137 NOTES, PART 2:
THE AFFINE AND PROJECTIVE PLANES

Z. RAN

1. The affine plane

Our aim now is to begin to extend the constructions we discussed for the case of 1 dimension, i.e. the line and 1-variable polynomials, to the 2-dimensional case. To start with, the affine plane $\mathbb{A}^2_{\mathbb{F}}$ or $\mathbb{A}^2$ is just a copy of $\mathbb{F}^2$ (the difference is that $\mathbb{F}^2$ is viewed as a vector space while $\mathbb{A}^2$ is a 'geometric' object). We note that $\mathbb{A}^2$ admits a set (in fact, a group) of *affine transformations* defined as follows: let C be an invertible 2x2 matrix and $\vec{b} \in \mathbb{F}^2$ a 2-vector. Then define

$$T = T_{C,\vec{b}}$$

by

$$T(x_1, x_2) = C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{1}$$

We call C and $\vec{b}$ the *linear* and *translation* parts of T, respectively. The 3x3 matrix

$$M = M(C, \vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ b_1 & c_{11} & c_{12} \\ b_2 & c_{21} & c_{22} \end{bmatrix} \tag{2}$$

is called the *augmented matrix* of T. We shall see more of this when we discuss projective transformations, but for now we notice that M can be used to simplify the 'action rule' of T as follows: write $T(x_1, x_2) = (y_1, y_2)$. Then

$$\begin{bmatrix} 1 \\ y_1 \\ y_2 \end{bmatrix} = M \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}. \tag{3}$$

The following is easy to see:

Date: March 18, 2006.

Exercise 1.1. *(i) Prove that if T_1, T_2 are affine transformations with respective augmented matrices M_1, M_2, then the augmented matrix of $T_1 \circ T_2$ is $M_1 M_2$.*

(ii) Prove that

$$M(C, \vec{b})M(C', \vec{b'}) = M(CC', \vec{b} + C\vec{b'}).$$

(iii) Deduce that the composition of two affine transformations, and the inverse of an affine transformation, are affine.

Thus, the collection of all affine transformations of $\mathbb{A}^2$ is a group, denoted $\mathrm{Aff}_2(\mathbb{F})$ or Aff_2.

Exercise 1.2. *(i) Prove that for any affine transformation T and any line*

$$L = \{(x_1, x_2) : a_0 + a_1 x_1 + a_2 x_2 = 0\},$$

the inverse image

$$T^{-1}(L) = \{p : T(p) \in L\}$$

is another line and determine its equation.

(ii) Deduce that in this situation $T(L)$ is a line as well

(iii) Prove that if L_1, L_2 are parallel lines, then so are $T(L_1), T(L_2)$ and conversely.

Exercise 1.3. *(i) Prove that given any 3 non-collinear points $P, Q, R \in \mathbb{A}^2$, there exists a unique affine transformation T such that $T(e_1) = P, T(e_2) = Q, T(0,0) = R$. Here as usual $e_1 = (1,0), e_2 = (0,1)$.*

(ii) Conclude that for any 2 pairs of non-collinear points P, Q, R, P', Q', R', there exists a unique affine transformation T such that $T(P) = P', T(Q) = Q', T(R) = R'$.

The 'polynomial' side of the story is much more involved and will occupy us for quite some time, starting shortly. For now let's just discuss a few basic notions. We will deal with polynomials in 2 variables x_1, x_2 with coefficients in our field $\mathbb{F}$. The set of all these polynomials forms an $\mathbb{F}$-vector space, denoted $\mathbb{F}[x_1, x_2]$; in fact, $\mathbb{F}[x_1, x_2]$ is what's called a *ring* or $\mathbb{F}$-*algebra*, because it makes sense to multiply polynomials besides adding them. The set of polynomials of degree $\leq m$ is a vector subspace (but not a subring) of $\mathbb{F}[x_1, x_2]$ that we will denote by $F[x_1, x_2]_{m]}$.

Exercise 1.4. *Prove that the dimension of $\mathbb{F}[x_1, x_2]_{m]}$ as $\mathbb{F}$-vector space is $\binom{m+2}{2}$.*

As before, it is important to consider the action of affine transformations on polynomials. Thus let $T = T_{C,\vec{b}}$ be an affine transformation and $f \in \mathbb{F}[x_1, x_2]$ a polynomial. We define a polynomial $g = T^*(f)$ by

$$g(x_1, x_2) = f(T(x_1, x_2)) = f(c_{11}x_1 + c_{12}x_2 + b_1, c_{21}x_1 + c_{22}x_2 + b_2).$$

Some important properties of this operation are

- If T_1, T_2 are affine, then
$$(T_1 \circ T_2)^*(f) = T_2^*(T_1^*(f)).$$
- $\deg(T^* f) = \deg(f)$.

As in the 1-variable case, two polynomials f, g are said to be *affine equivalent* if there exists an affine transformation T such that $T^*(f) = cg$ for some $c \in \mathbb{F}^x = \mathbb{F} \setminus \{0\}$. As before, this is an equivalence relation on polynomials, whose equivalence classes are also called *affine orbits*. Some important properties of this operation are

Lemma 1.1. *If T_1, T_2 are affine, then*

$$(T_1 \circ T_2)^*(f) = T_2^*(T_1^*(f)), \tag{4}$$

$$\deg(T_1^* f) = \deg(f) \tag{5}$$

Proof. We prove the second equation. The inequality $\leq$ is obvious. Applying the same reasoning to the polynomial $T_1^*(f)$ and the affine transformation T_1^{-1} yields

$$\deg((T_1^{-1})^*(T_1^*(f))) \leq \deg((T_1)^* f).$$

However, by the first equation,

$$(T_1^{-1})^*(T_1^*(f)) = (T_1 \circ T_1^{-1})^*(f) = f$$

therefore

$$\deg(f) \leq \deg((T_1^*)(f))$$

so they are equal. □

As in the 1-variable case, two polynomials f, g are said to be *affine equivalent* if there exists an affine transformation T such that $T^*(f) = cg$ for some $c \in \mathbb{F}^\times = \mathbb{F} \setminus \{0\}$. As before, this is an equivalence relation on polynomials, whose equivalence classes are also called *affine orbits*. The zero-set of a polynomial is defined as before as

$$\text{Zeros}_{\mathbb{F}}(f) = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{F}} : f(x_1, x_2) = 0\} \tag{6}$$

The zero-sets of two affine equivalent polynomials are affine equivalent, in fact

$$T(\text{Zeros}_{\mathbb{F}}(T^*(f))) = \text{Zeros}_{\mathbb{F}}(f). \tag{7}$$

A subset $C \subset \mathbb{A}^2_{\mathbb{F}}$ of the form $C = \text{Zeros}_{\mathbb{F}}(f)$, for some polynomial f is called an *affine plane curve*.

Example 1.2. Some low-degree examples:

$m = 1$: A nonconstant linear polynomial has the form

$$f(x_1, x_2) = u_0 + u_1 x_1 + u_2 x_2$$

where $(u_1, u_2) \neq (0, 0)$; i.e. letting $\vec{u}$ denote the column vector $(u_0, u_1, u_2)^T$, we can write

$$f(x_1, x_2) = \vec{u}^T \vec{x} = \langle \vec{u} \cdot \vec{x} \rangle.$$

It is easy to check that if $T = T_{C,\vec{b}}$ has augmented matrix M, then $T^* f = v_0 + v_1 x_1 + v_2 x_2$ where

$$\vec{v} = M^T \vec{u}$$

(considering $\vec{u}, \vec{v}$ as column vectors). Indeed

$$\begin{aligned} T^* f(x_1, x_2) &= \vec{u}^T (M\vec{x}) = (\vec{u}^T M)\vec{x} \\ &= (\vec{u}^T (M^T)^T)\vec{x} = (M^T \vec{u})^T \vec{x} \\ &= \vec{v}^T \vec{x} \end{aligned}$$

From this it is easy to see that any two nonconstant linear polynomials are affine equivalent. Or more geometrically, any two lines in $\mathbb{A}^2$ are affine equivalent.

Exercise 1.5. *Let $(L_1, L_2), (M_1, M_2)$ be pairs of lines in $\mathbb{A}^2$.*

(1) *Prove that if either*
- $L_1 \parallel L_2$ *and* $M_1 \parallel M_2$ *or*
- $L_1 \nparallel L_2$ *and* $M_1 \nparallel M_2$

then (L_1, L_2) is affine equivalent to (M_1, M_2).

(2) *Prove that if $L_1 \parallel L_2$ but $M_1 \nparallel M_2$ then (L_1, L_2) and (M_1, M_2) are not affine equivalent.*

$m = 2$: Here we are essentially discussing the classification of (affine) conics, mentioned earlier. This depends on the field $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, the orbits are

(1) circles $x_1^2 + x_2^2 - 1$
(2) hyperbolas $x_1 x_2 - 1$
(3) parabolas $x_1^2 - x_2 = 0$
(4) line-pair $x_1 x_2$
(5) double line x_1^2
(6) point $x_1^2 + x_2^2$
(7) empty $x_1^2 + x_2^2 + 1$

If $\mathbb{F} = \mathbb{C}$, cases 1, 2 and 7 are equivalent, as are 4 and 6, and there are just 4 orbits.

The proof is a bit tedious, but can be streamlined somewhat from the viewpoint of projective equivalence (see below).

Exercise 1.6. *Prove that Cases (1), (2), (7) above are equivalent over* $\mathbb{C}$ *but not over* $\mathbb{R}$.

2. The projective plane

The basic definitions are just as in the case of the projective line. Consider the 3-dimensional vector space $\mathbb{F}^3$ with coordinates X_0, X_1, X_2, and define $\mathbb{P}^2_{\mathbb{F}}$, or $\mathbb{P}^2$ when $\mathbb{F}$ is understood, to be the set of 1-dimensional vector subspaces (or lines through the origin) in $\mathbb{F}^3$. Since specifying a 1-dimensional subspace ℓ of $\mathbb{F}^3$ is the same as specifying a nonzero vector in $\mathbb{F}^3$ up to proportionality, we can identify $\mathbb{P}^2$ with the set of proportionality classes $[X_0, X_1, X_2]$ where $(X_0, X_1, X_2) \neq (0, 0, 0) \in \mathbb{F}^3$. Let

$$\pi : \mathbb{F}^2 \setminus \{\vec{0}\} \to \mathbb{P}^2_{\mathbb{F}}$$

be the obvious (tautological) map sending v to $[v]$. For $P = [X_0, X_1, X_2] \in \mathbb{P}^2$, the vector (X_0, X_1, X_2) is said to be a *lift* or *representative* of P. Set

$$L_0 = \{[0, X_1, X_2] : (X_1, X_2) \neq (0, 0)\} \subset \mathbb{P}^2.$$

Then L_0 may be identified in an obvious way with $\mathbb{P}^1_{\mathbb{F}}$ and we call it the *line at infinity* in $\mathbb{P}^2$. Why? Set

$$U_0 = \mathbb{P}^2_{\mathbb{F}} \setminus L_0 = \{[X_0, X_1, X_2] : X_0 \neq 0\} = \{[1, x_1, x_2] : (x_1, x_2) \in \mathbb{A}^2_{\mathbb{F}}\}$$

Thus, U_0, which is known as the *finite plane* can be identified naturally with $\mathbb{A}^2_{\mathbb{F}}$. Geometrically, U_0 is the set of lines through the origin which are not contained in the plane $X_0 = 0$; such a line will meet the plane $X_0 = 1$ in a unique point, and therefore U_0 can be identified with the set of points on the plane $X_0 = 1$.

Now again as in the case of $\mathbb{P}^1$, there is nothing special about infinity: indeed we can define

$$L_i = \{[X_0, X_1, X_2] : X_i \neq 0\}, \quad U_i = \mathbb{P}^2 \setminus L_i$$

and these are exactly analogous to L_0, U_0, and can be identified respectively with $\mathbb{P}^1, \mathbb{A}^2$.

Example 2.1. One nice thing about the case $\mathbb{F} = \mathbb{R}$ is that we can make an explicit 'topological' model of $\mathbb{P}^2_{\mathbb{R}}$, as follows. Note that any representative v of $P \in \mathbb{P}^2_{\mathbb{R}}$ can be scaled by the length $\|v\|$, therefore P admits precisely 2 representatives $v = (x, y, z)$ with $x^2+y^2+z^2 = 1$, i.e. with $v \in S^2$, the unit sphere in $\mathbb{R}^3$. In other words, the map $\pi : S^2 \to \mathbb{P}^2_{\mathbb{R}}$ is 2:1. We can say that $\mathbb{P}^2_{\mathbb{R}}$ is obtained from S^2 by identifying antipodal pairs. Let

$$H_\epsilon = \{(x, y, z) \in S^2 : z \geq -\epsilon\}$$

(i.e a small thickening of the upper hemisphere). Then $\pi(H_\epsilon) = \mathbb{P}^2_{\mathbb{R}}$. Let 'cut up' H_ϵ further in

$$D = \{(x, y, z) \in S^2 : z \geq \epsilon\}, B = \{(x, y, z) \in S^2 : -\epsilon \leq z \leq \epsilon\}$$

Then π maps D 1-1 to its image $R \subset \mathbb{P}^2_{\mathbb{R}}$ and maps B 2-1 to a Möbius strip $M \subset \mathbb{P}^2_{\mathbb{R}}$. Thus

$$\mathbb{P}^2_{\mathbb{R}} = R \cup M$$

is a union of a disc and a Möbius strip, joined along the circle that is the boundary of R and of M.

Thus, to make a paper-and-glue model of $\mathbb{P}^2_{\mathbb{R}}$, take a Möbius strip of radius r and a disc of radius $2r$ (sic!) and glue along boundary (this will require making some further cuts, but that is the fault of our 3-dimensional world that cannot accommodate a model of $\mathbb{P}^2_{\mathbb{R}}$ without self-intersection (this has to do with the fact that $\mathbb{P}^2_{\mathbb{R}}$ is a so-called nonorientable (1-sided) closed surface).

For $\mathbb{F} = \mathbb{C}$, the projective plane has 4 real dimensions and is essentially undrawable topologically (indeed, it is topologically rather sophisticated); we will nonetheless draw it 'geometrically' as a 2-dimensional plane with the coordinate lines indicated.

Again as in the case of $\mathbb{P}^1$, it makes sense to talk about zero sets of homogeneous polynomials in $\mathbb{P}^2$: indeed if $F \in \mathbb{F}[X_0, X_1, X_2]_m$ is a homogeneous polynomial of degree m then

$$F(\lambda X_0, \lambda X_1, \lambda X_2) = \lambda^m F(X_0, X_1, X_2)$$

so given a point $P \in \mathbb{P}^2$ and any two representative v, v' of P, $F(v) = 0$ iff $F(v') = 0$. Therefore it makes sense to say that P is a *zero* of F and write $F(P) = 0$ if $F(v) = 0$ for one or any representative v of P. The set of all zeros of F in $\mathbb{P}^2$ is denoted Zeros(F). A subset $C \subset \mathbb{P}^2$ of the form Zeros(F) for some homogeneous polynomial F is called a *projective plane curve* and it is the sort of object this course is all about. We defer for now the important question of multiplicities. It is important to note that a homogeneous polynomial in 3 or more variables will not, in general, split into linear factors (unlike in the 2-variable case). This explains why, going up from 1 to 2 dimensions, the theory of plane curves is much more complicated than that of polynomials in 1 variable (or that of homogenous polynomials in 2 variables, which is more or less the same).

Note that for a projective curve $C = \text{Zeros}(F)$, the intersection $C_0 = C \cap U_0$, called the *affine* or *finite* part of C, may be identified with a subset of $\mathbb{A}^2$, under the usual identification of U_0 with $\mathbb{A}^2$. As such, C_0

is none other than the zero-set of the dehomogenization

$$f(x_1, x_2) = \mathrm{deh}(F) = F(1, x_1, x_2).$$

Classically, given a projective curve $C = \mathrm{Zeros}(F)$ and the corresponding affine curve $C_0 = C \cap U_0 = \mathrm{Zeros}(f)$, C_0 is called the *finite part* of C while $C \cap L_0 = C \setminus C_0$ is known as the set of *asymptotes* of C_0 or the set of *points at infinity belonging to* C_0 (though they are not actually *on* C_0, but are limiting positions of $\overline{OP}$ where O is the origin and P is a point on C_0 going out to infinity). Note that if we write

$$f = f_m + f_{m-1} + ... + f_1 + f_0$$

, with f_i homogenous of degree i, then $C \cap L_0$ is the zero-set of $f_m(X_1, X_2)$.

Example 2.2.
- $m = 1$: if F is homogeneous linear, the zero-set is naturally called a (projective) line. We can write $F = u_0X_0 + u_1X_1 + u_2X_2$ with the u's not all zero. If, say, $u_0 \neq 0$, then projection to the X_1, X_2 coordinates gives a 1-1, onto mapping $L := \mathrm{Zeros}(F) \to \mathbb{P}^1$, we we can identify L with $\mathbb{P}^1$; similarly in case u_1 or $u_2 \neq 0$. Thus any line is 'intrinsically the same as' (or as we shall say, projectively isomorphic to) $\mathbb{P}^1$.
- $m = 2$ Let's consider the real conics we saw above in Example 1.2. They homogenize respectively to
  (1) $X_1^2 + X_2^2 - X_0^2$
  (2) $X_1X_2 - X_0^2$
  (3) $X_1^2 - X_0X_2$
  (4) X_1X_2
  (5) X_1^2
  (6) $X_1^2 + X_2^2 + X_0^2$

Note that in cases (1) and (6), the equation restricted on the line at infinity $L_0 \simeq \mathbb{P}^1$ gives the irreducible quadratic with real zeros $X_1^2 + X_2^2$; in cases (3) and (5), restriction on L_0 gives a square X_1^2.

Next we discuss projective transformations of the plane. Analogously as in the case of $\mathbb{P}^1$, we can define for any nonsingular 3x3 matrix M with entries in $\mathbb{F}$ a transformation

$$T = T_M : \mathbb{P}^2_{\mathbb{F}} \to \mathbb{P}^2_{\mathbb{F}}$$

by

$$T([X_0, X_1, X_2]) = [Y_0, Y_1, Y_2],$$
$$\begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \end{bmatrix} = M \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

Comparing (1), we see that if we identify the affine plane $\mathbb{A}^2$ with the subset $U_0 \subset \mathbb{P}^2$, any affine transformation $T_{C,\vec{B}}$ is a projective one T_M where M is the augmented matrix $M(C, \vec{b})$. In fact, this T takes the line at infinity L_0 to itself.

In fact, we claim that any projective transformation $T = T_M$ taking L_0 to itself is affine, in other words M is, up to scalar, of the form (2). Indeed, note that $P = T[0,1,0], Q = T[0,0,1]$ are just the second and third columns of M, respectively. Since $P, Q \in L_0$ have 0th component $= 0$, we have that $m_{01} = m_{02} = 0$. Now $m_{00} \neq 0$ because M is nonsingular. Therefore up to scalar, M is of the form (2).

There are, of course, plenty of projective transformations that are not affine.

Exercise 2.1. *Give the matrix of a projective transformation that takes the line at infinity $X_0 = 0$ to the line $X_1 = 0$ and the 'origin' $[1,0,0]$ to the point $[0,0,1]$ on the line at infinity.*

Exercise 2.2. *(i) Prove that if M is a nonsingular 3x3 matrix such that T_M is the identity, then M is a scalar matrix.*

(ii) Conclude that if M_1, M_2 are nonsingular 3x3 matrices such that $T_{M_1} = T_{M_2}$ then $M_2 = cM_1$ for some nonzero scalar c.

We denote by $\mathrm{GL}_3(\mathbb{F})$ the set of all nonsingular 3x3 matrices with entries in $\mathbb{F}$ and by $\mathrm{PGL}_3(\mathbb{F})$ the set of all projective transformations of $\mathbb{P}^2_{\mathbb{F}}$. By the last exercise, there is a surjective (onto) mapping $\mathrm{GL}_3(\mathbb{F}) \to \mathrm{PGL}_3(\mathbb{F})$ such that 2 matrices $M_1, M_2 \in \mathrm{GL}_2(\mathbb{F})$ have the same image in $\mathrm{PGL}_3(\mathbb{F})$ iff they are proportional.

What is the analogue of Proposition 1-4.5 for the plane ? Let's indulge in a little 'count of parameters', a favorite technique of classical algebraic geometers. A projective transformation T is given by a 3x3 matrix, but scaling doesn't matter, so effectively T depends on $3 \times 3 - 1 = 8$ parameters. Each point in $\mathbb{P}^2$ requires 2 parameters. So roughly speaking, we can expect to find T taking $4 = 8/2$ given points to 4 given points. This idea obviously needs some refinement: T necessarily takes a line to a line, so if we start with a collinear quadruple of points, T can only take them to another collinear quadruple; similarly with a quadruple that contains a collinear triple. It turns out, interestingly, that this is the only 'obstruction'. First let's make a definition:

Definition 1. *A collection of distinct points $P_1, ..., P_n \in \mathbb{P}^2$ is said to be* in general position *if no 3 of them are collinear.*

Now let's set up some notation: $e_0 = [1,0,0], e_1 = [0,1,0], e_2 = [0,0,1], f = [1,1,1]$.

Proposition 2.3. *Given 4 distinct points in general position $P_0, ..., P_3 \in \mathbb{P}^2$, there exists a unique projective transformation T such that*

$$T(e_i) = P_i, i = 0, 1, 2, T(f) = P_3.$$

Proof. Let's write $T = T_M$ with M a 3x3 matrix to be determined, and represent $P_i, i = 0, 1, 2$ by a column vector denoted p_i. Then the condition $T(e_i) = P_i$ simply means that the ith column of M is proportional to p_i. Thus, letting P be the matrix with columns p_0, p_1, p_2 these 3 conditions mean that

$$M = PD$$

for some diagonal matrix D, corresponding to a vector

$$d = [d_0, d_1, d_2], d_0 d_1 d_2 \neq 0.$$

Note that P is nonsingular because its columns are linearly independent: a linear dependence relation between p_0, p_1, p_2 is precisely the same thing as a line containing the points P_0, P_1, P_2! Is remains to account for the last condition $T(f) = P_3$. Note that $Df = d$, so to achieve $Mf = PDf = p_3$ it suffices to set $d = P^{-1}p_3$– provided, that is, that this vector has all coordinates $\neq 0$. Suppose, say, that $d_0 = 0$. Then d, e_1, e_2 are linearly dependent. Applying P, it follows that p_3, p_1, p_2 are linearly dependent, i.e. P_3, P_1, P_2 are collinear, which contradicts our general position hypothesis. Thus, $d_0 d_1 d_2 \neq 0$ and our projective transformation $T = T_M$ exists.

To show uniqueness, suppose T_{M_1}, T_{M_2} both take e_0, e_1, e_2, f to $P_0, ..., P_3$. Let $M = M_1^{-1} M_2$. Then T_M fixes e_0, e_1, e_2, f. Now analyzing the condition that $T(e_i) = e_i, i = 0, 1, 2$ as above, we see that it is equivalent to M being a diagonal matrix, say with entries d_0, d_1, d_2. But then $T(f) = [d_0, d_1, d_2]$. Since this equals f, we conclude that M is a scalar c, therefore $M_2 = cM_1$ and $T_{M_1} = T_{M_2}$.

□

Corollary 2.4. *If $P_0, ..., P_3$ are in general position and so are $Q_0, ..., Q_3$, there is a unique projective transformation T such that $T(P_i) = Q_i, i = 0, ..., 3$.*

Exercise 2.3. *Prove that, given points $P_0, ..., P_3 \in \mathbb{P}^2$, the existence of T as in Prop 2.3 implies they are in general position.*

Next note that as before, given a nonsingular matrix M and a homogeneous polynomial F of degree m, we can define another such polynomial $T_M^*(F)$ by

$$T_M^*(F)(\vec{X}) = F(M\vec{X}).$$

Changing M by a multiplicative scalar c will only change $T^*_M(F)$ by the multiplicative factor c^m so does not change the zero-set. Two homogeneous polynomials F, G are said to be *projectively equivalent* if $G = T^*_M(F)$ for some nonsingular matrix M.

Example 2.5. $m = 1$: Note that any homogeneous polynomial of degree 1 has the form

$$\ell_u(X) = u_0X_0 + u_1X_1 + u_2X_2 = u^TX$$

where u is the nonzero column vector $[u_0, u_1, u_2]$. Thus, there is a 1-1 correspondence between linear polynomials ℓ_u and vectors u.

Lemma 2.6. *Two linear polynomials ℓ_u, ℓ_v have the same line L as zero-set iff u, v are proportional.*

Proof. 'If' is obvious. For 'only if', we may assume by permuting coordinates that L does not contain $[e_2] = [0, 0, 1]$. This means $u_2 \neq 0 \neq v_2$ so we may as well assume $u_2 = v_2 = 1$. Then a parametrization of $L = \text{Zeros}(\ell_u)$ is given by

$$\mathbb{P}^1 \to \mathbb{P}^2$$

$$[t_0, t_1] \mapsto [[t_0, t_1, -u_0t_0 - u_1t_1].$$

Since this this is also a parametrization of $\text{Zeros}(\ell_v)$ it follows that

$$v_0t_0 + v_1t_1 - u_0t_0 - u_1t_1 = (v_0 - u_0)t_0 + (v_1 - u_1)t_1 \equiv 0,$$

so $v_0 = u_0, v_1 = u_1$ and we are done. □

Thus, there is a 1-1 correspondence between the set of lines in $\mathbb{P}^2_{\mathbb{F}}$ and the set of points in another copy of $\mathbb{P}^2_{\mathbb{F}}$, known as the *dual projective plane*, denoted $\mathbb{P}^{2*}_{\mathbb{F}}$. Moreover, the action of projective transformations on $\mathbb{P}^2$ yields a *dual action* on $\mathbb{P}^{2*}$: if $F(X) = u^TX$, then

$$(T^*_MF)(X) = F(MX) = u^T(MK) = u^T(M^T)^TX = (M^Tu)^TX.$$

Therefore $T^*_M(F)$ is the line with coefficient vector M^Tu.

Exercise 2.4. *Show that any two nonzero homogenous polynomials of degree 1 are projectively equivalent.*

Now because $\mathbb{P}^{2*}$ is essentially 'just another copy of $\mathbb{P}^2$, what we prove about $\mathbb{P}^2$ is generally valid for $\mathbb{P}^{2*}$ as well, and sometimes this has an interesting interpretation in terms of the original $\mathbb{P}^2$. In fact, one can set up a sort of 'dictionary' between $\mathbb{P}^2$ and $\mathbb{P}^{2*}$: some entries in the dictionary are as follows

$\mathbb{P}^2$	$\mathbb{P}^{2*}$
point	line
line	point
k points on a line	k lines through a point

Continuing with this dictionary, we may say that a collection of lines $L_1, ..., L_n \subset \mathbb{P}^2$ is *in general position* if no 3 of them are concurrent; this is the same as saying that, as points in $\mathbb{P}^{2*}$, $L_1, ..., L_n$ are in general position.

Exercise 2.5. *Prove that any 2 quadruples on lines in general position are projectively equivalent. (Hint: emulate the proof of Prop 4 above, or use its statement)*

A convenient way to express the relation between $\mathbb{P}^2$ and $\mathbb{P}^{2*}$ is by means of the *incidence correspondence*: let

$$I = \{(P, L) : P \in L\}$$
$$= \{([X_0, X_1, X_2], [u_0, u_1, u_2]) \quad : u_0X_0 + u_1X_1 + u_2X_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$$

This is the set of pairs (point, line) such that the point lies on the line. From its equation description, I is completely symmetric in u and X. We have a diagram

$$\begin{array}{ccc} & I & \\ \swarrow p_1 & & p_2 \searrow \\ \mathbb{P}^2 & & \mathbb{P}^{2*} \end{array}$$

The following result is already some indication of the 'complete' nature of the projective plane:

Proposition 2.7. *Any two lines in $\mathbb{P}^2$ intersect.*

Proof. Consider two lines L, M with equations $u^T.X = 0, v^T.X = 0$. Their intersection comes from the solutions to a system of 2 homogeneous linear equations in the 3 unknowns X_0, X_1, X_2. By linear algebra, such a system always admits a nonzero solution, and this yields a point in $L \cap M$. □

In fact, much more is true, for example

Proposition 2.8. *If $\mathbb{F} = \mathbb{C}$, any line in $\mathbb{P}^2$ meets any curve.*

Proof. Let L be a line and $C = \text{Zeros}(F)$ a curve. There exists a projective transformation T such that $T(L)$ is the line with equation X_2. Then $T(C) = \text{Zeros}(G)$ for some homogeneous polynomial G. So it suffices to prove that G has a zero of the form $[X_0, X_1, 0], (X_0, X_1) \neq (0, 0)$.

But $G(X_0, X_1, 0)$ is a homogenous polynomial in X_0, X_1 (possibly zero), therefore by the fundamental theorem of algebra it is a product of linear factors, hence always admits a zero. □

3. CONICS

Here we give a general discussion of homogeneous quadratic polynomials. It will be convenient to start with a more general notion. Let V be a finite-dimensional $\mathbb{F}$-vector space, and $f : V \to \mathbb{F}$ a function. We will say that f is a *homogeneous function* or *a form* of degree m if, given any collection of elements $v_1, ..., v_k \in V$, the functions $g : \mathbb{F}^k \to \mathbb{F}$ given by

$$g(x_1, ..., x_k) = f(x_1v_1 + ... + x_kv_k)$$

is a homogeneous polynomial of degree m in $x_1, ..., x_k$.

Proposition 3.1. *There is a 1-1 correspondence between quadratic forms q on V and symmetric bilinear forms*

$$b : V \times V \to \mathbb{F}$$

given by

$$q = \phi(b), q(v) = b(v, v), \tag{8}$$

$$b = \psi(q), b(u, v) = \frac{1}{2}(q(u + v) - q(u) - q(v)). \tag{9}$$

Proof. To start with, it is clear that, given a symmetric bilinear from b, the function $q = \phi(b)$ is indeed a quadratic form. Next, we claim that given q quadratic, $b = \psi(q)$ is bilinear, symmetric. Indeed symmetry is obvious from the definition, so to prove bilinearity is a matter of proving

$$b(u + u', v) = b(u, v) + b(u', v), \tag{10}$$

$$b(ru, v) = rb(u, v), r \in \mathbb{F}. \tag{11}$$

Let

$$f(x_0, x_1, x_2) = q(x_0u + x_1u' + x_2v)$$

By assumption, this is a homogeneous quadratic form in x, hance has the form

$$f(x_0, x_1, x_2) = a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + 2c_{01}x_0x_1 + 2c_{12}x_1x_2 + 2c_{02}x_0x_2$$

Plugging into the formula defining b, we see that

$$c_{01} = b(u, u'), c_{02} = b(u, v), c_{12} = b(u', v).$$

On the other hand, again by plugging into the defining formula, we have

$$\begin{aligned} b(u+u',v) &= \frac{1}{2}(q(u+u'+v) - q(u+u') - q(v)) \\ &= \frac{1}{2}(f(1,1,1) - f(1,1,0) - f(0,0,1)) = \\ \frac{1}{2}&((a_0 + ... + 2c_{02}) - (a_0 + a_1 + 2c_{01}) - a_2) \\ &= c_{02} + c_{12} = b(u,v) + b(u',v) \end{aligned}$$

The proof of (5) is similar. Now we claim that for any symmetric bilinear form b and any quadratic form q, we have that

$$\psi(\phi(b)) = b \tag{12}$$

$$\phi(\psi(q)) = q \tag{13}$$

We will prove (9) as the proof of (8) is similar and simpler. Given q, $\psi(q) = b$ is defined by ? and then $q' = \phi(b)$ is defined by

$$q'(u) = b(u,u) = \frac{1}{2}(q(2u) - 2q(u)) = q(u)$$

by homogeneity of q. □

Now if q is a quadratic form on V and $B = (v_1, ..., v_n)$ is a basis of V, there is defined a quadratic form q_B on $\mathbb{F}^n$ by

$$q_B(x_1, ..., x_n) = q(x_1 v_1 + ... + x_n v_n) \tag{14}$$

Note that if $V = \mathbb{F}^n$, then B corresponds to a nonsingular matrix A with columns $v_1, ..., v_n$ and then, with the notation introduced earlier,

$$q_B = T_A^*(q)$$

is a quadratic form projectively equivalent to q.

Proposition 3.2. *Let q be a quadratic form on an n-dimensional $\mathbb{F}$-vector space V. Then there exists a basis B of V such that*

$$q_B = a_1 x_1^2 + ... + a_n x_n^2, \quad a_1, ..., a_n \in \mathbb{F}.$$

Proof. Use induction on $n = \dim(V)$. If $n = 1$ the result certainly holds. For the induction step, assume the result is true for $n-1$ and that $q \neq 0$. Pick any $v_1 \in V$ with $a_1 := q(v_1) \neq 0$. Let b be the bilinear form associated to q, and set

$$V' = v_1^{\perp} = \{u \in V : b(u, v_1) = 0\}$$

This is clearly an $(n-1)$-dimensional subspace of V. Applying the induction hypothesis to the restriction of q on V', there is a basis $B' = (v_2, ..., v_n)$ of V' such that

$$q_{B'} = a_2x_2^2 + ... + a_nx_n^2$$

in other words,

$$q(x_2v_2 + ... + x_nv_n) = a_2x_2^2 + ... + a_nx_n^2$$

But now note that for any $v \in V'$, we have

$$q(x_1v_1 + v) = q(x_1v_1) + q(v)$$

because $b(v_1, v) = 0$. Therefore

$$q(x_1v_1 + x_2v_2 + ... + x_nv_n) = a_1x_1^2 + a_2x_2^2 + ... + a_nx_n^2,$$

proving the proposition.

□

Note that by permuting our basis we may always assume that

$$a_1, ..., a_r \neq 0, a_{r+1} = ... = a_n = 0$$

for some integer r which is called the *rank* of q. If $r = n$ q and the associated bilinear form b are said to be *nondegenerate*. A basis as in Proposition 3.2 is called a *diagonalizing* basis for q. It can be shown that the rank r coincides with the rank of the linear map

$$r : V \to V^*$$

where $V^* = L(V, \mathbb{F})$ is the dual vector space, defined by

$$r(v)(u) = b(u, v)$$

and in particular, r is independent of the choice of diagonalizing basis (of which there are, in general, many).

Corollary 3.3. *Assumptions as in the previous Proposition. If* $\mathbb{F} = \mathbb{R}$, *there is a basis* B *of* V *such that*

$$q_B = \pm x_1^2 \pm ... \pm x_r^2.$$

Proof. Start with a diagonalizing basis $(v_1, ..., v_n)$ and for each i such that $a_i = q(v_i) \neq 0$ let

$$v_i' = \sqrt{|a_i|}^{-1} v_i.$$

For each i such that $q(v_i) = 0$ let $v_i' = v_i$. Then

$$q(x_1v_1' + ... + x_nv_n') = q(\frac{x_1}{\sqrt{|a_1|}}v_1 + ... + \frac{x_r}{\sqrt{|a_r|}}v_r + ... + x_nv_n)$$
$$= a_1(\frac{x_1}{\sqrt{|a_1|}})^2 + ... + a_r(\frac{x_r}{\sqrt{|a_r|}})^2 = \pm x_1^2 \pm ... \pm x_r^2$$

□

Corollary 3.4. *Any real conic in* $\mathbb{P}^2_{\mathbb{R}}$ *is projectively equivalent either to* $X_0^2 + X_1^2 + X_2^2$ *(empty) or* $X_0^2 + X_1^2 - X_2^2$ *(hyperbola) or* $X_1^2 - X_2^2$ *(line-pair) or* $X_1^2 + X_2^2$ *(single point) or* X_1^2 *(double line). Only the first two are nondegenerate.*

Proof. The rank r can be 3, 2, or so the result is immediate from the above classification.

□

Note that in particular the circle (homogeneous equation $-X_0^2+X_1^2+X_2^2$), the hyperbola as above, and the parabola $X_0X_1 + X_2^2$ are projectively equivalent, despite the fact that their affine (inhomogeneous) parts are not affine equivalent! The reason behind this is that the 3 curves intersect the line at infinity $X_0 = 0$ differently: for the circle the intersection is empty with equation $X_1^2 + X_2^2$, for the hyperbola it is 2 distinct points corresponding to the reducible quadratic $X_1^2 - X_2^2$, while for the parabola it is the double-root quadratic X_2^2.

The situation is even simpler for the complex case:

Corollary 3.5. *If* $\mathbb{F} = \mathbb{C}$*, there is a basis* B *of* V *such that*

$$q_B = x_1^2 + ... + x_r^2.$$

Proof. Similar to the above, with the square roots, using the fact that every complex number admits a square root. □

Corollary 3.6. *Any conic in* $\mathbb{P}^2_{\mathbb{C}}$ *is projectively equivalent to* $X_0^2+X_1^2+X_2^2$ *(nondegenerate) or* $X_1^2 + X_2^2$ *(line-pair) or* X_2^2 *(double line)*

Note that in both the real and complex cases the affine classification is more complicated than the projective one. This may be explained by the remark that to specify an affine conic is essentially to specify a projective conic C plus the position of C relative to a fixed line (which one may think of as the line at infinity, though, as we've seen, in $\mathbb{P}^2$ all lines are 'the same', i.e. projectively equivalent. This lends support to the viewpoint that the correct ambient space in which to consider curves in the projective, rather than the affine plane.

Given a projective curve C, a *homogeneous parametrization* of C is by definition a map

$$G = (g_0, g_1, g_2) : \mathbb{P}^1 \to \mathbb{P}^2$$

where g_0, g_1, g_2 are homogenous polynomials of the same degree with no common zeros, such that the image of G coincides with C. One nice consequence of the classification of conics is the following

Corollary 3.7. *Any nondegenerate conic over $\mathbb{C}$ admits a homogeneous parametrization.*

Proof. Consider the conic D with equation $X_0X_1 - X_2^2$. The explicit substitution

$$X_0 \mapsto X_0 + iX_1, X_1 \mapsto X_0 - iX_1$$

shows that this equation is diagonalizable and nondegenerate. Therefore any conic is projectively equivalent to D, and it suffices to show D admits a homogenous parametrization. Indeed consider the map

$$G : \mathbb{P}^1 \to \mathbb{P}^2$$

given by

$$G(U_0, U_1) = [U_0^2, U_1^2, U_0U_1].$$

Clearly, the image of G is contained in D. Conversely, given a point $P = [a_0, a_1, a_2] \in D$, either $a_0 \neq 0$ or $a_1 \neq 0$. If $a_0 \neq 0$ then we may assume $a_0 = 1$ and then $a_1 = a_2^2$ and $P = G[1, a_2]$. Similarly if $a_1 \neq 0$. Therefore G maps onto D. In fact, a similar argument shows G is one to one. □

Corollary 3.8. *Every projective curve over $\mathbb{C}$ intersects every conic.*

Proof. Consider a curve $V = \mathrm{Zeros}(F) \subset \mathbb{P}^2$ where F is homogeneous of degree m, and a conic C. If C is degenerate (a double line or line pair our result follows from Prop. 2.7. If C is nondegenerate, it admits a parametrization by G, Then

$$F' = F \circ G = F(g_0(U_0, U_1), ...)$$

is homogeneous (in fact, of degree $2m$) in U_0, U_1, hence admits a nontrivial zero (b_0, b_1). Then

$$P = G(b_0, b_1) \in C \cap V.$$

□

The foregoing argument suggests that if V is given by a polynomial of degree m then it will generally meet a conic in $2m$ points. In fact, Bézout's Theorem to be discussed later shows that two projective curves over $\mathbb{C}$ with equations of degree m, n will in general intersect in mn points, and in particular their intersection is always nonempty. On the other hand, the analogue of Cor. 3.7 is decidedly false in general for curves of degree 3 or more, as follows from the next result.

Proposition 3.9. *If $\lambda \neq 0, 1$, then there are no nonconstant rational functions $f = f(t), g = g(t)$ such that*

$$f^2 = g(g-1)(g-\lambda)$$

Note this implies the curve with equation

$$X_0X_1^2 = X_2(X_2 - X_0)(X_2 - \lambda X_0)$$

has no homogeneous parametrization.

Proof. Write

$$f = r/s, g = p/q$$

with r, s relatively prime and p, q relatively prime polynomials. Then

$$r^2q^3 = s^2p(p - q)(p - \lambda q)$$

As r, s are relatively prime, it follows that $s^2|q^3$. Similarly, because p, q are relatively prime, we have $q^3|s^2$. Therefore

$$s^2 = aq^3, a \in \mathbb{C}.$$

Hence

$$aq = (s/q)^2$$

so q is a square in $\mathbb{C}[t]$. Plugging this into the equation above yields

$$r^2 = ap(p - q)(p - \lambda q)$$

and the relative primeness of p, q implies that $p, q, p - q, p - \lambda q$ are all squares. Generally, a pair of polynomials u, v is said to have the *4-square property* if there exist 4 distinct, non-proportional, nontrivial linear combinations $au + bv$ that are squares (in $\mathbb{C}[t]$). Now use:

Lemma 3.10. *Let $p, q \in \mathbb{C}[t]$ be relatively prime with the 4-square property. Then p, q are constant.*

Proof. By contradiction. If false, let p, q be a counterexample (i.e. a nonconstant relatively prime pair with the 4-square property), such that $M := \max(\deg(p), \deg(q))$ is smallest among all counterexamples. Now linear combinations of p, q correspond 1-1 to points $[a, b] \in \mathbb{P}^1$. We may assume 3 of our combinations correspond to the points $0, -1, \infty$. Therefore we may assume $p, q, p - q, p - \lambda q, \lambda \neq 1$, are squares. Write

$$p = u^2, q = v^2.$$

Thus

$$\max(\deg(u), \deg(v)) < M.$$

Now

$$\square \ni p - q = (u - v)(u + v)$$

and as u, v are relatively prime is follows that $u - v, u + v$ are squares. Then

$$\square \ni u^2 - \lambda v^2 = (u - \mu v)(u + \mu v), \mu = \sqrt{\lambda} \neq \pm 1$$

and again it follows that $u - \mu v, u + \mu v$ are squares. Thus we have the 4 distinct squares $u - v, u + v, u - \mu v, u + \mu v$, so u, v have the 4-square property. This contradicts minimality of M.

$\square$

$\square$