Type I Almost-Homogeneous Manifolds of Cohomogeneity One—III

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Abstract

This paper is one of a series of papers in which we generalize our results in [Guan 2003] on the equivalence of existence of Calabi extremal metrics to the geodesic stability for any type I compact complex almost homogeneous manifolds of cohomogeneity one. In this paper, we actually carry all the results in [Guan 2003] to the type I cases. As requested by earlier referees of this series of papers, in this third part, we shall first give an updated description of the geodesic principles and the classification of compact almost homogeneous Kähler manifolds of cohomogeneity one. Then, we shall give a proof of the equivalence of the geodesic stability and the negativity of the integral in the first part. The major tool is from [Guan 1999]. Finally, we shall address the relation of our result to Ross-Thomas version of Donaldson's K-stability. One should easily see that their result is a partial generalization of our integral condition in the first part. And we shall give some further comments on the Fano manifolds with the Ricci classes. In Theorem 4, we give a result of Nadel type. We define the strictly slope stability. In our case, it is stronger than Ross-Thomas slope stability. We strengthen two Ross-Thomas results in Theorem 5 and 6. The similar proofs of the results other than the existence for the type II cases are more complicated and will be done in [Guan 2012+a].

Key Words: Kähler manifolds, Einstein metrics, Ricci curvature, constant scalar curvature, fibration, almost-homogeneous, cohogeneity one, semisimple Lie group, semisimple elements, root vectors, geodesic stability, strictly slope stability, anticanonic divisor.

AMS Subject Classification: 53C10, 53C25, 53C55, 34B18, 14J45.

1. Introduction

This paper is one of a series of papers in which we finished the project of the existence of extremal metrics in any Kähler class on any compact almost homogeneous manifolds of cohomogeneity one.

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In [Guan 2011a], [Guan 2011b] we proved that for the type I compact almost homogeneous Kähler manifolds of cohomogeneity one, the existence of Calabi extremal metrics is the same as the negativity of a topological integral. We also proved in [Guan 2011b] that for any two Káhler metrics in the Mabuchi moduli space of Kähler metrics there is a smooth geodesic connecting them. That is, the geodesic principle I is true for these manifolds.

As in [Guan 2003], the major tool is from [Guan 1999]. Although the problem of existence of the extremal metrics can be reduced to an ordinary differential equation for our manifolds, the problem of the existence of the geodesics has two variables. Thanks to the Legendre transformation that we can carry it out for the type I manifolds. But for a general type II manifolds, this method does not work any more. And we need a new method, which will be carried out in [Guan 2012+a].

Even for the Kähler Einstein equation, our method in [Guan 2011a] is different from [Guan-Chen 2000]. We used a semisimple method in [Guan 2011a]. One notices that our exponential map there is not the one for the geodesics. No geodesic in that situation could have infinite length. It was well-known for many years, there were many nonsmooth solutions for even a real homogeneous Monge-Ampére equations. In [Chen-Tian 2008] Professor Chen gave some example which looks like nonsmooth solution for one dimensional toric case, i.e., $\mathbf{C}P^1$. He also mentioned earlier it to me 1999 in Princeton. Mabuchi also mentioned it to me in Pisa Italy 2004. However, we already solved the smoothness question for the toric manifolds in [Guan 1999]. In this simple case, his own method should also produce the smooth solution, see [Guan-Phong 2012]. The content of this note was presented in the AMS meeting in Pomona California May 2008. Recently, L. Lempert and L. Vivas claimed (also mentioned by the referee) that they found a counterexample for our geodesic principle I on torus. However, their examples are not very explicit and not published yet. We are not able to check their examples in this paper. As we know that there is no much equivariant geometry on the torus. The geodesic problem was trivial on the torus. However, see also [Feng 2012]. We checked that all the geodesic principles hold on compact cohomogeneity one Kähler manifolds. We conjucture that the geodesic principles hold for all the spherical manifolds. We take them as working principles in our research. *For our safety, we just require that every thing is analytic*. For example, for any analytic initial value in the tangent space of the equivariant Mabuchi moduli space at a given metric, there is a geodesic ray. That is, *the geodesic principle I is not really needed for the geodesic stability.* In [Guan-Chen 2000], there are some possible obstructions emerged that I worked alone and eventually treated them in [Guan 2002], which led to the strictly slope stability. After a long run, we are able to overcome all the difficulties. To solve the extremal metrics cases, we have to deal with a fourth order ordinary differential equation, which in our cases is fortunately reduced to a second order nonlinear equation and was successfully treated.

All the solutions we find in the cohomogeneity one cases are not explicit except those in [Guan 1995a] and [Guan 2007].

In this paper, we shall prove that the negativity of the integral is actually the same as the geodesic stability.

A classification which we refer to in this paper can be found in [Gu3 section 12].

Here we shall describe our updated *Geodesic Stability Principles*. We conclude these principles by following the cumulation of other people's observations and the evidences from our examples. See [Guan 2003]. We do not assume that these principles due to us completely, in particular the first principle.

In [Mabuchi 1986], motivated by the Donaldson's functional in the vector bundle case, Mabuchi defined a functional on the Mabuchi moduli space of the Kähler metrics (see also a conjecture therein). It was later on modified independently by several people to fit the situation of Calabi extremal metrics (see [Guan 1999], [Guan-Chen 2000] etc.) on the equivariant Mabuchi moduli space of Kähler metrics, which we call the modified Mabuchi functional.

Principle I. *For any two Kähler metrics in a given Kähler class, there is a unique (smooth) geodesic in the Mabuchi moduli space of Kähler metrics connecting them.*

This principle has been tested for toric bundles in [Guan 1999]. We also found that the same method applies to the Kähler metrics on the type I compact almost homogeneous Kähler manifolds of cohomogeneity one in [Guan 2003], [Guan 2011b] (see also [Guan 2007]). It seems to us that there is not any complete geodesic except the ones induced by the holomorphic vector fields. In [Chen 2000], X. X. Chen proved the existence of an unique $C^{1,1}$ solution in general.

We shall concentrate on the maximal geodesic rays. It turns out that the majority of the maximal geodesic rays are of finite length (this is different from holomorphic vector bundle theory on vector bundles, Cf. [Kobayashi 1987] p.197 and also from the picture showed in [Semmes 1991] p.544). The maximal geodesic rays with infinite length are very special with some *strong convex* property, which we call "effective" maximal geodesic rays. The direction of the effective geodesic rays at each metric might form a *convex cone* $\mathcal{C}$.

Principle II. *The limit metrics of the maximal geodesics are concentrations;*

A. Finite ray: cone concentration—partial concentration.

B. Infinite ray: blow up caused by some subvarieties outside a compact set—complete concentration outside the compact set, the metric on this compact set does not change.

We call the limit of the ratio of the modified Mabuchi functional the **Generalized Futaki Invariants of the maximal geodesic rays**. *The generalized Futaki invariant is positive infinite for finite ray, i.e., the only interesting generalized Futaki invariants come from the effective maximal geodesic rays.*

The second principle is based on our work on the toric manifolds and the cohomogeneity one manifolds, see [Guan 2003], [Guan 2007] for examples.

For all the examples we considered in this paper the Mabuchi equivariant moduli space is flat (see [Guan 1999]), this is similar to the vector bundle case and is not true in general (see [Mabuchi 1987]). For two maximal geodesic rays, the generalized Futaki invariants might be the same if there is a curve connecting the beginning points such that there is a parallel vector field along this curve which connects the two tangent vectors at these two points. This observation make the definition of the generalized Futaki invariants independent of the initial Kähler metrics.

The generalized Futaki invariants define a function of the effective geodesic cone which is probably a linear function $F_{M,\omega}$, which is continuous on certain given Banach space. Therefore, F can be defined on the closure $\bar{\mathcal{C}}$ of the effective cone $\mathcal{C}$ in the Banach space. We call the $F|_{\bar{\mathcal{C}}}$ the *Generalized Futaki Invariant Functional* or simplely the *Generalized Futaki Invariant*. There is a seminorm $||\ ||_*$ which is locally equivalent to the given norm except on some subvarieties and is zero on the functions induced by the holomorphic vector fields.

Principle III. *There is a unique extremal metric in a given Kähler class up to the automorphism group if and only if the Kähler class is geodesic stable, i.e., with* **positive** *generalized Futaki invariant which is bounded below by the given seminorm.*

The next principle came a little bit later than the other ones, Therefore, in many of our papers, we shall call this principle the fourth principle and the next the third principle instead.

In general, the Mabuchi moduli space might not be flat. We might have some way to relate the Futaki invariants for two infinite maximal geodesic rays starting from different points. Let $\gamma_i(t),\ i = 1, 2$ be two maximal geodesic rays. We say that they have the same infinite points if

$$d(\gamma_1, \gamma_2) = \sup_{t\in[0,+\infty)} d(\gamma_1(t), \gamma_2(t))$$

is finite. Then we have (see also [Guan 2007] Remark 4):

Principle IV. *The Futaki invariants of two maximal geodesic rays with the same infinite point are the same.*

In the last section, we shall see that our stability in this case is the same as a version of the slope stability which is stronger than that in [Ross-Thomas 2006].

2. The Preliminary

Here we summarize some known results about the compact complex almost-homogeneous manifolds of cohomogeneity one. In this paper, we only consider manifolds with a Kähler structure. For earlier results one might check with [Akhiezer 1983] and [Huckleberry 1982].

We call a compact complex manifold an almost homogeneous manifold if its complex automorphism group has an open orbit. We say that a manifold is of cohomogeneity one if the maximal compact subgroup has a (real) hypersurface orbit. In [Guan-Chen 2000] and [Guan 2003], we reduced the compact complex almost homogeneous manifolds of cohomogeneity one into three types of manifolds.

We denote the manifold by M and let G be a complex subgroup of its automorphism group which has an open orbit on M.

Let us assume first that M is simply connected. Let the open orbit be G/H, K be the maximal connected compact subgroup of G, L be the generic isotropic subgroup of K, i.e., K/L be a generic K orbit. We have [Guan-Chen 2000] Theorem 1:

Proposition 1. *If G is not semisimple, then M is a completion of a $\mathbf{C}^*$ bundle over a projective rational homogeneous space.*

If a compact almost homogeneous Kähler manifold is a completion of a $\mathbf{C}^*$ bundle over a product of a torus and a projective rational homogeneous space, we call it a *manifold of type III*. We have dealt with this kind of manifolds in our dissertation [Guan 1995a], [Guan 1995b]. There always exists an extremal metric in any Kähler class. Recently, we generalized this existence result to a family of metrics, which connects the extremal metric in [Guan 1995a] and the generalized quasi-einstein metric [Guan 1995b], called the extremal-soliton metrics in [Guan 2007]. The existence of the extremal-soliton is the same as the geodesic stability with respect to a generalized Mabuchi functional.

More recently in [Guan 2012], we even generalized the extremal-solitons to the generalized extremal solitons, which also include Nakagawa's generalized Kähler-Ricci solitons [Nakagawa 2011] as a special case. We proved the existence of both generalized extremal-soltons and the generalized Kähler-Ricci solitons on these manifolds. In a forthcoming paper [Guan 2012+b], we proved the existence of the so called m-extremal metrics on these manifolds.

In general, if M is a compact almost homogeneous Kähler manifold and O is the open orbit, then $D = M - O$ is a proper closed submanifold. Moreover, D has at most two components. We call each component of D an end. If D has two components (or one component), we say M is an almost homogeneous manifold with two ends (or one end). We have [Huckleberry 1982] Theorem 3.2:

Proposition 2. *If M is a compact almost homogeneous Kähler manifolds with two ends, then M is a manifold of type III.*

Therefore, we only need to deal with the case with one end. In [Guan-Chen 2000], we treated the first example, i.e., the blowup of the diagonal of the product of two copies of $\mathbf{C}P^n$. We treated another series in [Guan 2003]. We treated much more of them in [Guan 2011b] and [Guan 2009], [Guan 2011c], etc.. Again, in the case of M being simply connected, we only need to take care of the case in which G is semisimple. If G is semisimple and M has two G orbits, one open and one closed, and moreover if the closed orbit is a complex hypersurface, there are two possibilities. Let $\mathcal{K}, \mathcal{L}$ be the Lie algebras of K, L. Then the centralizer of $\mathcal{L}$ in $\mathcal{K}$ is a direct sum of the center of $\mathcal{L}$ and a Lie subalgebra $\mathcal{A}$ with $\mathcal{A}$ being either one dimensional or a 3-dimensional Lie algebra $su(2)$. If $\mathcal{A}$ is one dimensional, we call M a *manifold of type I*. If $\mathcal{A}$ is $su(2)$, we call M a *manifold of type II*.

In general, if the closed orbit has a higher codimension, we can always blow up the closed orbit to obtain a manifold $\tilde{M}$ with a hypersurface end. we call the manifold M a *manifold of type I (or II)* if $\tilde{M}$ is of type I (or II).

There is a special case of the type II manifolds. If the open orbit is a $\mathbf{C}^k$ bundle over a projective rational homogeneous manifold, we call M an *affine type manifold* (not to be confused with the closed complex submanifolds of $\mathbf{C}^m$).

Then we have (see [Guan 2003] section 12):

Proposition 3. *Any compact almost homogeneous Kähler manifold M of cohomogeneity one is an $Aut_0(M)$ equivariant fibration over a product of a rational* *projective homogeneous manifold Q and a complex torus T with a fiber F. Therefore, M can be regarded as a fiber bundle over T with a simply connected fiber M_1. One of following holds:*

(i) M is a manifold of type III.

(ii) M_1 is of type II but not affine.

(iii) M_1 is affine.

(iv) M_1 is of type I.

We say that M is a *manifold of type I (or type II, affine)* if M_1 is a manifold of type I (or type II, affine).

We actually can also obtain a structure of the M_1 bundle over T from [Huckleberry 1982]. We only need to understand the bundle structure for the open orbit. By [Huckleberry 1982] Corollary 4.4 we have that the bundle structure is a product unless when we apply Proposition 3 to $\tilde{M}$ $F = Q^k$. In the latter case, there is an unbranched double covering $\bar{M}$ of M such that the bundle structure is a product. We have:

Proposition 4. *The M_1 bundle over T is a product except in the case with which the open orbit is a F_0 bundle over $Q \times T$ such that F_0 is in the second, sixth and eighth cases in [Akhiezer 1983] p.67. In the latter cases, the M_1 bundle has an unbranched double covering which is a product of M_1 and T.*

In [Guan 2011a], [Guan 2011b], we dealt with the type I cases.

One updated remark is that since we are dealing with the Kähler metrics it is more convenient to separate the type II case into two cases in [Guan 2009] and [Guan 2011c]. We call the cases in [Guan 2009] (and the papers between [Guan 2009] and [Guan 2003]) the *type IV* cases. They are the affine cases such that the group $\pi(G_F)$, the restriction of the subgroup G_F of G fixing a given fiber F, is not of type A. Therefore, one might also call them the *non-type-A type II* cases. All of them are Fano.

One might call the rest (in [Guan 2011c]) of the type II cases the new type II cases (or simply the type II cases). They are those type II cases such that $\pi(G_F)$ is of type A. Therefore, one might also call them the *type-A type II* cases.

This note is a continuation of the first part and the second part of this paper [Guan 2011a], [Guan 2011b]. We shall carry all the notations here.

3. The Complex Structures of the Type I Almost Homogeneous Manifolds

In this section, we shall deal with the complex structure of the type I almost homogeneous manifolds. We retain the notations in [Guan 2011a], [Guan 2011b]. Let us recall some basic notations of the Lie algebras.

Let G be the complex Lie group action and S be the connected complex Lie subgroup acting on a given fiber. According to [Guan 2003] p.283 Theorem 12.1(ii), a compact complex almost homogeneous manifold of cohomogeneity one is type I if and only if the fiber F is one of (1) the second and third case with $n \geq 3$, (2) the fourth case, (3) the eight and ninth cases, (4) the fifth case in [Akhiezer 1983] p.67.

The fiber F in (4) has $S = \pi(G_F) = F_4$, so $G = F_4 = S$, that is, $M = F$ is homogeneous. Therefore, every Kähler class of M has a metric with constant scalar curvature. So, we do not need to do anything with (4).

In [Guan 2011a], we look at three special possible fiber cases [Akhiezer 1983] p.67 first:

(1) $F = F(OP_n)$: The third case in [Akhiezer 1983] p.67 with $n \geq 3$. $F = \mathbf{C}P^n$. $S = \pi(G_F) = SO(n, \mathbf{C})$ with regarding $\mathbf{C}P^n$ as a completion of $\mathbf{C}^n$. The corresponding compact rank one symmetric space is the real n dimensional real projective space. It has an equivariant branched double covering Q^n of the second case. We denote the latter case by $F(OQ_n)$.

(2) $F = F(Gr_k)$: The fourth case with a standard $S = Sp(k, \mathbf{C})$ action on the manifold $F = Gr(2k, 2)$. The corresponding compact rank one symmetric space is the quarterion projective space.

(3) $F = F(Sp_7^p)$: The ninth case with an $S = Spin(7, \mathbf{C})$ action on $F = \mathbf{C}P^7$. This is the restriction of (1) with $n + 1 = 8$ to the complex Lie subgroup $Spin(7, \mathbf{C})$. It has an equivariant branched double covering Q^7 of the eighth case. In [Guan 2011a], we also denote the latter case by $F(Sp_7^q)$ and denote both of them by $F(Sp_7)$ whenever there is no confusion.

In [Guan 2011a], we defined a certain basis of the Lie algebra α, F_α and G_α for positive roots α. And we considered a fixed point p_0 and its orbit p_s generated by a semisimple element $-iH$ in the Lie algebra. Let T be the tangent vector of p_s and p_∞ be the limit point in the closed orbit.

In the case (1), we obtained:

Proposition 5. *For $F(OP_n)$ and $F(OQ_n)$, along p_s we have:*

$$J(F_{e_1+e_i} \pm F_{e_1-e_i}) = -(\tanh s)^{\mp 1}(G_{e_1+e_i} \pm G_{e_1-e_i})$$

(and $JF_{e_1} = -(\tanh s)G_{e_1}$). We also have that

$$F_{e_i \pm e_k} = G_{e_i \pm e_k} = 0$$

(and $F_{e_i} = G_{e_i} = 0$) for $i > 1$. In particular, at p_∞, $JF_\alpha = -G_\alpha$ for $\alpha \neq e_i \pm e_k$ (and e_i) $1 < i < k$.

In the case of (2), we obtained:

Proposition 6. *For $F(Gr_k)$, we have*

$$JF_{\alpha_1} = -(\tanh 2s)G_{\alpha_1},$$

$$J(F_{2e_1} \pm F_{2e_2}) = -(\tanh 2s)^{\mp 1}(G_{2e_1} \mp G_{2e_2}),$$
$$J(F_{e_1-e_k} \pm G_{e_2-e_k}) = -(\tanh s)^{\mp 1}(G_{e_1-e_k} \pm F_{e_2-e_k}),$$
$$J(F_{e_1+e_k} \pm G_{e_2+e_k}) = -(\tanh s)^{\mp 1}(G_{e_1+e_k} \pm F_{e_2+e_k}).$$

$$F_\alpha = G_\alpha = 0$$

for

$$\alpha = e_1 + e_2, e_i - e_k, 2e_i, e_i + e_k$$

with $i > 2$.

At p_∞, *we have* $F_\alpha = G_\alpha = 0$ *if* $\alpha = e_1 + e_2, 2e_i, e_i \pm e_k,\ i > 2$; *and* $JF_\alpha = G_\alpha$ *if* $\alpha = 2e_2,\ e_2 \pm e_k$. *Otherwise* $JF_\alpha = -G_\alpha$.

Before we consider the isolated case (3), we can look at the general cases in which $G \neq S = \pi(G_F) \subset Aut(F)$, where G_F is the subgroup that acts on the fiber F and $\pi:\ G_F \to Aut(F)$ is the induced map from G_F to $Aut(F)$. As in [Akhiezer 1983], G is semisimple, $U_G = H$ is the 1-subgroup. There is a parabolic subgroup $P = SS_1R$ with S, S_1 semisimple and R solvable such that $U_G = US_1R$ where $U = H \cap S$ is a 1-subgroup of S. The manifold is a fibration over G/P with the completion of $P/U_G = S/U$ as the isotropic open orbit of the almost homogeneous fiber. In this case, the root system of S is a subsystem of the root system of G. In the Lie algebra of G, we also have some other F_α, G_α outside $\mathcal{S}$. Let K be a maximal connected compact Lie subgroup of G and L be the isotropic subgroup of K at a generic orbit. Let $\mathcal{K}, \mathcal{L}$ be the corresponding Lie algebras. The tangent space of G/U_G along p_s is decomposed into irreducible $\mathcal{L}$ representations. These F_α, G_α are in the complement representation of the Lie algebra $\mathcal{S}$ of S. $JF_\alpha = -G_\alpha$ (mod $\mathcal{S}$) as it is in the tangent space of G/P. Therefore, we have $JF_\alpha = -G_\alpha$ for any α which is not in the root system of S.

If S is B_2, G can be B_n, C_n, F_4. If S is B_3, G can be B_n, F_4. If S is C_3, G can be C_n, F_4. If S is B_n with $n > 3$, G can only be B_{m+n}. If S is C_n with $n > 3$, then G can be C_{n+m}. The case of B_2 action which has an isotropic group of $SO(4, \mathbf{C})$ generated by roots $\pm e_1 \pm e_2$ is exactly the same as the case of $Sp(2, \mathbf{C})$ action, which has an isotropic subgroup of $Sp(1, \mathbf{C}) \times Sp(1, \mathbf{C})$ generated by $\pm 2e_1, \pm 2e_2$.

We have a few more possibilities. If $S = D_k,\ k > 3$, G can only be $D_n,\ n > 3$ or $E_n\ n > k$. If $S = D_3$, that is an A_3, G can be $A_n\ n > 2$, $B_n,\ n > 3$, $C_n\ n > 3$, $D_n\ n > 2$ and E_n. If $S = D_2$, G can be any simple group or product of simple groups other than G_2.

We then treated the isolated case (3) of the $Spin(7, \mathbf{C})$ action on $\mathbf{C}P^7$ in [Guan 2011a]. This case is the restriction of the case (1) with an $G = S = SO(8, \mathbf{C})$ action to the $Spin(7, \mathbf{C})$ action induced by the spinor representation.

We obtained:

Proposition 7. *For* $F(Sp_7)$, *we have*

$$J(\sqrt{2}F_{h_i} \pm F_{h_j+h_k}) = -(\tanh \frac{\sqrt{3}}{2}s)^{\mp 1}(\sqrt{2}G_{h_i} \pm G_{h_j+h_k})$$

and

$$JH = -T,$$

$$F_{e_i-e_j} = G_{e_i-e_j} = 0\ 0 < i < j < 4.$$

At p_∞, $JF_{h_i} = -G_{h_i}$, $JF_{h_j+h_k} = -G_{h_j+h_k}$, $F_{h_i-h_k} = G_{h_i-h_k} = 0$.

However, in this case $S = B_3$, G can only be B_n or F_4.

4. The Kähler Structures

In [Guan 2011a], we examined the Kähler structure for the $S = SO(n, \mathbf{C})$ actions and obtained that for any possible G and $S = SO(n, \mathbf{C})$ we always have Kähler metric: $\omega([X, Y]) = (aH + I, [X, Y])$ with the I in the C center of $\mathcal{L}$ and a a nonpositive function of s.

See [Guan 2011a], section 3.

Therefore, we have the volume formula:

$$V = -Ma'a^{2(n-1)} \prod_1^r (a_i - a) \prod_1^s (b_j + a)$$

$$(\text{or } V = Ma'a^{2n-1}(\tanh s) \prod_1^r (a_i - a) \prod_1^s (b_j + a))$$

with some positive numbers a_i and b_j.

Then in [Guan 2011a], we dealt with the Kähler metrics with $Sp(k, \mathbf{C})$ and $Spin(7, \mathbf{C})$ actions. We have the volume form:

$$V = Ma'a^{4k-5}(\tanh 2s) \prod_1^r (a_i - a) \prod_1^s (b_j + a)$$

for the $Sp(k, \mathbf{C})$ actions.

For $S = Spin(7, \mathbf{C})$ action, we obtained the volume form:

$$V = -Ma'a^6 \prod_{i=1}^r (a_i - a) \prod_{j=1}^s ((b_j + a).$$

We also observe that a_i and b_j come in pairs, and $b_{j(i)} = a_i$.

Altogether, we have:

Proposition 8. *For the type I case the volume is*

$$V = -Ma'a^{2m} \prod (a_i^2 - a^2) \tag{1}$$

for the cases $S = D_k$ or $Spin(7, \mathbf{C})$ and

$$V = Ma'a^{2m+1}(\tanh bs) \prod (a_i^2 - a^2)$$

for the cases $S = B_k$ (or C_k) with $b = 1$ (or 2), where M and a_i are positive numbers, m are nonnegative integers. We also have that $2m + 1$ (or $2m + 2$) are the dimensions of the fiber. Moreover, the vectors in Proposition 5, 6 and 7 are orthogonal to each other.

Let $h = \log V$, in [Guan 2011a] section 5 (Theorem 2) we obtained:

Proposition 9. *If the fiber with the S action is of type I of complex dimension n, then the function a for the Ricci form ρ is*

$$a_\rho = \frac{1}{2}((\log(a'a^{n-1} \prod_1^r (a_i^2 - a^2)))' - 2 \sum_1^{n-1} N_i \coth 2N_i s).$$

Moreover, (1) N_i are 1 for $S = SO(n+1, \mathbf{C})$ and (2) 1 except three of them being 2 for S of type C_k, (3) $\frac{\sqrt{3}}{2}$ for the case $S = Spin(7, \mathbf{C})$. Other coefficients come from the Ricci curvature of G/P which is $-(q_{G/P}, [X, Y])$ with $q_{G/P} = \sum_{\alpha \in \Delta^+ - \Delta_P} H_\alpha$ with the standard inner product.

Then we calculated the scalar curvature in [Guan 2011a] section 6 (Theorem 3). We write

$$V = -Ma'\tilde{Q}(a) = -Ma'(-a)^{n-1}Q_1(a)g(s)$$

with $g(s) = 1$ for $S = D_k$ or $Spin(7, \mathbf{C})$ and $g(s) = \tanh bs$ for $S = B_k$ or C_k. We write $Q(a) = (-a)^{n-1}Q_1(a)$ and obtained: $\rho \wedge \omega^{N-1} = M((-a_\rho Q(a))' + p_0 a')$.

Proposition 10. *The scalar curvature is $R = \frac{2(-a_\rho Q)' + pa'}{-a'Q}$. Moreover, $p(a) = (-a)^{n-1}p_1(a)$ with $p_1(a)$ a polynomial of a and is a positive linear sum of Q_1 and product of $\deg Q_1 - 1$ factors of Q_1. The contribution of each constant factor k_j (i.e., the vector F_α such that the corresponding metrics $\omega(F_\alpha, JF_\alpha) = k_j$ is a constant along p_s) is $\frac{2k_{\rho,j}}{k_j}$ for the Q_1 factor. The contribution of each $a_i \pm a$ is $2\frac{a_{\rho,i}Q_1}{q_i}$.*

Therefore, we have

$$R_0 = \frac{\int_0^{-l}[(2u_\rho Q)_x + p]dx}{\int_0^{-l} Qdx} = \frac{2u_\rho(-l)Q(-l) + \int_0^{-l} pdx}{\int_0^{-l} Qdx},$$

where we let $u = -a$ and $l = \lim_{s \to +\infty} a$. We also obtained in [Guan 2011a] that $a_\rho(0) = 0$.

5. Geodesic Stability and Existence of the Calabi Extremal Metrics

In [Guan 2011b] section 2, for any metric we obtained a function $\Gamma(s)$ such that $-4a = 4u = \Gamma'$ and the geodesic equation is

$$\ddot{\Gamma}\Gamma'' = (\dot{\Gamma}')^2,$$

where $'$ is the derivative with respect to s the parameter from the manifold and $\dot{}$ is the derivative with respect to t the parameter for the geodesic. We obtain the smooth geodesics and so the uniqueness. Therefore, we might regard $U = 4u$ as g in [Guan 2011a].

We also have:

$$4u_s(+\infty) = \Gamma_{ss}(+\infty) = 0$$

since u is increasing and bounded by $-l$ (see the end of last section).

We shall apply the method in [Guan 2003] to prove the second and third *geodesic stability principles* for all the type I Kähler almost homogeneous manifolds of cohomogeneity one.

The proof is parallel to what we have in [Guan 2003] but even simpler (with our advanced notations).

Let H be the Legendre transformation of Γ as in [Guan 2003], then a path Γ_t represents a geodesic in the Mabuchi moduli space of the equivariant Kahler metrics in a given Kähler class is a geodesic if and only if H_t is linear on t. We denote $h = \dot{H}$.

Recall the R is the scalar curvature, HR its average, Q the volume function appeared right before the Proposition 10. Applying the scalar curvature formula in the Proposition 10, we have that with a positive constant C the derivative of Mabuchi functional is:

$$
\begin{aligned}
-\int_M \dot{\Gamma}(R - HR)\omega^{2n} &= -C\int_0^{-l} \dot{\Gamma}(s,t)(2u_\rho Q + \int (p - R_0 Q)du)_x dx \\
&= C\int_0^{-l} \dot{H}(x,t)(2u_\rho Q - \int (R_0 Q - p)du)_x dx \\
&= C(2h(-l)u_\rho(-l)Q(-l) - 2h(0)u_\rho(0)Q(0) - R_0 h(-l)\int_0^{-l} Qdx \\
&+ R_0\int_0^{-l} h'(\int_0^x Qdu)dx + h(-l)\int_0^{-l} pdx - \int_0^{-l} h'(\int_0^x pdu)dx \\
&- 2\sum_1^{n-1}\int_0^{-l} N_i \coth(2N_i s)h'Qdx + \int_0^{-l} h'(\log(Qu_s))_s Qdx) \\
&= C(R_0\int_0^{-l} h'(\int_0^x Qdu)dx - \int_0^{-l} h'(\int_0^x pdu)dx \\
&- 2\sum_1^{n-1} N_i\int_0^{-l} \coth(2N_i s)h'Qdx + \int_0^{-l} h'(Qu_s)_x dx) \\
&= C(R_0\int_0^{-l} h'(\int_0^x Qdu)dx - \int_0^{-l} h'(\int_0^x pdu)dx \\
&- 2\sum_1^{n-1} N_i\int_0^{-l} \coth(2N_i s)h'Qdx - \int_0^{-l} Qu_s h'' dx) \\
&= C(R_0\int_0^{-l} h'(\int_0^x Qdu)dx - \int_0^{-l} h'(\int_0^x pdu)dx \\
&- 2\sum_1^{n-1} N_i\int_0^{-l} \coth(2N_i H_x)h'Qdx - \int_0^{-l} Q(H_{xx})^{-1}h'' dx).
\end{aligned}
$$

The changing of sign in the second equality comes from $\dot{\Gamma}(s,t) = -\dot{H}(x,t)$ for the Legendre transformation as in [Guan 2003].

If h'' is negative somewhere, then the geodesic is finite and the limit is a cone metric. The point $-l$ can not be a singular point. At the singular points h'' is negative. Therefore, the last term of the right hand side is positive infinite. The second term from the right hand side is finite if 0 is not a singular point and positive if 0 is a singular point since in that case $h''(0) < 0$ and $h'(0) = s(0) - s_0(0) = 0, h' < 0$ near 0.

If h'' is nonnegative, then the geodesic ray is infinite and h' is increasing. s turns to infinite at each point with $h' > 0$, so $\coth(2N_i s)$ turns to 1 at such points. It is not difficult to see that $(H_{xx})^{-1}$ turns to zero whenever h'' is not zero. The limit of the derivative is:

Theorem 1. *For type I compact Kähler almost homogeneous manifolds of cohomogeneity one, the generalized Futaki invariant of a maximal geodesic ray with a convex function h is*

$$C(\int_0^{-l} h'(\int_0^x (R_0Q - p)du - 2\sum_1^{n-1} N_iQ)dx)$$

with a constant $C > 0$.

According to [Guan 2011a] (14), this is proportional to the negative of

$$\int_0^{-l} h' g_l dx.$$

We notice that all the generalized Futaki invariants of the maximal geodesic rays do not depend on the initial metrics and they are positive if there is an extremal metric.

Moreover, if there is a Kähler metric with a constant scalar curvature, then at the corresponding H_0 we have that the slopes of Mabuchi functionals are zeros. Therefore,

$$\int_0^{-l} [h'[\int_0^x (R_0Q - p)du - 2\sum_1^{n-1} N_iQ\coth(2N_iH_{0,x})] - Q(H_{0,xx})^{-1}h'']dx = 0$$

for any h.

In general, the slope of the Mabuchi functional is

$$\begin{aligned}
C\int_0^{-l} Q((2\sum_1^{n-1} N_i(\coth(2N_iH_{0,x}) - \coth(2N_iH_x))h' + ((H_{0,xx})^{-1} - (H_{xx})^{-1})h'')dx \\
= \quad C\int_0^{-l} Q(4\sum_1^{n-1} N_i\frac{e^{2N_iH_{0,x}}(e^{2N_ith'} - 1)}{(e^{2N_i(H_{0,x}+th')} - 1)(e^{2N_iH_{0,x}} - 1)}h' \\
+ \quad ((H_{0,xx})^{-1} - (H_{xx})^{-1})h'')dx.
\end{aligned}$$

It turns to

$$C\int_0^{-l} Q(\sum_1^{n-1}\frac{4N_i}{e^{2N_iH_{0,x}} - 1}h' + H_{0,xx}^{-1}h'')dx.$$

Therefore, using this formula as a hint we can define

$$||h||_*^{2,1} = \int_0^{-l} Q(\sum_1^{n-1}\frac{4N_i}{e^{2N_iH_{0,x}} - 1}|h'| + H_{0,xx}^{-1}|h''|)dx$$

to be the norm of $W_*^{2,1}$. A calculation shows that this is related to

$$\int_0^{-l} |\Delta_0 h|Q dx$$

and also

$$\int_0^{-l} sup\{|\partial^2 h(v)|_0/|v|_0\}dV$$

with dV the volume element. The generalized Futaki functional is positive on the closure of the effective cone in $W_*^{2,1}$.

The generalized Futaki functional is positive if and only if they are positive for

$$h' = \{ \begin{array}{ll} 1 & \text{if } x > x_0 \\ 0 & \text{if } x \leq x_0 \end{array}$$

with $x_0 \in [0, -l)$. These functions h' correspond to functions of h in $W_*^{2,1}$ which are the extremal rays of the effective cone. As what we see in the sentence right after Theorem 1 that this is the same as the partial integral

$$\int_{x_0}^{-l} g_l du = \int_{x_0^2}^{l^2} f_l dx < 0$$

for the g_l, f_l in [Guan 2011a]. That is the same as the necessary and sufficient condition in [Guan 2011a] (see (7) and (16) there) for the existence of the Kähler metrics with constant scalar curvatures.

Therefore, we obtain:

Theorem 2. *For type I Kähler compact almost homogeneous manifolds of cohomogeneity one manifolds, there is a unique extremal metric in a Kähler class on the manifold up to the automorphism group if and only if the Kähler class is geodesic stable.*

The same method works for some of Kähler classes on type II compact Kähler almost homogeneous manifolds of cohomogeneity one. But in general, we will use a different method. A result similar to Theorem 1 and the same result of Theorem 2 are true for general compact almost homogeneous manifolds of cohomogeneity one. But it take us some more time to publish the related results and proofs. We also expect that Theorem 2 is true for any Kähler class on any compact Kähler manifold.

Theorem 1 also gives another proof for the stability (the necessary condition) in [Guan 2011a]. However, the integral itself and its partial integrals *do not occur directly* as generalized Futaki invariants of any (smooth) geodesic.

A generalization of our argument is essential to prove the necessary condition for the type II cases (and the type IV case in [Guan 2009]). However, since we have not seen any example with a zero value of the integral for the Ricci classes, for all the known cases so far in [Guan 2009] etc., the corresponding result in the next section is enough for the necessary part for the Kähler-Einstein case.

6. Geodesic Stability and Strictly Slope Stability

In this section, we shall discuss our result and the strictly slope stability. This is something also similar to the holomorphic vector bundle case and can be defined on any Kähler class of any compact Kähler manifolds.

6.1 To make the things simpler, first we assume that the Kähler class is the anticanonic class $-K_M$, N be a smooth subvariety and $M(N)$ be the blow-up of M along N. Let E be the exceptional divisor and e be the largest number such that $-K_M - aE > 0$ on $M(N)$ with regarding K_M as the pullback line bundle for any a such that $0 < a < e$,

$$m(N) = \int_0^e (-K_M - (n - \dim N)E)(-K_M - aE)^{n-1} da,$$

$$m = \int_0^e (-K_M - aE)^n da.$$

We say that M is *strictly* slope stable if for any subvariety N (not necessary smooth) *which is not a component of the fixed point set of a holomorphic vector field* we have $m(N) < m$. That is

$$\int_0^e (a - (n - \dim N))E(-K_M - aE)^{n-1} da < 0.$$

Notice that there is only one possible zero for $a - (n - \dim N)$, we see that if $m(N) - m < 0$ then

$$\int_0^c (a - (n - \dim N))E(-K - aE)^{n-1} da < 0$$

for any $0 < c < e$. That is, when N is smooth, our stability is stronger than Ross-Thomas' *slope stability* in [Ross-Thomas 2006], which only require the inequality for rational c with $0 < c < e$. While our inequality is true for any c with $0 < c \leq e$. If N is not smooth, we do not know that the slope stability in [Ross-Thomas 2006] implies these inequalities or not.

And a smooth N destables M only if $-K_M - (n - \dim N)E$ is ample, therefore, $-K(E)$ is ample on E if E is smooth, and is kind of ample even if E is singular. When N is smooth, we see that E is Fano. By [Futaki 1987], we see that N is Fano also. This is quite similar to the calculation in [Guan 2003], [Guan 2011a].

Actually, when $F = \mathbf{C}P^k$ or $Gr(2k, 2)$, we have in [Guan 2011b] section 3 that $D(F) = 2$ in the Theorem 15 in [Guan 2011b]. Therefore, for the closed orbit N, $e = -2^{-1} l_\rho$ and the codimension can only be 1 (see [Guan 2011b] section 3). Let $y = -l_\rho - 2a$, then above integral is

$$\int_0^{-l_\rho} (-2^{-1}(y + l_\rho) - 1)E(\omega + 2^{-1}(y + l_\rho)E)^{n-1} 2^{-1} dy$$

$$= C \int_0^{-K(F)} (-K(F) - D(F) - y)Q dy$$

with a positive number C. That is exactly the same condition in the Theorem 15 there.

When $F = Q^k$, $D(F) = 1$. Therefore, $e = -l_\rho$. Let $y = -l_\rho - a = -K(F) + m - 1 - a$ with $m = n - \dim N$. The above integral is

$$\int_0^{-l_\rho} (-l_\rho - y - m)E(\omega + (y + l_\rho)E)^{n-1} dy = C \int_0^{-K(F)+m-1} (-K(F) - D(F) - y)Q dy$$

with a $C > 0$. Again, that is exactly the Theorem 15 in [Guan 2011b].

6.2 In general, for any given Kähler class ω we let

$$m_c(N) = \int_0^c (-K_M - (n - \dim N)E)(\omega - aE)^{n-1} da$$

and

$$m_c = \int_0^c (\omega - aE)^n da$$

with $0 < c \leq e$ and e the largest number such that $\omega - aE > 0$ for $0 < a < e$. We let $\mu_c(N) = \frac{m_c(N)}{m_c}$. If $N = M$, we let $m(M) = (-K_M)\omega^{n-1}$ and $\mu = \frac{m(M)}{\omega^n}$. Then the strictly slope stability says that $\mu_c(N) - \mu < 0$ for all $0 < c \leq e$. Similar obstructions appeared in [Guan 2003]. At that time I was not able to understand the general meaning of this obstruction and related it to the Ding-Tian generalized Futaki invariant forcibly. But it was clear in [Guan 2003] it was not the Ding-Tian generalized Futaki invariant. I also talked on this at Pisa Italy in 2004. Ross-Thomas [Ross-Thomas 2006] partially generalized this obstruction but without the *strictly* part for a smooth N, i.e. they assume that $0 < c < e$. Also, they assume that c is rational, that make their slope stability much weaker. For a nonsmooth subvariety N, I am not sure that their stability implies these inequality or not. For our case, our strictly slope stability is equivalent to the existence. But the Ross-Thomas slope stability is only a necessary condition. Therefore, *a Kähler class with the integral equal to zero when $c = e$ or c irrational would give a counterexample for the equivalence between the Ross-Thomas slope stability or Donaldson K-stability and the existence.* See also [Guan 2003], [Guan 2007].

It is very easy to check that if K_M is the Kähler class and we replace $-K_M - aE$ by $K_M - aE$, let

$$m_c(N) = \int_0^c (-K_M - (n - \dim N)E)(K_M - aE)^{n-1} da,$$

the strictly slope stability means that $m_c(N) + m_c < 0$ and holds automatically. Moreover, if $K_M = 0$, for any Kähler metric ω we replace $-K_M - aE$ by $\omega - aE$ and let

$$m_c(N) = \int_0^c (-n + \dim N)E(\omega - aE)^{n-1} da,$$

the strictly slope stability means that $m_c(N) < 0$ and holds automatically. These strengthen the Theorem 5.4 in [Ross-Thomas 2006], which only concerned when N is smooth and $0 < c < e$ is rational.

In the following of this section, we want to see that the strictly slope stability is the same as the existence for type I manifolds.

To make the things simpler, let us take care of the $F(OP_n)$ fiber case first. In our setting, we only need to deal with the case in which N is the closed orbit. In this case, by [Guan 2011b] section 3 we have $\dim N = n - 1$. Let us calculate the number e for our case. By [Guan 2011b] section 3 we see that the curvature of the exceptional divisor has eigenvalues $D(\mathbf{C}P^n) = 2$ multiple of the coefficient of u.

Therefore, $\omega - aE$ has the first zero eigenvalues when $a = (D(F))^{-1}(-l)$. That is, $e = -2^{-1}l$.

$$\begin{aligned}
&\omega^n m_c(N) - m(M)m_c \\
= &\int_0^{-l} Qdu[\int_0^c (-K_M)((\omega - xE)^{n-1} - \omega^{n-1}) \\
- &E(\omega - xE)^{n-1} - R_0((\omega - xE)^n - \omega^n)dx].
\end{aligned}$$

This is proportional to

$$\int_0^c [\int_0^x [(n-1)K_M E(\omega - uE)^{n-2} + nR_0 E(\omega - uE)^{n-1}]du - E(\omega - xE)^{n-1}]dx.$$

Let $y = -l - 2x$ and $v = -l - 2u,\ d = -l - 2c$ we obtain that the integral is proportional to

$$\begin{aligned}
&\int_d^{-l} [\int_y^{-l} [(n-1)K_M E(\omega + 2^{-1}(v+l)E)^{n-2} + nR_0 E(\omega + 2^{-1}(v+l)E)^{n-1}]dv \\
- &\ 2E(\omega + 2^{-1}(y+l)E)^{n-1}]dy \\
= &\int_d^{-l} h_l dy.
\end{aligned}$$

By taking the derivative twice we have

$$h_l' = -(n-1)K(E)E(\omega + 2^{-1}(y+l)E)^{n-2} - nR_0 E(\omega + 2^{-1}(y+l)E)^{n-1}.$$

By the argument in [Guan 2011a] after (14) and in the proof of Lemma 6, we see that h_l' is proportional to g_l' there. Therefore, we only need to check for a point 0, the function h_l is right. To prove our conclusion, we only need to check that

$$h_l(0) = \int_0^{-l} [(n-1)K_M E(\omega + 2^{-1}(v+l)E)^{n-2} + nR_0 E(\omega + 2^{-1}(v+l)E)^{n-1}]dv = 0$$

since $g_l(0) = 0$. Notice that $nE(\omega + 2^{-1}(v+l)E)^{n-1}$ is related to ω^n there.

The exactly same argument works for the case in which the fiber $F = Gr(2k, 2)$.

For the case in which the fiber $F = Q^n$, we have $D(F) = 1$. Therefore, we could let $y = -l - x,\ v = -l - u,\ d = -l - c$ instead and we notice that $-K(E) = -K_M - (n - \dim N)E$. The same proof goes through.

Theorem 3. *On a type I compact almost homogeneous manifold of cohomogeneity one here is a Kähler metric of constant scalar curvature in a given Kähler class if and only if the Kähler class is strictly slope stable with respect to the closed orbit.*

This is also true for general compact Kähler almost homogeneous manifolds of cohomogeneity one. But it take some times for us to publish the detail results and proofs.

6.3 In the case of Fano manifolds, our discussion at the beginning of this section (6.1) shows that:

Theorem 4. *Let M be any Fano manifold, if a smooth submanifold N destables the Ricci class, then N, the blowing-up manifold $M(N)$ of M along N and the exceptional divisor E are all Fano manifolds.*

One could also consider the case that N is a union of smooth submanifolds. We expect that each of them should be Fano also. Similarly, it should be easy to obtain some results similar to those of Nadel's in [Nadel 1990] and to check out the unstable Fano threefolds.

For the compact Kähler manifolds with a zero or negative first Chern class we have shown at the beginning of 6.2 that:

Theorem 5. *Let M be any compact Kähler manifold with a negative first Chern class, then the negative Ricci class is strictly slope stable.*

Theorem 6. *Let M be any compact Kähler manifold with a zero first Chern class, then any Káhler class is strictly slope stable.*

Theorems 4, 5, 6 give a good reason why the Calabi conjecture is true for the negative and zero case but not true in general for the positive case.

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