim_{t\to+\infty} f''/f = a^2$ and $\lim_{t\to-\infty} f''/f = b^2$. And by $f''/f > g^2$, we see that both A and B are finite. $A > B$. By the fact that the orbits do contract in one real dimension, we see that $A = a$ and $B = b$ when we choose the sign of a, b properly. If the choice of a and b are attainable, i.e., there are actually examples with this pair of numbers, we replace w by lw in our construction and we see that the pair la and lb is also attainable. Therefore, if a, b are rational, then the pair is attainable. We notice that in the earlier examples we have $\theta_0 = -2dx$ instead of dx.

Therefore, it is reasonable to believe that for any pair of a, b it is attainable.

Also, the metric is equivalent to $|dz - gdw|^2 + g'|dw|^2$. And at the ends $g' = 0$. The metrics are $dx^2 + (dy - adv)^2$ and $dx^2 + (dy - bdv)^2$. Therefore, there are two vectors $k(ay + v)$ and $l(by + v)$ in the lattice. Since $g' = 0$ at the ends, we have the metrics $|dz - adw|^2$ and $|dz - bdw|^2$ at the ends. We know that the ends are complex one-dimensional torus. The kernels of G acts on them are $y - av = 0$ and $y - bv = 0$. Therefore, there are two linearly independent elements in the discrete subgroup Γ of $\mathbf{R}^3$ such that $G = \mathbf{R}^3/\Gamma$. Therefore, the complex two-dimensional group $G^{\mathbf{C}}$ is a quotient of $U = \mathbf{C}^* \times \mathbf{C}^*$. We can understand that if $2\pi k(a, 1)$, $2\pi l(b, 1)$ are two minimums of them, then the action on U is $(e^{k(az+w)}, e^{l(bz+w)})$. We want to choose the right signs of k and l in the way to make the coefficients of z non-negative.

Then, either M can be the quotient of $(\mathbf{C}^2 - \{0\})/(c, d)$ with nonzero c and d such that both $|c| = e^{kas}$, $|d| = e^{lbs} > 1$. They are the generalized Hopf surfaces. The existence of the locally conformal Kähler structures was established earlier by many authors. For example, see [11]. We just note here that the metrics constructed in [11], page 1114 are G invariant locally conformal Kähler metrics. We have in page 1111, $\phi(c^t u, d^t v) = \phi(u, v) + t$. Then, in page 1112 $\Phi(c^t u, d^t v) = (cd)^t \Phi(u, v)$. $[|u|^2 \Phi^{-\frac{2k_1}{k_1+k_2}}](c^t u, d^t v) = [|u|^2 \Phi^{-\frac{2k_1}{k_1+k_2}}](u, v)$, similarly for the one with v. Δ in (16) of page 1113 is invariant under this action also. We have $\Phi^{\frac{k_2-k_1}{k_2+k_1}}(c^t u, d^t v) = |d/c|^t \Phi(u, v)$, similarly for $\partial^2_{u,\bar{u}}\Phi$.

Or, one of $|c|, |d|$ is 1. Say, for example, $|c| = 1$ but $|d| > 1$. What we mean is that in the other case, the construction for M would be the same. Now, we assume that $|c| = e^{kas} = 1$ with $ks \neq 0$. Then, $a = 0$. The generic G orbits are $(S^1 \times \mathbf{C}^*)/(c, d)$.

We now have that the open orbit is U with the action $(e^{kw}, e^{l(bz+w)})$. Again, since we only need to deal with the existence of the complex structure near any one of the ends, by changing w we can assume that $k = 1$. We recall that near the end, the metric is $|dz - gdw|^2 + g'|dw|^2$ with $\lim_{t\to-\infty} g = \lim_{t\to-\infty} g' = 0$. Therefore, we could let $g = h\frac{2e^{2t}}{1+e^{2t}} = hF$. We only need to prove that the complex structure is attainable for a function $g = F$. Let $W = e^w$. Then, $F = \frac{2|w|^2}{1+|W|^2}$ and $F' = \frac{4|W|^2}{(1+|W|^2)^2}$. The metric is $|dz - \frac{2\bar{W}dW}{1+|W|^2}|^2 + \frac{4|dW|^2}{(1+|W|^2)^2}$ near $W = 0$. This is obviously attainable. The other side could be also down by regarding it as the end of Hopf surfaces. Indeed, even in this case, one might change the first $\mathbf{C}^*$ by multiplying the second $\mathbf{C}^*$, therefore, it reduces to the earlier generalized Hopf surfaces case. In fact, in this case, we have $|c| = |d|$ and the standard Kähler metrics on $\mathbf{C}^2$ would produce the required locally conformal Kähler metric, as pointed out by [11] on page 1110.

Theorem 3. *Let M be a compact cohomogeneity one locally conformal Kähler manifold. If $\theta_0 \neq 0$ and the normalization fibration on the $G^{\mathbf{C}}$ open orbit has a homogeneous base, then the manifold is a generalized Hopf surface bundle over a rational projective homogeneous manifold.*

4. Proof of the Main Theorem

Now, we can concentrate ourselves on the situations with $\theta_0 = 0$.

Before we go any further, let us make an observation that the Tits fibration [10] in this case is a complex torus bundle over a rational projective homogeneous space. According to Tits, the fiber is a complex compact parallelizable homogeneous space. That is, it is a quotient of a complex Lie group by a discrete subgroup. In our case, The complex Lie group has a codimension one compact subgroup. Then first, the semi-simple part is trivial. Second, the compact subgroup is a torus. Therefore, the complex Lie group itself is abelian and could be a complex torus. This is similar to the homogeous case in [7], in which we had a Hano–Kobayashi torus fibration.

Let t be the map from M to S^1. We denote $M_s = t^{-1}(s)$ and $h = g = dt^2 + g_t$ by choosing a right coordinate for S^1. Then, we have:

Theorem 4. *Let M be a compact cohomogeneity one locally conformal Kähler manifold with $\theta_0 = 0$, then $\Omega = f(t)(\theta \wedge J\theta) + d_t\alpha_t(\mathrm{mod}\mathcal{H})$, where $\mathcal{H}$ is the Lie algebra of H, each hypersurface orbit is an S^1 bundle over a same given rational projective homogeneous space and the bundle maps are submersions. Moreover, the semi-simple part S of G has a cohomogeneity one action.*

Proof. By a local argument used in the proof of Theorem 1, we see that $\theta = da$ for a function a of t.

By orthogonal decomposition, we obtain

$$\Omega = \varphi(t)\theta \wedge J\theta + \Omega_1$$

with $J\theta \perp \theta, \Omega_1 \perp \theta$.

Then $J\theta,\ \Omega_1 \in \Lambda T^* M_t$ and

$$d\Omega = -\varphi(t)\theta \wedge d_t(J\theta) + \frac{d\Omega_1}{dt} \wedge dt + d_t\Omega_1 = \theta \wedge \Omega_1.$$

We have $d_t\Omega_1 = 0$ and

$$\Omega_1 = d_t\alpha_t + \Omega_0(\mathrm{mod}\mathcal{H})$$

with $\Omega_0 \in \wedge^2 C^*$ if we denote C the center. □

Lemma 4. $\Omega_0 = 0$.

Proof. We have:

$$d\Omega = -\varphi(t)\theta \wedge d_t(J\theta) + \frac{d(d_t\alpha_t + \Omega_0)}{dt} \wedge dt = \theta \wedge (d_t\alpha_t + \Omega_0)(\text{mod}\mathcal{H})$$

We now consider the Tits fibration from [10]:

$$M = G^{\mathbf{C}}/H^{\mathbf{C}} \to G^{\mathbf{C}}/\text{Norm}_{G^{\mathbf{C}}}(H_0^{\mathbf{C}}),$$

where $\text{Norm}_{G^{\mathbf{C}}}(H_0^{\mathbf{C}}) = N^{\mathbf{C}}$ is the normalization of the identity component $H_0^{\mathbf{C}}$ of $H^{\mathbf{C}}$ in $G^{\mathbf{C}}$. The fiber is a complex parallelizable manifold, i.e., a quotient of a complex Lie by a discrete subgroup. It is abelian by our discussion at the beginning of this section.

Now we restrict our attention to the complex fiber N/H with $N = G \cap N^{\mathbf{C}}$, which acts on the fiber as a torus. We have $C = CH/H \subset N/H_0$ as an abelian subgroup. Therefore, on the (complex) Lie algebra (of N/H_0) level, we have that Ω_0 (the corresponding Hermitian metrics) is non-negative and

$$\frac{d\Omega_0}{dt} \wedge dt = \theta \wedge \Omega_0 = (\varphi(t))^{-\frac{1}{2}}\Omega_0 \wedge dt,$$

since all the $d_t(J\theta), d_t\alpha_t$ are zeros on the semi-simple part of H and therefore are zeros on the fiber according to [12]. That is,

$$\frac{d\Omega_0}{dt} = (\varphi(t))^{-\frac{1}{2}}\Omega_0.$$

By the non-negativity and the periodicity, we determine that $\Omega_0 = 0$. □

Now, we continue the proof of our Theorem 4:

We now have:

$$\Omega = \varphi(t)\theta \wedge J\theta + d_t\alpha_t(\text{mod}\mathcal{H}).$$

That is, $d_t\alpha_t$ has complex rank $n-1$. This means that Tits fibration of M in [10] is a map from M to the rational projective homogeneous space $S/C(\alpha_t^*)$, where α_t^* is the dual of α_t through Killing form and $C(\alpha_t^*)$ is the centralizer of α_t^* in the semi-simple part S of the Lie group G (for this construction, see [12] for example). And therefore, the fiber has a complex-dimensional one, which can only be a complex one-dimensional torus. We obtain our Theorem 4.

From now on, we assume that $G = S$.

Furthermore, $J\theta = J_t\theta = b(t)\beta \in T^*M_t$ for a constant element β.

$$d\Omega = -\varphi(t)\theta \wedge d_t(J\theta) + d_t(\frac{d_t\alpha_t}{dt}) \wedge dt = \theta \wedge d_t\alpha_t,$$

that is,

$$-(\varphi(t))^{\frac{1}{2}}dt \wedge d_t(J\theta) + d_t(\frac{d\alpha_t}{dt}) \wedge dt = (\varphi(t))^{-\frac{1}{2}}dt \wedge d_t\alpha_t.$$

Therefore,

$$-(\varphi(t))^{\frac{1}{2}}d_t(J\theta) + d_t(\frac{d\alpha_t}{dt}) = (\varphi(t))^{-\frac{1}{2}}d_t\alpha_t.$$

We have

$$-\phi(t)b(t)\beta + \frac{d\alpha_t}{dt} = (\phi(t))^{-1}\alpha_t,$$

where $\phi(t) = (\varphi(t))^{\frac{1}{2}}$.

Here, α_t, β are the elements in the center of the centralizer of the isotropic subgroup. Therefore, we have a standard decomposition for this equation with $\alpha_t = a(t)\beta + \gamma_t$:

$$\frac{d\gamma_t}{dt} = (\phi(t))^{-1}\gamma_t,$$

$$-\phi(t)b(t)\beta + a'(t)\beta = (\phi(t))^{-1}a(t))\beta.$$

By the maximal principle, we obtain $\gamma_t = 0$. Therefore, $\alpha_t = a(t)\beta$.

$$-(\phi(t))^2 b(t) = -\phi(t)a'(t) + a(t).$$

Given a positive $\alpha_t = a(t)\beta$ and a $\varphi(t)$, we could obtain a $b(t)$ (not necessarily negative).

In the homogeneous case, we could have $\varphi = 1$, $a(t) = 1$, $\beta = -J\theta$ and therefore, $b(t) = -h(t) = -1$. In general, let s be the t for a homogeneous case. Then, $Jds = -\beta$. If $\theta = k(t)ds$, we have $J\theta = k(t)J(ds) = -k(t)\beta$. That is, $b(t) = -k(t)$.

$$a'(t) - (\phi(t))^{-1}a(t) = -\phi(t)k(t).$$

It takes us some simple exercises to determine some interesting periodic $\varphi(t)$, $a(t)$, then $k(t)$. For example, one might let $\varphi(t) = 1$ for the simplicity. Then we could have:

A. $a(t) = 1 - a\sin t$, then $k(t) = a(t) - a'(t) = 1 - a(\sin t - \cos t)$;
B. $a(t) = 1 - a\cos t$, then $k(t) = 1 - a(\cos t + \sin t)$;
C. $a(t) = 1 - a(\cos t + \sin t)$, $k(t) = 1 - 2a\sin t$.

One might go the other way around. Given a periodic $k(t)$, then find some $a(t)$ with the same period by solving the equation.

Although this might look complicated, we obtain our Main Theorem anyway. More precisely, we obtained following:

Theorem 5. *Let M be a compact cohomogeneity one locally conformal Kähler manifold. If $\theta_0 = 0$, the manifold itself admits a compact homogeneous locally conformal Kähler structure*

$$\Omega = \theta \wedge J\theta - d(J\theta)$$

with $\theta = ds$. The cohomogeneity one locally conformal Kähler structure has the form

$$\Omega' = \Phi(s)\theta \wedge d\theta - A(s)d(J\theta)$$

with $\Phi(s), A(s)$ positive functions. On the other hand, all the Hermitian metrics in this form are locally conformal Kähler. In particular, M is cohomogeneity one under the action of the semi-simple part S of the Lie group, i.e., it has hypersurface orbits. $M = N \times S^1$ as a homogeneous space (but not necessary as a Riemannian manifold) with N the S orbits. The related Kähler metrics are cohomogeneity one under the S action. Moreover, M is a complex one-dimensional torus bundle over a projective rational homogeneous space $Q = G/K$ and a finite quotient of a quotient of a $\mathbf{C}^*$ bundle by some action e^a with $\mathrm{Re}a \neq 0$.

Notice that the equation involved $\phi(t), a(t)$ and $k(t)$ before the statement of this Theorem is too complicated. To see that there are a lot of cohomogeneity one but not homogeneous locally conformal Kähler structures, we simply check the Hermitian forms Ω' with positive $\Phi(s)$ and $A(s)$.

The homogeneous ones correspond to with Φ, A being constants.

5. Some Further Observations for the Compact Homogeneous or Cohomogeneity One Locally Conformal Kähler Structures

Our earlier Theorems basically say that the compact cohomogeneity one locally conformal Kähler manifolds are exactly the same as complex manifolds as those compact homogeneous locally conformal Kähler manifolds if the restriction of θ on the generic G orbit is cohomologous zero.

One observation we had in [7] for the homogeneous case that was not in the earlier published results in [6] or [8] is the following:

Theorem 6. *The compact homogeneous (cohomogeneity one with $\theta_0 = 0$) locally conformal Kähler manifolds are exactly those compact complex manifolds that are compact quotients of a positive homogeneous $\mathbf{C}^*$ bundle over a projective rational homogeneous space.*

Proof. We just notice that $-d(J\theta)$ is positive. □

Another observation is the following: If the readers are not very comfortable with our quotient of positive $\mathbf{C}^*$ bundle structure, one might look at the complex two-dimensional case first, which was given in the introduction of [7]. Then, take a point q in the base Q of the Tits fibration

$$T : M \to Q.$$

According to [12], we have $Q = S^{\mathbf{C}}/P$ with $S^{\mathbf{C}}$ a complex semi-simple Lie group and P a parabolic subgroup. A complex subgroup P is a parabolic subgroup if it contains a Borel subgroup B, i. e., $B \subset P$. A connected complex subgroup is the Borel subgroup if it contains the maximal Cartan torus and all the corresponding one parameter subgroups with negative roots. Let L be an $SL(2, \mathbf{C})$ subgroup such that its Borel subgroup B_L is in B. Then, the orbit of L through q induces an L homogeneous rational curve Q_L on Q. Then $T^{-1}(Q_L)$ is a compact locally conformal Kähler submanifold of complex dimension two. And as illustrated in the introduction of [7], it has a reasonable positive $\mathbf{C}^*$ bundle structure.

In general, we have following:

Theorem 7. *Let $L \subset S^{\mathbf{C}}$ be a complex semi-simple Lie subgroup, if the Borel subgroup B_L of L is in B, the orbit Q_L of L through q is a rational projective homogeneous subspace of Q. Moreover, $T^{-1}(Q_L)$ is a compact locally conformal Kähler submanifold.*

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