CANONICAL INFINITESIMAL DEFORMATIONS

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0. Introduction

Our purpose here is to develop a 'canonical' approach to infinitesimal and formal deformation theory. For simplicity we shall focus in the paper mainly to one fundamental-and somewhat typical-case, that of a compact complex manifold X without global vector fields. Our starting point, and model , is the classical (first-order) Kodaira-Spencer formalism: this associates to any deformation $\mathfrak{X}/S$ with special fiber X a 'Kodaira-Spencer' map

$$\kappa : T_0 S \to H^1(X, T)$$

where $T = T_X$ is the tangent sheaf, and consequently obtains a canonical identification between the set of first-order deformations of X and the cohomology group $H^1(X, T)$. It is then natural to seek a higher-order analogue of this, for n-th order tangent spaces and n-th order deformations. At a minimum, one would like an n-th order analogue of κ:

$$\kappa_n = \kappa_n(\mathfrak{X}/S) : T^{(n)} S \to (?)_n$$

where $(?)_n$ is an explicit (and preferably computable) cohomological functor of X which, at least in favorable cases (e.g. when a global moduli space exists), should be canonically identifiable with the n-th order tangent space at a smooth point of the moduli space. Put another way, one knows, when $H^0(T) = 0$, the *existence* of an universal formal deformation

$$\hat{X}/\hat{S} = \lim_{n} X_n/S_n.$$

The problem is to write each X_n/S_n, i.e, the universal n-th order deformation, as an explicit cohomological functor of X , extending the above Kodaira-Spencer identification of first-order deformations.

Our approach to this is to combine some earlier constructions from $[R_1]$ with an important and very novel insight coming out of some work of Beilinson, Drinfeld and Ginzburg, cf. [BG]. The latter is, among other things, concerned specifically with deformations of vector bundles and principal bundles on a fixed complex curve X. It gives a formula for the n-th order cotangent space to the moduli of such bundles as H^0 of a suitable sheaf on the Knudsen-Mumford space $\hat{X}^n$ parametrizing n-tuples of points on X. This suggests the simple but stunning–to the author–and very broad philosophy that n-th order deformations should be related to n-tuples: e.g. that $T^{(n)}M$ or something similar ought to be writable in terms of cohomology, at least on some sort of parameter space for n-tuples on X (notwithstanding that a good analogue of $\hat{X}^n$ is not known if $\dim X > 1$).

Here we will realize this philosophy as follows. First, we construct certain spaces $X < n >$, related to symmetric products, which we call the *very symmetric products* of X. To be precise, $X < n >$ parametrizes the nonempty subsets of X of cardinality $\leq n$. These naturally form a tower:

$$X = X < 1 > \subset X_2 = X < 2 > \subset X < 3 > \cdots \subset X < n > \cdots \subset X < \infty > = \lim_{\rightarrow} X < n > .$$

Then on $X < \infty >$ we construct a certain complex $J^{\cdot} = J^{\cdot}(T_X)$ which we call the *Jacobi complex* of X: this is essentially just a multivariate version of the standard complex used to compute the *Lie algebra homology* of T_X. cf. [F] (indeed the latter homology coincides with the cohomology of $J^{\cdot}$ along $X < 1 >$). The subcomplex $F^n J^{\cdot} =: J^{\cdot}_n$ is natually supported on $X < n > \subset X < \infty >$. This material is developed in Sect. 2. With it, we will prove the following

Theorem 0.1. *Let X be a compact complex manifold with $H^0(T_X) = 0$ and let J be the Jacobi complex of X. Then*

(i) for each n there is a canonical ring structure on

$$R^u_n = \mathbb{C} \oplus \mathbb{H}^0(J_n)^*$$

and a canonical flat deformation X^u_n/R^u_n, and these fit together to form a direct system with limit

$$\hat{X}^u/\hat{R}^u = \lim_n X^u_n/R^u_n ;$$

(ii) for any artin local $\mathbb{C}-algebra$ R_n of exponent n and flat deformation X_n/R_n of X, there is a canonical Kodaira-Spencer ring homomorphism

$$\alpha_n = \alpha_n(X_n/R_n) : R^u_n \to R_n$$

and an isomorphism

$$X_n/R_n \xrightarrow{\sim} \alpha^*_n X^u_n = X^u_n \times_{R^u_n} R_n ;$$

(iii) if $\hat{R} = \lim\limits_{\leftarrow} R_n$ is a complete local noetherian $\mathbb{C}-algebra$ and $\hat{X} = \lim X_n/R_n$, then $\hat{\alpha} = \lim_n \alpha_n : \hat{R}^n \to \hat{R}$ exists and $\hat{X}/\hat{R} = \hat{\alpha}^*(\hat{X}^u/\hat{R}^u)$

Note that, effectively, constructing the n-universal deformation X^u_n/R^u_n amounts to associating to any local homomorphism $\beta : R^u_n \to R_n$ to another artin local algebra of exponent n a deformation $X_n/R_n = (X_n/R_n)(\beta)$: indeed given X^u_n/R^u_n

exists, of course $(X_n/R_n)(\beta) = \beta^*(X_n^u/R_n^u)$; conversely, if the $(X_n/R_n)(\beta)$ exist compatibly for all β, just set $X_n^u/R_n^u := (X_n/R_n)(id)$ where id:$R_n^u \to R_n^u$ is the identity. Of course, the latter remark is just a tautology, but a useful one, as we shall find it most convenient to construct the $(X_n/R_n)(\beta)$ directly.

The proof of the Theorem is given mainly in Sect. 4, after a different and more concrete argument in the second- order case is given in Sect. 3. The proof is based on the idea that a deformation of X, i.e. of its structure sheaf $\mathcal{O}_X$, can be obtained by taking a suitable (e.g. Dolbeault or Čech) resolution of $\mathcal{O}_X$ and appropriately deforming the differentials in the resolution. We will show that such deformations of the differentials can be measured by the Jacobi complex. The familiar *integrability condition* of Kodaira-Spencer on the one hand is the condition that the deformed differential have square 0, i.e. yields a complex, and on the other hand is a cocycle condition in the Jacobi (bi)complex.

The result naturally generalizes (cf. Sect. 5). If g is a sheaf of $\mathbb{C}$- Lie algebras on X and E a g-module (both assumed reasonably 'tame'), let $\bar{g}$ be the unique quotient of g acting faithfully on E and assume that $H^0(\bar{g}) = 0$. For any artin local $\mathbb{C}$-algebra (R, m) of exponent n we may define a sheaf of groups G_R on X by

$$G_R = \exp(g \otimes m)$$

(i.e. G_R is $g \otimes m$ with multiplication determined by the Campbell- Hausdorff formula) and similarly $\bar{G}_R = \exp(\bar{g} \otimes m)$. Then g-deformations of E over R are locally trivial deformations with transitions in $\bar{G}_R$, and are naturally classified by the nonabelian Čech cohomology set $\check{H}^1(X, \bar{G}_R)$. Our construction yields a bijection $v = v_{R,E}$ between these and a certain subset of $\mathbb{H}^0(J_n(\bar{g})) \otimes m$ (i.e. the set of 'morphic' elements). For $n \geq 3$ this correspondence is given somewhat indirectly and in particular does not come from an explicit correspondence on the cocycle level. v apparently depends on both E and g, though its source and target depend only on $\bar{g}$. It is unknown to the author whether (say for E a faithful g-module) v is independent of E, a fortiori whether it can be defined in terms of g alone. For another, perhaps more 'conceptual' interpretation of our construction of the universal deformation of E, see [R3], Theorem 3.1.

A further generalization, to the case of a sheaf of differential graded Lie algebras, will be considered in [R3] (though the formal construction generalises directly and indeed is already stated in this generality here, its interpretation in the dgla case requires some care).

As indicated above, the existence of the universal formal deformation $\hat{X}^u/\hat{R}^u$ was known before, thanks to the work of Grothendieck, Schlessinger, Kuranishi, et al.: our point is its explicit construction and description. As for applications and extensions of the method, these have been, and will be given elsewhere, but a few can be mentioned here.

(i) An analogous deformation theory for deformations of vector bundles (or more generally locally free sheaves over a fixed locally $\mathbb{C}$-ringed space) and, as one application, construction of a symplectic (closed) 2-form on the moduli space, generalising at the same time constructions of Hitchin (for local systems on Riemann surfaces) and Mukai (for holomorphic vector bundles over K3 surfaces)[R5].

(ii) A direct construction of the universal variation of Hodge structure associated to a compact Kähler manifold and resulting study of the (local) period map and characterisation of its image (local Schottky relations), especially for Calabi-Yau manifolds and curves [R3],[L].

(iii) A theory of semiregularity for submanifolds and embedded, as well as relative deformations and resulting dimension bounds for Hilbert schemes and relative deformation spaces [R2][R4].

Higher-order Kodaira-Spencer maps, especially associated to 'geometric' (reduced) families, and a version of the Jacobi complex, were discovered independently and about the same time as this author (Winter 1992) by Esnault and Viehweg , cf. [EV]; their paper defines higher- order (additive) Kodaira-Spencer classes, but does not construct the universal family. Some important antecedents (albeit employing a different viewpoint) are in the work of Goldman-Millson [GM].

1. Coalgebra

The purpose of this section is to characterize the vector space m^* dual to the maximal ideal of an artin local $\mathbb{C}$-algebra (R, m). While the general concept of coalgebra is well known, our application in the artin local case assigns a special role to the m-adic filtration and its dual, the 'order' filtration,not present in the general case. Moreover, it is convenient to state the definition in a more general categorical setting. Consequently, it will be convenient to give a brief self-contained treatment here.

By an *admissible category* we shall mean a small abelian category $\mathcal{V}$ admitting an initial element 0, an internal product, denoted $\otimes$, which is symmetric and such that each $V \otimes V$ decomposes canonically into eigenobjects under the natural $\mathbb{Z}_2$-action, denoted $\mathrm{sym}^2(V), \bigwedge^2(V)$. Examples of admissible categories include modules over a ring S containing 1/2 and complexes of S-modules over a topological space.

By a ($\mathcal{V}$-compatible) *Order-Symbolic (OS) structure* of order $n \leq \infty$ on an object V of an admissible category $\mathcal{V}$ we mean an increasing filtration (i.e. chain of injections)

$$V^0 = 0 \subseteq V^1 \subseteq \cdots\cdots \subseteq V^n = V$$

and mutually compatible 'symbol' or 'comultiplication' maps

$$\sigma^{i,j} : V^i/V^j \to \mathrm{sym}^2(V^{i-j}), \qquad j < i.$$

(Sometimes we shall use the same notation to denote the induced map $V^i \to \mathrm{sym}^2(V^i)$; actually, a moment's thought shows that $\sigma := \sigma^{n,1}$ is sufficient to determine the rest) . These are assumed to satisfy the natural (co)associativity condition which states that the following diagram should commute

$$\begin{array}{ccccc} & & \mathrm{sym}^2(V^{n-1}) \otimes V^{n-1} & & \\ & \overset{\varphi}{\nearrow} & & \searrow & \\ V/V^1 \to \mathrm{sym}^2 V^{n-1} \subset V^{n-1} \otimes V^{n-1} & & & & V^{n-1} \otimes V^{n-1} \otimes V^{n-1} \\ & \underset{\psi}{\searrow} & & \nearrow & \\ & & V^{n-1} \otimes \mathrm{sym}^2(V^{n-1}) & & \end{array}$$

$$\varphi = \sigma^{n-1,1} \otimes id, \qquad \psi = id \otimes \sigma^{n-1,1}$$

An OS structure V is said to be *standard* if σ is injective (hence $\sigma^{n,j}$ is injective for all $j < n$).

We can now state the basic result about OS structures on finite-dimensional vector spaces, which relates them with artin local algebras.

Proposition 1.1. *There is an equivalence of categories between $\mathcal{O}S_n$, the category of OS structures of order n on finite dimensional $\mathbb{C}$-vector spaces, and $\mathcal{F}R_n$, the category of commutative artin local $\mathbb{C}$-algebras (R, m) of exponent n together with a super-m-adic filtration $(m_i \supseteq (m)^i)$, where standard structures correspond with m-adically filtered algebras. The correspondence is given by*

$$(V, V^{\cdot}, \sigma) \mapsto (\mathbb{C} \oplus V^*, V^{i\perp} = (V/V^i)^*, \sigma_n^*)$$
$$(S, m, m.) \mapsto (m^*, m_{i-1}^{\perp} = (m/m_{i-1})^*, \text{comultiplication}).$$

Proof. Basically trivial. Given V etc. define

$$R = \mathbb{C} \oplus V^*, m = V^*, m_i = (V^{i-1})^{\perp} = (V/V^{i-1})^* \subset V^*$$

Dualising σ yields the multiplication map

$$sym^2(m) \to \text{sym}^2(m/m_n) \xrightarrow{\sigma^*} m_2 \subset m$$

This extends in an obvious way to a commutative associative multiplication map $\text{sym}^2 R \to R$. By construction, σ^* descends to a map

$$\text{sym}^2(m/m_i) \xrightarrow{\sigma^{i+1,2*}} m_2/m_{i+1}$$

hence $m \cdot m_i \subset m_{i+1}$. So inductively m_i is firstly an ideal and then $m_i \supseteq m^i$ by induction. The rest is similar. □

Thus in particular, to an artin local $\mathbb{C}$-algebra (R, m) of exponent n, we have a uniquely determined standard OS structure on $T^n R = m^*$, which conversely determines (R, m). For later use it is convenient to explicate and amplify the morphism part of the above equivalence.

Corollary 1.2. *Let $(R, m), (R', m')$ be artin local algebras of exponent n. Then the following are mutually interchangeable:*
(i) a local homomorphism $\eta : R' \to R$;
(ii) an OS morphism $\kappa : T^n R \to T^n R'$;
(iii) a compatible collection of elements

$$v_i \in m^{n+1-i} \otimes T^n R'/T^{n-i} R'$$

such that $\quad (id \otimes \sigma)(v_n) = v_n \cdot v_n \in m^2 \otimes \text{sym}^2(T^n R') \qquad$ *(1.1)*

Proof. Only (iii) may require comment. v_n evidently determines κ as well as $v_1, \cdots\cdots, v_{n-1}$; it is the existence of the latter that ensures that κ is filtration-preserving, while (1.1) makes κ compatible with comultiplication. □

Let us call an element $v \in m_R \otimes T^n R'$ as above *morphic.*

2. Products

2.1. **Very symmetric products.** Fix a Hausdorff topological space X. For any $n \geq 1$, we denote by X^n and X_n the Cartesian and symmetric products, respectively. The system $(X^n, n \in \mathbb{N})$ forms essentially a *simplicial* configuration (while the X_n's are related to one another in even more complicated ways). On the other hand, the system of the n-th order neighborhoods of a point(say on a moduli space), $n \in \mathbb{N}$, is simply a *tower.* This indicates that the 'right' spaces of point-configurations to work with in deformation theory are neither X^n or X_n but a suitable modifications thereof which form a tower. We now proceed to define these spaces which we call the *very symmetric products* (powers) of X and denote by

$X < n >$. A word to the wise: defining $X < n >$ may appear to be a fastidious bother as (sheaf) cohomology behaves simply with respect to finite maps; however, it is *complexes* that we must work with, and even to define the coboundary maps in appropriate complexes, a certain minimum amount of 'quotienting' must be effected , e.g. it seems that the Jacobi complexes defined below on $X < n >$ cannot be defined on any natural space strictly 'above' $X < n >$ (and this certainly includes X_n).

As a set , we define
$X < n >= X^n / \sim$
$(x_1, \cdots, x_n) \sim (y_1, \cdots, y_n)$ iff $\{x_1, \cdots, x_n\} = \{y_1, \cdots, y_n\}$.

Thus $X < n >$ parametrises precisely the nonempty subsets of X of cardinality $\leq n$. We endow $X < n >$ with the quotient topology induced by the projection $\pi_n : X^n \to X < n >$. Note that we indeed have a tower of closed embeddings

$$X = X < 1 > \subset X_2 = X < 2 > \subset X < 3 > \cdots \subset X < n > \cdots \subset X < \infty >= \lim_{\to} X < n > .$$

One basis for the topology of $X < n >$ consists of all open subsets $U < n >$ where $U \subseteq X$ is open. Another, perhaps more useful basis may be described as follows. Take $\underline{x} = \{x_1, ..., x_k\} \in X < n >$ with $x_1, ..., x_k$ distinct and pick mutually disjoint neighborhoods $U_i \ni x_i, i = 1, ..., k$. Then define the basic open set

$$U. = U_1 \cdots U_k = \pi_n(\coprod_{\sigma \in S_k} \coprod_{\sum n_i = n} U_{\sigma(1)}^{n_{\sigma(1)}} \times ... \times U_{\sigma(k)}^{n_{\sigma(k)}}) \qquad (2.0)$$

(note that the set in parentheses is a π_n-saturated open subset of $X < n >$). These $U.$ clearly form a basis for the topology of $X < n >$. If X is not Hausdorff, e.g. a variety with its Zariski topology, a formula such as (2.0) is impossible, therefore in this case $X < n >$ should be defined as a Grothendieck topology (see below; however we shall not need this here).

For an alternative, inductive construction of $X < n >$, let

$$diag_{n-1} : X^{n-1} \to X^n \to X_n$$

be a 'diagonal' map, e.g. $(x_1, \cdots, x_{n-1}) \mapsto (x_1, \cdots, x_{n-1}, x_{n-1})$,whose image D_{n-1} is the big 'diagonal' in X_n (and is independent of the choice of which point gets doubled). Then

$$X < n > \quad = \quad X_n \bigcup_{X^{n-1}} X < n - 1 > \qquad (2.1)$$

(i.e. the quotient of $X_n \coprod X < n - 1 >$ by the relation $\pi_{n-1}(\underline{x}) \sim diag_{n-1}(\underline{x})$ (i.e. the topological pushout of the pair $(diag_{n-1}, \pi_{n-1})$. Indeed it is easy to see inductively that π_{n-1} factors through a map $D_{n-1} \to X < n-1 >$, so we may also write

$$X < n >= X_n \bigcup_{D_{n-1}} X < n - 1 > .$$

Via (2.1), very symmetric products may be defined in more general settings, e.g. when X is a Grothendieck topology.

For future reference, we let $u_{a,b} : X < a > \times X < b > \to X < a + b >$ be the natural 'union' map, which is clearly continuous, and likewise $u_{a,b,c}$ etc; note that this makes sense even if a or $b = \infty$. Under $u_{\infty,\infty}$, $X < \infty >$ forms a topological semigroup. We also let $v_a^b : X < a >< b > \to X < ab >$ be the natural union map. Note the factorization

$$\pi_{a+b} = u_{a,b} \circ (\pi_a \times \pi_b).$$

It is not hard to see that if X has a structure of (separated) analytic space, then so does $X < n >$ in a natural way. However, we shall not need this fact. Rather, the sheaves on $X < n >$ relevant to us will be alternating products of sheaves induced from X, which now proceed to define. Let S be a ring assumed for simplicity to contain $\mathbb{Q}$, and A a sheaf of S-modules on X. Let $\pi_n : X^n \to X < n >$ as above be the natural map, and set

$$\tau_S^n(A) = \pi_{n*}(A \boxtimes_S \cdots \boxtimes_S A)$$

(When S is understood, e.g. $S = \mathbb{C}$, we may suppress it). Note that the symmetric group Σ_n acts in a natural way on $\tau_S^n(A)$ and let $\sigma_S^n(A)$ (resp. $\lambda_S^n(A)$) denote the invariant and antiinvariant factors. Note that this definition makes sense on the symmetric product X_n already, and may also be extended to mixed (Schur) tensors in an obvious way.

More explicitly, on an open set $U. = U_1 \cdots U_k$ as in (2.0), we can write, e.g.

$$\lambda^n(A)(U.) = \sum_{n.} \bigwedge^{n_1}(A(U_1)) \otimes ... \otimes \bigwedge^{n_k}(A(U_k)) =: \sum_{n.} \lambda^{n\cdot} A(U.) \qquad (2.2)$$

We also agree that $\lambda^{n\cdot} A(U.) = 0$ if some $n_i = 0$.

When A is replaced by a complex $A^\cdot$ of S-modules, these constructions extend in a natural way to make $\tau_S^n(A^\cdot), \sigma_S^n(A^\cdot), \lambda_S^n(A^\cdot)$ into complexes (see [FH] for details, especially on sign rules); for instance

$$\lambda_S^2(A^\cdot) = \lambda_S^2(A^{even}) \oplus \pi_{2*}(A^{even} \boxtimes A^{odd}) \oplus \sigma_S^2(A^{odd}).$$

The cohomology of $\tau_S^n(A)$ can be computed by the Künneth formula, at least if A is S-free, i.e.

$$H^m(\tau_S^n(A)) = [\otimes_1^n H^\cdot(A)]^m$$

In fact the n-th tensor power of a Čech complex for A (with respect to an acyclic cover of X) yields one for $\tau_S^n(A)$. As everything decompose into $\pm$ eigenspaces under the action of Σ_n, analogous comments apply to $\sigma_S^n(A)$ and $\lambda_S^n(A)$ (one must take into account the usual sign rules for cup products, e.g. $a \cup b = (-1)^{\deg a \deg b} b \cup a$). For instance, in the case of principal interest to us, we have $H^0(A) = 0$ and then

$$H^i(X < n >, \lambda_S^n(A)) = H^i(X < n >, \tau_S^n(A)) = 0, i < n;$$

$$H^n(X < n >, \lambda_S^n(A)) = \mathrm{sym}_S^n H^1(A) \ :$$

in fact, the symmetric power of the Čech complex for A may be used to compute the cohomology of $\lambda_S^n(A)$.

Note that the operation

$$(A, B) \mapsto u_{\infty,\infty*}(A \boxtimes B)$$

defines an 'exterior' product on sheaves (or complexes) on $X < \infty >$ (likewise with (∞, ∞) replaced by any (a, b), the product being defined on $X < a + b >$). Similarly, if $v : X < \infty >< 2 > \to X < \infty >$ is the union map,

$$A \mapsto v_*(\sigma^2(A)$$

defines an exterior symmetric product. These operations endow the category of S-modules over $X < \infty >$ with an admissible structure that is different from the 'plain' one (involving ordinary tensor and symmetric products); a similar such structure exists for any topological semigroup.

Remark The spaces $X < n >$ have recently appeared in the work of Beilinson and Drinfeld on 'Chiral Algebras'; I am grateful to V. Ginzburg for pointing this out.

2.2. **Jacobi complex.**

2.2.1. *Definition.* Let $\mathcal{L}^{\cdot}$ be a sheaf of complex differential graded Lie algebras (DGLAs) on X. Thus $\mathcal{L}^{\cdot}$ is a "Lie object' in the category of complexes of $\mathbb{C}$-modules on X, which means there is a morphism $bt : \Lambda^2(\mathcal{L}^{\cdot}) \to \mathcal{L}^{\cdot}$, whose natural extension as a derivation of degree -1 on the Grassmann algebra $\oplus\Lambda^i(\mathcal{L}^{\cdot})$ satisfies $bt^2 = 0$; or course 'Grassmann algebra' and wedge must be understood in the graded sense, compatible with the gradation on $\mathcal{L}^{\cdot}$.Note that bt induces a map

$$br : \lambda^2(\mathcal{L}^{\cdot}) \to \Lambda^2(\mathcal{L}^{\cdot}) \to \mathcal{L}^{\cdot}$$

(i.e. restriction followed by bt). Now we associate to $\mathcal{L}^{\cdot}$ a complex $J^{\cdot}(\mathcal{L}^{\cdot})$ on $X < \infty >$ called the Jacobi complex of $\mathcal{L}^{\cdot}$, as follows. Set

$$J^{-n}(\mathcal{L}) = \lambda^n(\mathcal{L}^{\cdot}), \qquad n \geq 1$$

(where the latter is viewed as a sheaf on $X < \infty >$ via $X < n > \subset X < \infty >$ and λ is understood in the graded sense); the differential $d_n : \lambda^n(\mathcal{L}^{\cdot}) \to \lambda^{(n-1)}(\mathcal{L}^{\cdot})$ is defined as follows. First, let $alt : \tau^2(\mathcal{L}^{\cdot}) \to \lambda^2(\mathcal{L}^{\cdot})$ be the alternation or skew-symmetrization map, where $\lambda^2(\mathcal{L}^{\cdot})$ is viewed as a complex on $X < 2 >$ via the diagonal embedding $X \to X < 2 >$, and set

$$a = u_{n-2,2*}(id \boxtimes alt) : \pi_{n*}(\boxtimes^n \mathcal{L}^{\cdot}) = u_{n-2,2*}(\tau^{n-2}\mathcal{L}^{\cdot} \boxtimes \tau^2(\mathcal{L}^{\cdot}))$$
$$\to u_{n-2,2*}(\tau^{n-2}\mathcal{L}^{\cdot} \boxtimes \lambda^2(\mathcal{L}^{\cdot})).$$

Next, set

$$b = u_{n-2,2*}(id \boxtimes br) : u_{n-2,2*}(\tau^{n-2}\mathcal{L}^{\cdot} \boxtimes \lambda^2(\mathcal{L}^{\cdot})) \to u_{n-2,2*}(\tau^{n-2}\mathcal{L}^{\cdot} \boxtimes \mathcal{L}^{\cdot})$$
$$= u_{n-2,1*}(\tau^{n-2}\mathcal{L}^{\cdot} \boxtimes \mathcal{L}^{\cdot}) = \tau^{n-1}(\mathcal{L}^{\cdot}).$$

Finally let $p : \tau^{n-1}(\mathcal{L}^{\cdot}) \to \lambda^{n-1}(\mathcal{L}^{\cdot})$ be the natural alternation map and $i : \lambda^n(\mathcal{L}^{\cdot}) \to \tau^n(\mathcal{L}^{\cdot})$ the inclusion. Then define

$$d_n = p \circ b \circ a \circ i.$$

More explicitly, in terms of a decomposition (2.2) over an open subset $U.$ as in (2.0), $d_n(U.)$ is defined on each summand $\lambda^{n.}A(U.)$ as the differential on the tensor-product complex of the standard complexes of the dgla's $\mathcal{L}^{\cdot}(U_i), i = 1, ..., k$, i.e. $d_n(U.)$ on a summand $\lambda^{n.}A(U.)$ is a direct sum of maps

$$\lambda^{n.}A(U.) \to \lambda^{n_1,...n_i-1,...n_k}A(U.)$$

each given by $(-1)^{\sum\limits_{j<i} n_j} id \otimes \delta^i \otimes id$, where δ^i is the differential of the standard complex for $\mathcal{L}^{\cdot}(U_i)$ [F] which, as we recall, is given by

$$\delta(t_1 \wedge \cdots \wedge t_n) = \frac{1}{(n-1)!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)[t_{\sigma(1)}, t_{\sigma(2)}] \wedge t_{\sigma(3)} \wedge \cdots \wedge t_{\sigma(n)}$$

$$= [t_1, t_2] \wedge t_3 \wedge ... \wedge t_n + [t_2, t_3] \wedge ... \wedge t_n \wedge t_1 + ... + [t_n, t_1] \wedge t_2 \wedge ... \wedge t_{n-1}.$$

At this point let us introduce some notation. For a multiindex $I = (i_1 < ... < i_a) \subset \{1, ..., n\} =: [n]$, set $|I| = a, \mathrm{sgn}(I) = (-1)^{\sum(i_j - j)}$, $t_I = t_{i_1} \wedge ... \wedge t_{i_a}$, $[n] - I =$ complementary multiindex, and if $I = (i < j), [t_I] = [t_i, t_j]$. Then we can also write

$$\delta(t_J) = \sum_{|I|=2} \mathrm{sgn}(J)\,\mathrm{sgn}(I)\,\mathrm{sgn}(J - I)[t_I] \wedge t_{J-I}.$$

I claim next that with these d_n, $J(\mathcal{L}^{\cdot})$ forms a complex, i.e. $d_{n-1} \circ d_n = 0$. indeed this follows immediately from the fact the δ form a complex (for a dg Lie algebra), plus the standard fact that the tensor product of complexes is a complex; alternatively, note first that for $n > 3$, $d_{n-1} \circ d_n$, like d_n, admits a factorization as $p \circ c \circ i$ where $c = u_{n-3,3*}(id \boxtimes (d_2 \circ d_3))$, so it suffices to prove $d_2 \circ d_3 = 0$, where, NB, $d_2 \circ d_3$ factors through a map

$$\delta_2 \circ \delta_3 : \bigwedge^3(\mathcal{L}^{\cdot}) \to \mathcal{L}^{\cdot}$$

on $X = X < 1 >$=supp$J_1(\mathcal{L}^{\cdot})$. Indeed,

$$\begin{aligned}\delta_2(\delta_3(t_1 \wedge t_2 \wedge t_3)) &= d_2([t_1, t_2] \wedge t_3 + [t_2, t_3] \wedge t_1 + [t_3, t_1] \wedge t_2) \\ &= [[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2] = 0\end{aligned}$$

by the Jacobi identity for $\mathcal{L}^{\cdot}$. Put $J^{\cdot}_n(\mathcal{L}^{\cdot}) = J^{\geq -n}(\mathcal{L}^{\cdot})$, which may be viewed as complex on $X < n >$.

2.2.2. *OS structure.* Our next goal is to define an OS structure on $V^n(\mathcal{L}^{\cdot}) = \mathbb{H}^0(J_n(\mathcal{L}^{\cdot}))$. Indeed this will be induced by an OS structure on $J(\mathcal{L}^{\cdot})$ itself and the $J_n(\mathcal{L}^{\cdot})$, in the sense of the admissible structure on the category of complexes of sheaves on $X < \infty >$ discussed in Sect 2.1. To this end consider first the complex $\sigma^2(J_{n-1}(\mathcal{L}^{\cdot}))$ on $X < n - 1 >< 2 >$ (σ^2 = signed symmetric square), and its image $K^{\cdot}$ on $X < 2n - 2 >$ by the natural map

$$v = v^2_{n-1} : X < n - 1 >< 2 >\to X < 2n - 2 >$$

(or, what is the same, by v^2_∞) $v_*\sigma^2(J_{n-1}(\mathcal{L}^{\cdot}) =: K^{\cdot}$, which takes the form :

$$... \to \lambda^3(\mathcal{L}^{\cdot}) \boxtimes \mathcal{L}^{\cdot} \oplus \sigma^2(\lambda^2(\mathcal{L}^{\cdot})) \to \lambda^2(\mathcal{L}^{\cdot}) \boxtimes \mathcal{L}^{\cdot} \to \lambda^2(\mathcal{L}^{\cdot})$$

Now I claim that the natural direct summand inclusions

$$\begin{gathered}\lambda^2(\mathcal{L}^{\cdot}) \to \lambda^2(\mathcal{L}^{\cdot}) \\ \lambda^3(\mathcal{L}^{\cdot}) \to \lambda^2(\mathcal{L}^{\cdot}) \boxtimes (\mathcal{L}^{\cdot}) \\ \lambda^4(\mathcal{L}^{\cdot}) \to \lambda^3(\mathcal{L}^{\cdot}) \boxtimes \mathcal{L}^{\cdot} \oplus \sigma^2(\lambda^2(\mathcal{L}^{\cdot})) \\ ...\end{gathered}$$

yield a morphism of complexes

$$(J_n/J_1)(\mathcal{L}^{\cdot}) \to K^{\cdot}$$

which means that the following diagrams commute for $r \equiv 1 \mod 4$

$$\begin{array}{ccc} \lambda^r(\mathcal{L}^{\cdot}) & \to & \lambda^{r-1}(\mathcal{L}^{\cdot}) \\ \downarrow & & \downarrow \\ \sum\limits_{a<r/2} u_{a,r-a*}(\lambda^a(\mathcal{L}^{\cdot}) \boxtimes \lambda^{r-a}(\mathcal{L}^{\cdot})) & \to & \sum\limits_{1<a<r/2} u_{a,r-a*}(\lambda^{a-1}(\mathcal{L}^{\cdot}) \boxtimes \lambda^{r-a}(\mathcal{L}^{\cdot})) \oplus v_*(\sigma^2(\lambda^{(r-1)/2}(\mathcal{L}^{\cdot}))) \end{array} \qquad (2.3)$$

with similar diagrams for other congruence classes of r mod 4. Now for a basic open set $U.$ (cf. (2.0)), $K^{\cdot}(U.)$ is clearly just the symmetric square of $J^{\cdot}_{n-1}(\mathcal{L}^{\cdot})(U.)$, which itself is a tensor product of standard complexes associated to the $\mathcal{L}^\cdot(U_i)$. It therefore suffices to prove the commutativity of the analogue of (2.3) for the standard complex of a Lie algebra L, which is

$$\begin{array}{ccc} \bigwedge^r(L) & \to & \bigwedge^{r-1}(L) \quad (2.4) \\ \downarrow & & \downarrow \\ \sum\limits_{a<r/2} \bigwedge^a(L)\boxtimes\bigwedge^{r-a}(L) & \to & \sum\limits_{1<a<r/2}\bigwedge^{a-1}(L)\boxtimes\bigwedge^{r-a}(L)\oplus \mathrm{sym}^2(\bigwedge^{(r-1)/2}(L)) \end{array}.$$

where, e.g. the left vertical map takes an element $t_1\wedge ...\wedge t_r$ to

$$\sum_a (-1)^{a(r-a)}\sum_{|I|=a}\mathrm{sgn}(I)\,\mathrm{sgn}([r]-I)t_I\otimes t_{[r]-I}, \qquad (2.5)$$

and the bottom arrow takes the latter to

$$\sum_{a\leq (r-1)/2}(-1)^{a(r-a)}(\sum_{|I|=a}\sum_{J\subset I,|J|=2}\mathrm{sgn}(J)\,\mathrm{sgn}(I-J)\,\mathrm{sgn}([r]-I)[t_J]\wedge t_{I-J}\otimes t_{[r]-I}$$

$$+(-1)^a\sum_{|I|=a}\sum_{J\subset[r]-I,|J|=2}\mathrm{sgn}(J)\,\mathrm{sgn}(I)\,\mathrm{sgn}([r]-I-J)t_I\otimes[t_J]\wedge t_{[r]-I-J}). \quad (2.6)$$

where we have used $\otimes$ to denote multiplication in $\mathrm{sym}^2(\bigwedge^{(r-1)/2}(L))$ which occurs in the second double sum for $a=(r-1)/2$.

Going clockwise, $t_1\wedge ...\wedge t_r$ first maps to

$$\sum_{J\subset[r],|J|=2}\mathrm{sgn}(J)\,\mathrm{sgn}([r]-J)[t_J]\wedge t_{[r]-J},$$

then the right vertical arrow sends each summand to a sum analogous to (2.5) with $[r]$ replaced by the ordered set $(\{J\}<[r]-J)$ (i.e. $\{J\}\coprod([r]-J)$, ordered by putting $\{J\}$ first) , which may be written as

$$\sum_{a\leq(r-1)/2}(\sum_{J\subset I,|I|=a}\mathrm{sgn}(J)\,\mathrm{sgn}(I-J)\,\mathrm{sgn}([r]-I)(-1)^{a-1}(-1)^{(a-1)(r-a)}[t_J]\wedge t_{I-J}\otimes t_{[r]-I}$$

$$+\sum_{J\subset[r]-I,|I|=a}\mathrm{sgn}(J)\,\mathrm{sgn}(I-J)\,\mathrm{sgn}([r]-I)(-1)^{a(r-1-a)}t_I\otimes t_J\wedge t_{[r]-J}).$$

As $(-1)^{(r-a+1)(a-1)}=(-1)^{a(r-a)}, (-1)^{a(r-1-a)}=-(-1)^{a(r-a)}$, we see that by summing over J we get the same thing as (2.6), so the diagram commutes. Other congruence classes of r mod 4 are treated similarly.

Thus we have direct summand embeddings

$$\sigma^n: J_n(\mathcal{L}^\cdot)/J_1(\mathcal{L}^\cdot) \;=\; J^{-n\leq\cdot\leq -2}(\mathcal{L}^\cdot)\to v_*\sigma^2(J_{n-1}(\mathcal{L}^\cdot),$$

$$\sigma^\infty: J(\mathcal{L}^\cdot)/J_1(\mathcal{L}^\cdot)\to v^2_{\infty *}\sigma^2(J_(\mathcal{L}^\cdot),$$

I claim next that with this σ^n, $J_n(\mathcal{L}^\cdot)$ forms an OS structure (and likewise with $n=\infty$). Indeed commutativity and standardness are obvious, and it suffices to check the associativity condition, which follows from the equality of the composite maps (where we have omitted the $\mathcal{L}^\cdot$)

$$\lambda^r\to\sum_{a+b=r}u_{a,b*}(\lambda^a\boxtimes\lambda^b)\to\sum_{a+b=r}\sum_{b'+b"=b}u_{a,b*}(\lambda^a\boxtimes u_{b',b"*}(\lambda^{b'}\boxtimes\lambda^{b"}))=\sum_{a+b+c=r}u_{a,b,c*}(\lambda^a\boxtimes\lambda^b\boxtimes\lambda^c)$$

$$\lambda^r\to\sum_{a+b=r}u_{a,b*}(\lambda^a\boxtimes\lambda^b)\to\sum_{a+b=r}\sum_{a'+a"=a}u_{a,b*}(u_{a',a"*}(\lambda^{a'}\boxtimes\lambda^{a"})\boxtimes\lambda^b)=\sum_{a+b+c=r}u_{a,b,c*}(\lambda^a\boxtimes\lambda^b\boxtimes\lambda^c).$$

Working over a basic open set $U.$ and decomposing as in (2.2), we are again reduced to proving the corresponding assertion for the exterior powers of a vector space L, which is standard: indeed it boils down to the fact that the duality isomorphism

$$\bigwedge^{\cdot}(L) \simeq (\bigwedge^{\cdot}(L^*))^*$$

endows the Grassmann algebra $\bigwedge^{\cdot}(L)$ with a coalgebra structure that is (co)associative because, as is well known, $\bigwedge^{\cdot}(L^*)$ is associative; alternatively, the assertion can be verified by a simple computation. This completes the verification that $(J(\mathcal{L}^{\cdot}), \sigma^\infty)$ is indeed an OS structure, with substructures $(J_n(\mathcal{L}^{\cdot}), \sigma^n)$. It follows immediately that $V(\mathcal{L}^{\cdot}) = \mathbb{H}^0(J(\mathcal{L}^{\cdot})$ and $V^n(\mathcal{L}^{\cdot}) = \mathbb{H}^0(J_n(\mathcal{L}^{\cdot})$ inherit OS structures, which are standard provided $\mathbb{H}^{\leq 0}(\mathcal{L}^{\cdot}) = 0$.

By Section 1 then we obtain an inverse system of artin local algebras

$$R_n(\mathcal{L}^{\cdot}) = \mathbb{C} \oplus V^n(\mathcal{L}^{\cdot})^*$$

and their limit $\hat{R}(\mathcal{L}^{\cdot})$ which might be called the deformation ring associated to $\mathcal{L}^{\cdot}$.

In particular, if X is a compact complex manifold, its tangent sheaf $T = T_X$ forms a Lie algebra under Lie bracket of vector fields, and we denote the associated Jacobi complexes by $J_{n,X}$ or J_n, the corresponding OS structure by V^n_X or V^n, and the corresponding ring by $R^u_{n,X}$ or R^u_n, As we shall see, when $H^0(T) = 0$ the latter turns out to be the base ring of the n-universal deformation of X.

2.3. **Obstructions.** Assume $\mathcal{L}^{\cdot}$ is a dgla with $\mathbb{H}^{\leq 0}(\mathcal{L}^{\cdot}) = 0$. Note that the long cohomology sequence associated to

$$0 \to J_{n-1}(\mathcal{L}^{\cdot}) \to J_n(\mathcal{L}^{\cdot}) \to \lambda^n(\mathcal{L}^{\cdot})[n] \to 0, n \geq 2,$$

gives rise to a *'big obstruction'* map

$$Ob_n : \mathrm{sym}^n\, \mathbb{H}^1(\mathcal{L}^{\cdot}) \to \mathbb{H}^1(J_{n-1}(\mathcal{L}^{\cdot})).$$

Let $K^n = \ker(Ob_n)$, so that we have an exact sequence

$$0 \to V^{n-1} \to V^n \to K^n \to 0.$$

Let $K^{n-1}.\mathbb{H}^1(\mathcal{L}^{\cdot})$ denote the intersection of $\mathrm{sym}^n\, \mathbb{H}^1(\mathcal{L}^{\cdot})$ and $K^{n-1} \otimes \mathbb{H}^1(\mathcal{L}^{\cdot})$ considered as subspaces of $\otimes^n \mathbb{H}^1(\mathcal{L}^{\cdot})$. In view of the natural direct summand embedding

$$(J_n/J_1)(\mathcal{L}^{\cdot}) \to u_{n-1,1*}(J_{n-1}(\mathcal{L}^{\cdot}) \boxtimes \mathcal{L}^{\cdot}),$$

analogous to σ^n above, it is easy to see that $K^{n-1}.\mathbb{H}^1(\mathcal{L}^{\cdot})$ coincides with the image of the natural map

$$\mathbb{H}^0(J_n/J_1)(\mathcal{L}^{\cdot}) \to \mathrm{sym}^n\, \mathbb{H}^1(\mathcal{L}^{\cdot}).$$

Now the diagram

$$\begin{array}{ccccccccc} 0 & \to & J_1(\mathcal{L}^{\cdot}) & \to & J_n(\mathcal{L}^{\cdot}) & \to & (J_n/J_1)(\mathcal{L}^{\cdot}) & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & J_{n-1}(\mathcal{L}^{\cdot}) & \to & J_n(\mathcal{L}^{\cdot}) & \to & \lambda^n(\mathcal{L}^{\cdot})[n] & \to & 0 \end{array}$$

induces

$$\begin{array}{ccc} \mathbb{H}^0((J_n/J_1)(\mathcal{L}^{\cdot})) & \to & \mathbb{H}^2(\mathcal{L}^{\cdot}) \\ \downarrow & & \downarrow \\ \mathrm{sym}^n\, \mathbb{H}^1(\mathcal{L}^{\cdot}) & \to & \mathbb{H}^1(J_{n-1}(\mathcal{L}^{\cdot}) \end{array}$$

and we let $\overline{\mathbb{H}^2(\mathcal{L}^{\cdot})}$ denote the image of the right vertical arrow. Then we get a map, called the *small obstruction* map

$$ob_n : K^{n-1}.\mathbb{H}^1(\mathcal{L}^{\cdot}) \to \overline{\mathbb{H}^2(\mathcal{L}^{\cdot})}.$$

Since elements in the kernel of Ob_n lift to $\mathbb{H}^0(J_n(\mathcal{L}^\cdot))$ and in particular to $\mathbb{H}^0((J_n/J_1)(\mathcal{L}^\cdot))$, they automatically lie in $K^{n-1}.\mathbb{H}^1(\mathcal{L}^\cdot)$ and belong to the kernel of ob_n; hence

$$K^n = \ker ob_n.$$

Thus K^n may be described inductively, starting with $K^1 = \mathbb{H}^1(\mathcal{L}^\cdot), ob_1 = 0$.

2.4. **Coefficients.** The following remark will be useful in Sect 4. Let

$$m^\cdot = (m = m^1 \supset m^2 ...)$$

be a filtered vector space (for example, the set of powers of the maximal ideal of an artin local algebra). Then the complex $J_n(\mathcal{L}^\cdot) \otimes m$- whose cohomology coincides with $\mathbb{H}^\cdot(J_n(\mathcal{L}^\cdot)) \otimes m$- contains a natural subcomplex denoted $J_n(\mathcal{L}^\cdot, m^\cdot)$ whose term in degree i is $J_n^i(\mathcal{L}^\cdot) \otimes m^i$. By considering the exact triangle associated to the natural map

$$J_n(\mathcal{L}^\cdot, m^\cdot) \to J_n(\mathcal{L}^\cdot) \otimes m$$

and its long cohomology sequence, it is immediate that, provided $\mathbb{H}^{\leq 0}(\mathcal{L}^\cdot) = 0$, the induced map

$$\mathbb{H}^0(J_n(\mathcal{L}^\cdot, m^\cdot)) \to \mathbb{H}^0(J_n(\mathcal{L}^\cdot)) \otimes m$$

is injective.

3. Second order deformations

For $n = 1$, Theorem 0.1 reduces to standard first -order Kodaira-Spencer deformation theory. Before taking up the general n-th order case in the next section, we consider here the second-order case , where it is possible to give a rather direct proof which illustrates some- though by no means all- of the ideas: the proof is based on a cocycle- level construction which does not generalize to higher order. Thus this section is logically unnecessary for the rest of the paper.

Let us fix an artin local $\mathbb{C}$-algebra(R_2, m_2) with reduction $(R_1, m_1) = (R_2/m_2^2, m_2/m_2^2)$ as well as an acyclic (say polydisc) open cover (U_α) of X, to be used in computing Čech cohomology. To a flat deformation

$$X_2/R_2 = Spec(\mathcal{O}_2)$$

we seek to associate a Kodaira-Spencer homomorphism

$$\alpha_2 = \alpha_2(X_2/R_2) : R_2^u \to R_2$$

or equivalently (cf. Section 2) a morphic element

$$v_2 = v_2(X_2/R_2) \in m_2 \otimes \mathbb{H}^0(J_2)$$

which is to be described by a hypercocycle

$$v_2 = (u, \tfrac{1}{2}u^2) \in \check{C}^1(T) \otimes m_2 \oplus \mathrm{sym}^2\, \check{C}^1(T) \otimes m_2^2 \subset \check{C}^0(J_2) \otimes m_2 \tag{3.1}$$

where $u = (u_{\alpha\beta})$ is required to be a lifting of

$$v_1 = (v_{1\alpha\beta}) \in \check{Z}^1(T) \otimes m_1,$$

a cocycle representing $\mathcal{O}_1 = \mathcal{O}_2 \otimes_{R_2} R_1$ (where $\mathcal{O}_2$ is the structure sheaf of X_2), and u^2 means exterior cup product in the cochain sense,i.e

$$(u^2)_{\alpha\beta\gamma} = u_{\alpha\beta} \times u_{\beta\gamma} \in \mathrm{sym}^2\, \check{C}^1(T) \otimes m_2^2 \subset \check{C}^2(\lambda^2 T) \otimes m_2^2$$

Note that the particular form of (3.1) makes the morphicity of v_2 automatic provided it is a hypercocycle, which means explicitly

$$-\frac{1}{2}[u_{\alpha\beta}, u_{\beta\gamma}] = u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = \delta(u) \tag{3.2}$$

$$\delta = \check{C}\text{ech coboundary}$$

Note that the LHS of (3.2) depends only on the reduction $v_{1\alpha\beta}$ of $u_{\alpha\beta}$ mod m_2^2

Now to define (u) we proceed as follows. As $\mathcal{O}_2/R_2$ in a flat deformation of $\mathcal{O}$, it is locally trivial, hence we have isomorphisms of R_2-algebras

$$\psi_\alpha : \mathcal{O}_2(U_\alpha) \to \mathcal{O}(U_\alpha) \otimes_{\mathbb{C}} R_2$$

which give rise to a gluing cocyle given by

$$D^2_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1} \in Aut_{R_2}(\mathcal{O}(U_\alpha \cap U_\beta) \otimes R_2)$$

which reduces mod m_2^2 to

$$D^1_{\alpha\beta} = I + v_{1\alpha\beta} \in Aut_{R_1}(\mathcal{O}(U_\alpha \cap U_\beta) \otimes R_1),$$

a gluing cocycle defining $\mathcal{O}_1$.

Now it is easy to see that $D^2_{\alpha\beta}$ is uniquely expressible in the form

$$\begin{aligned} D^2_{\alpha\beta} = &\quad \exp(u_{\alpha\beta}) \qquad\qquad (3.3)\\ = &\quad I + u_{\alpha\beta} + \tfrac{1}{2}u^2_{\alpha\beta}, \qquad u_{\alpha\beta} \in m_2 \otimes T(U_\alpha \cap U_\beta): \end{aligned}$$

indeed starting with an arbitrary lift $u'_{\alpha\beta}$ of $v_{1\alpha\beta}$ to $m_2 \otimes T(U_a \cap U_\beta)$, $\exp(u'_{\alpha\beta})$ and $D^2_{\alpha\beta}$ are R_1-algebra homomorphisms which agree mod m_2^2, hence differ by an m_2^2- valued derivation $t_{\alpha\beta}$ and we may set $u_{\alpha\beta} = u'_{\alpha\beta} + t_{\alpha\beta}$. Now we simply plug (3.3) into the cocycle equation for D^2:

$$D^2_{\alpha\beta} D^2_{\beta\gamma} = D^2_{\alpha\gamma} \tag{3.4}$$

which becomes,

$$\begin{aligned} I + u_{\alpha\beta} + u_{\beta\gamma} + \frac{1}{2}u^2_{\alpha\beta} + u_{\alpha\beta}u_{\beta\gamma} + \frac{1}{2}u^2_{\beta\gamma} &= I + u_{\alpha\gamma} + \frac{1}{2}(u_{\alpha\gamma})^2 \\ &= I + u_{\alpha\gamma} + \frac{1}{2}(u^2_{\alpha\beta} + u_{\alpha\beta}u_{\beta\gamma} + u_{\beta\gamma}u_{\alpha\beta} + u^2_{\beta\gamma}) \end{aligned}$$

as $(u_{\alpha\beta})$ is a cocycle mod m_2^2. This is obviously equivalent to (3.2). Thus v_2 is a hypercocycle , as claimed.

Now the foregoing argument can essentially be read backwards: given a morphic element

$$v_2 \in m_2 \otimes \mathbb{H}^0(J_2),$$

choose a representative for v_2 of the form

$$((u_{\alpha\beta}), (u'_{\alpha\beta\gamma})) \in \check{C}'(T) \otimes m_2 \oplus \mathrm{sym}^2\, \check{C}'(T) \otimes m_2^2 \subset \check{Z}^0(J_2),$$

where $(u_{\alpha\beta})$ is a lifting of $(v_{1\alpha\beta})$; thus compatibility with comultiplication yields that v_2 may also be represented by

$$((u_{\alpha\beta}), \frac{1}{2}(u_{\alpha\beta})^2).$$

Then simply setting $D^2_{\alpha\beta} = \exp(u_{\alpha\beta})$, the cocycle condition (3.4) follows from the hypercocycle condition (3.2), so that $(D^2_{\alpha\beta})$ yields a locally trivial flat deformation $X_2/R_2 = Spec\mathcal{O}_2$, which we denote by $\Phi_2(\alpha_2)$ (though it is yet to be established that this is independent of choices).

This construction applies in particular to the identity map $R_2^u \to R_2^u$, thus yielding a flat deformation over R_2^u which we call an *universal second order deformation* and denote by $X_2^u = Spec(\mathcal{O}_2^u)$. It is moreover clear by construction that $\Phi_2(\alpha_2) = \alpha_2^*(X_2^\kappa/R_2^n)$ for any $\alpha_2 : R_2^u \to R_2$ and also that for any second-order deformation X_2/R_2,

$$X_2/R_2 \approx \alpha_2(X_2/R_2)^*(X_2^u/R_2^u) \approx \Phi_2(\alpha_2(X_2/R_2)).$$

Similarly,

$$\alpha_2(\Phi_2(\beta)) = \beta.$$

Thus α_2 and Φ_2 establish mutually inverse correspondences, albeit on the cocycle level. What has to be established is that this correspondence descends to cohomology , i.e. non-abelian cohomology of Aut-cocycles and hypercohomology respectively. Here we give a computational proof of this; in the next section we shall give a more 'conceptual' proof.

In one direction, consider two cohomologous Aut-cocycles

$$D_{\alpha\beta}^2 \sim D_{\alpha\beta}^{2\prime} = A_\beta D_{\alpha\beta}^2 A_\alpha^{-1}$$

$A_\alpha \in Aut_{R_2}(\mathcal{O}(U_\alpha) \otimes R_2)$, as above uniquely expressible in the form $\exp(w_\alpha), w_\alpha \in m_2 \otimes T(U_\alpha)$. Thus

$$\begin{aligned}
D_{\alpha\beta}^{2\prime} &= (I + w_\beta + \frac{1}{2}w_\beta^2)(I + u_{\alpha\beta} + \frac{1}{2}u_{\alpha\beta}^2)(I - w_\alpha + \frac{1}{2}w_\alpha^2) \\
&= I + (u_{\alpha\beta} + w_\beta - w_\alpha + \frac{1}{2}[w_\beta - w_\alpha, u_{\alpha\beta}] + \frac{1}{2}[w_\alpha, w_\beta]) + \frac{1}{2}(u_{\alpha\beta} + w_\beta - w_\alpha)^2 \\
&= \exp(u_{\alpha\beta} + w_\beta - w_\alpha + \frac{1}{2}[w_\beta - w_\alpha, u_{\alpha\beta}] + \frac{1}{2}[w_\alpha, w_\beta]) \\
&=: \exp(u'_{\alpha\beta})
\end{aligned}$$

Then $v'_2 = v_2(D^{2\prime}) = (u', \frac{1}{2}(u')^2)$ is cohomologous to v_2 because

$$v'_2 - v_2 = \partial((w_\alpha), \frac{1}{2}(w_\alpha \times u_{\alpha\beta}) + \frac{1}{2}(w_\alpha \times w_\beta))$$

where $\partial = \delta \pm b$ is the differential of the Čech bicomplex of $\check{C}(J_2)$. Conversely, supposing $v_2 = (u, \frac{1}{2}u^2), v'_2 = (u', \frac{1}{2}u'^2)$ are cohomologous,

$$v'_2 - v_2 = \partial((w_\alpha), (t_{\alpha\beta}))\ .$$

Now as $H^0(T) = 0, \delta(t) = \frac{1}{2}(u')^2 - \frac{1}{2}u^2$ determines (t) up to adding a Čech coboundary $s_\alpha - s_\beta$ and, using $b\delta = \pm\delta b$ this may be absorbed into (w_α). Thus we may assume

$$t_{\alpha\beta} = \frac{1}{2}w_\alpha \times u_{\alpha\beta} + \frac{1}{2}w_\alpha \times w_\beta,$$

so that $(D_{\alpha\beta}^2 = \exp(u_{\alpha\beta}))$ and $(D_{\alpha\beta}^{2\prime} = \exp(u'_{\alpha\beta}))$ are cohomologous as above. This finally completes the proof of Theorem 0.1 for n=2.

4. n-th order deformations

We now complete the proof of Theorem 0.1 in the general n-th order case, $n \geq 1$, following in part the pattern of the case n=2 and using induction (however, the results of Sect.3 are not used). The argument becomes a bit more involved and less direct. We shall in fact give two (parallel) proofs, using Dolbeault and Čech cohomology, respectively.

Fix an artin local $\mathbb{C}$-algebra (R_n, m_n) of exponent n, with reduction (R_{n-1}, m_{n-1}), etc, and an acyclic open cover (U_α) of X. The main point (in the Čech version) is to associate a morphic hypercocycle

$$v_n = v_n(\mathcal{O}_n/R_n) \in m_n \otimes \check{Z}^0(J_n) ,$$

hence a Kodaira-Spencer homomorphism $\alpha_n(\mathcal{O}_n/R_n)$ etc- to an R_n-flat deformation $\mathcal{O}_n = \mathcal{O}_{X_n}$ of $\mathcal{O}$ (and similarly in the Dolbeault version). As before we seek v_n of the form

$$v_n = \epsilon(u_n) := (u_n, \frac{1}{2}(u_n)^2, \cdots, \frac{1}{n!}(u_n)^n) \tag{4.0}$$

for some cochain $u_n = (u_{n\alpha\beta}) \in \check{C}^1(T) \otimes m_n$ which is a lift of $u_{n-1} \in \check{C}^1(T) \otimes m_n$ analogously defining v_{n-1}. We shall then show that the cohomology class of v_n is independent of choices. Conversely we shall associate a deformation $\mathcal{O}_n$ to a hypercocycle $v_n = \epsilon(u_n)$ as in (4.0).

Step 0

We start with a 'reference' set of isomorphisms of algebras

$$\psi^n_\alpha : \mathcal{O}_n(U_\alpha) \xrightarrow{\sim} \mathcal{O}(U_\alpha) \otimes R_n$$

which yield a gluing cocycle by

$$D^n_{\alpha\beta} = \psi^n_\alpha(\psi^n_\beta)^{-1} \in Aut_{R_n}(\mathcal{O}(U_\alpha \cap U_\beta) \otimes R_n), \tag{4.1}$$

which as above we express in the form

$$D^n_{\alpha\beta} = \exp(t_{n\alpha\beta}), \tag{4.2}$$

This can be done because, assuming inductively that (4.2) holds for $n-1$ and letting t'_n be an arbitrary lift of t_{n-1} and $t_n = t'_n + \eta_n, \eta_n \in \check{C}^1(T) \otimes m_n^n$, (4.2) can be rewritten as

$$D^n_{\alpha\beta} = \exp(t'_{n\alpha\beta}) + \eta_n.$$

so just take $\eta_n = D^n_{\alpha\beta} - \exp(t'_{n\alpha\beta})$, which is in $\check{C}^1(T) \otimes m_n^n$ precisely because (4.2) holds for $n-1$.

We now proceed with the Dolbeault version of the proof, leaving the Čech version till later.

Step 1: definition of Kodaira-Spencer cocycle

Consider the DGLA sheaf

$$g^\cdot = (\mathcal{A}^{0,\cdot}(T), \bar{\partial}, [,])$$

($\Gamma(g^\cdot)$ is sometimes called the Frohlicher-Nijenhuis algebra); as $g^\cdot$ is a soft resolution of T, $J_n(g^\cdot)$ is a soft resolution of $J_n(T)$ which may be used to compute $\mathbb{H}^0(J_n(T))$. We view $J_n(g^\cdot)$ as a bicomplex with vertical differentials induced by $\bar{\partial}$ and horizontal ones induced by Lie bracket. As g^0 is soft it is easy to see that, up to shrinking our cover (U_α) we may assume

$$D^n_{\alpha\beta} = \exp(s_\alpha)\exp(-s_\beta)$$

$$s_\alpha \in g^0(U_\alpha) \otimes m_n.$$

Put another way, we may view ψ^n_α above as a holomorphic local trivialisation

$$U^n_\alpha \simeq U_\alpha \times \mathrm{Spec}(R_n)$$

U_α^n = open subset of X_n corresponding to U_α; on the other hand there is a global 'C^∞ trivialisation' $C : X \times \mathrm{Spec}(R_n) \to X_n$, and we may set

$$\exp(s_\alpha) = (\psi_\alpha^n) \circ C \qquad (4.3)$$

Now note that $\bar\partial$ extends formally as a derivation on the universal enveloping algebra $U(g^\cdot)$ and we set

$$\phi_\alpha = \exp(-s_\alpha)\bar\partial(\exp(s_\alpha)) = D(ad(s_\alpha))(\bar\partial s_\alpha) \qquad (4.4)$$

where D is the function

$$D(x) = \frac{\exp(x) - 1}{x} = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}.$$

Note that

$$0 = \bar\partial D_{\alpha\beta}^n = \bar\partial \exp(s_\alpha)\exp(-s_\beta) + \exp(s_\alpha)\bar\partial\exp(-s_\beta),$$

hence

$$\exp(-s_\alpha)\bar\partial\exp(s_\alpha) = -\bar\partial\exp(-s_\beta)\exp(s_\beta);$$

since moreover $\bar\partial(\exp(-s_\beta)\exp(s_\beta) = 0$ we have similarly

$$-\bar\partial\exp(-s_\beta)\exp(s_\beta) = \exp(-s_\beta)\bar\partial\exp(s_\beta), \qquad (4.5)$$

which means precisely that the ϕ_α glue together to a global section

$$\phi = \phi_n \in \Gamma(g^1) = A^{0,1}(T) \otimes m_n.$$

Next, note using (4.4) that

$$\bar\partial\phi_\alpha = \bar\partial\exp(-s_\alpha)\bar\partial\exp(s_\alpha) = \bar\partial\exp(-s_\alpha)\exp(s_\alpha)\exp(-s_\alpha)\bar\partial\exp(s_\alpha) = -\phi_\alpha\phi_\alpha;$$

recalling that for odd-degree elements $\phi, \psi \in g^\cdot, [\phi,\psi] = \phi.\psi + \psi.\phi$, we conclude that the integrability equation

$$\bar\partial\phi = \frac{-1}{2}[\phi,\phi] \qquad (4.6)$$

is satisfied, and consequently

$$\epsilon(\phi) = (\phi, \frac{1}{2}\phi\times\phi, ..., \frac{1}{n!}\phi\times ...\times\phi) \in \oplus\,\mathrm{sym}^i(\Gamma(g^1))\otimes m_n \subset \Gamma(J_n(g^\cdot))\otimes m_n$$

is a hypercocycle, which may be used to define a Dolbeault analogue of v_n (automatically morphic, due to the 'exponential' nature of ϵ).

Step 2: independence of choices

We claim next that the cohomology class

$$[\epsilon(\phi)] \in \mathbb{H}^0(J_n(g^\cdot))\otimes m_n$$

is independent of choices (i.e. of C, ψ_α^n, the covering (U_α) being fixed). This easily implies that the class $[\epsilon(\phi)]$ is canonically associated to the deformation $\mathcal{O}_n$. Clearly all possible ϕ's can be obtained by varying C only, replacing it by $C_1 = C\circ\exp(u)$ where $u \in \Gamma(g^0)$, which leads to

$$\phi_1 = \exp(-u)\bar\partial(\exp(u)) + \exp(-u)\phi\exp(u).$$

Now set

$$\phi_t = \exp(-tu)\bar\partial(\exp(tu)) + \exp(-tu)\phi\exp(tu).$$

Thus we are claiming that the cohomology class $[\epsilon(\phi_t)]$ is constant independent of t. To this end, note that if $g^\cdot$ is endowed with a suitable metric, $\mathbb{H}^0(J_n(g^\cdot))$ inherits a (quotient) topology as cocyles/coboundaries and by standard (real) Hodge theory this topology is *separated* (i.e. coboundaries are closed), hence coincides with the standard Euclidean topology (on a finite-dimensional vector space), and likewise for $\mathbb{H}^0(J_n(g^\cdot)) \otimes m_n$. Consequently, to show constancy it suffices to show that the derivative $d/dt(\epsilon(\phi_t))$ is null-cohomologous. Now we compute that

$$\phi_t' := d/dt(\phi_t) = \bar{\partial}(u) + \mathrm{ad}(u)(\phi_t)$$

and similarly

$$d/dt(\epsilon(\phi_t)) = (\phi_t', \phi_t' \times \phi_t, ..., \frac{1}{(n-1)!}\phi_t' \times \phi_t \times ...\phi_t).$$

As ϕ_t satisfies the integrability condition as in (4.6), a direct computation shows that the latter is the coboundary of

$$(u, u \times \phi_t, ..., \frac{1}{(n-1)!} u \times \phi_t ... \times \phi_t) \in \oplus \Gamma(g^0) \otimes \mathrm{sym}^i \, \Gamma(g^1)$$

(the integrability condition ensures the vanishing of the coboundary's components in $\Gamma(g^0) \otimes \mathrm{sym}^{i-1} \Gamma(g^1) \otimes \Gamma(g^2), i = 1, ..., n-1$.) Thus $[\epsilon(\phi_t)]$ is independent of t as claimed.

Step 3: from cocycles back to deformations

The key to going back is understanding the 'meaning' or interpretation of the tensor ϕ. To this end, note that, as operators,

$$\bar{\partial}(\exp(s_\alpha)) = [\bar{\partial}, \exp(s_\alpha)],$$

therefore clearly

$$\phi. = \exp(-s.)\bar{\partial}(\exp(s.)) - \bar{\partial}. \tag{4.7}$$

What (4.7) means is this: recall the map C above which yields a C^∞ trivialisation of the deformation X_n/R_n and in particular bundle isomorphisms

$$\mathcal{A}^{0,\cdot}(X) \otimes R_n \simeq \mathcal{A}^{0,\cdot}(X_n/R_n)$$

under which the canonical Dolbeault operator $\bar{\partial}_n$ on the RHS corresponds on the LHS precisely to $\bar{\partial}_0 \otimes 1 + \phi$. The integrability equation (4.6) reads, on the operator level

$$\bar{\partial}\phi + \phi\bar{\partial} = \phi\phi, \bar{\partial} := \bar{\partial}_0 \otimes 1,$$

i.e. is equivalent to $\bar{\partial}_n^2 = 0$.

Given this, it is now clear how to go backwards. Given $\phi = \phi_n \in A^{0,1}(T) \otimes m_n$ we may define an operator $d_n = d_n^\phi$ on $\tilde{\mathcal{A}}_n^{0,\cdot} := \mathcal{A}^{0,\cdot}(X) \otimes R_n$ by

$$d_n = \bar{\partial} + \phi,$$

and the integrability equation (4.6) guarantees that $(\tilde{\mathcal{A}}_n^{0,\cdot}, d_n)$ is a complex; by semicontinuity, this complex is clearly exact in positive degrees (because $\tilde{\mathcal{A}}_n^{0,\cdot} \otimes \mathbb{C}$ is) and we may define

$$\mathcal{O}_n = \ker(d_n, \tilde{\mathcal{A}}_n^{0,0}).$$

As d_n is an R_n- linear derivation, $\mathcal{O}_n$ is a sheaf of R_n- algebras. That $\mathcal{O}_n$ is R_n-flat is a consequence of the following easy observation.

Lemma 4.1. *Let R be an artin local ring with residue field k , M an R-module and $M \to N^\cdot$ a flat resolution such that $M \otimes k \to N^\cdot \otimes k$ is also a resolution. Then M is flat.*

Proof. Our assumption implies that $Tor_i(M,k) = 0, i > 0$. Now if P is any finite R-module then P admits a composition series with factors isomorphic to k, hence $Tor_i(M,P) = 0, i > 0$. Finally any R-module Q is a direct limit of its finite submodules and Tor commutes with direct limits, hence $Tor_i(M,Q) = 0$, so M is flat. □

Step 4: descent to cohomology

We claim finally that the isomorphism class of the deformation $\mathcal{O}_n$ is independent of the choice of $\phi = \phi_n$ yielding a given class $[\epsilon(\phi)]$. To prove this it suffices to prove that a first-order deformation of ϕ for which $[\epsilon(\phi)]$ remains constant leads to a family of operators d_n in a fixed conjugacy class (under the action of $\exp(\Gamma(g^0) \otimes m_n))$. Now recall the injection $\mathbb{H}^0(J_n(g^\cdot, m_n^\cdot)) \to \mathbb{H}^0(J_n(g^\cdot)) \otimes m_n$ whose image obviously contains $[\epsilon(\phi)]$; it will suffice to prove the corresponding assertion for $J_n(g^\cdot, m_n^\cdot)$. A deformation as above of ϕ is given by

$$\tilde{\phi} = \phi + \bar{\partial} u_0 + \sum \mathrm{ad}(v_i)(\phi_i) =: \phi + \phi'$$

where u_0, v_i may be taken in $\varepsilon\Gamma(g^0) \otimes m_n$ where $\varepsilon^2 = 0$, and, setting

$$u_1 = \sum v_i \times \phi_i \in \Gamma(g^0) \otimes \Gamma(g^1) \otimes m_n^2,$$

it will suffice to prove that

$$u_1 = u_0 \times \phi \in \varepsilon\Gamma(g^0) \otimes \Gamma(g^1) \otimes m_n^2. \tag{4.8}$$

(Note that conjugation by $\exp(u_0)$ takes d_n to $d_n + \bar{\partial} u_0 + \mathrm{ad}(u_0)(\phi)$, since $\varepsilon^2 = 0$). We will prove by induction on $j \geq 3$ that $u_1 \equiv u_0 \times \phi \mod m_n^j$.

Our assumption that the class $[\epsilon(\tilde{\phi})]$ is constant in $\mathbb{H}^0(J_n(g^\cdot, m_n^\cdot))$ yields a cochain

$$(u_0, u_1, ...u_n), u_i \in \varepsilon\Gamma(g^0) \otimes \mathrm{sym}^i\, \Gamma(g^1) \otimes m_n^{i+1}$$

with coboundary

$$(\phi', \phi' \times \phi, ..., \phi' \times \frac{1}{(n-1)!}\phi^{n-1}).$$

Now by considering the vertical, i.e. $\bar{\partial}-$ coboundary of $u_1 = \sum v_i \times \phi_i$ we see that

$$\sum \bar{\partial} v_i \times \phi_i \equiv \bar{\partial} u \times \phi \mod m_n^3.$$

Hence we can write $u_1 \equiv v \times \phi \mod m_n^3$ with $\bar{\partial} v \equiv \bar{\partial} u_0 \mod m_n^2$. By our assumption that $H^0(T) = 0$ is follows that

$$u_1 \equiv u_0 \times \phi \mod m_n^3.$$

Next, applying a similar argument to u_2 and its vertical (i.e. $\bar{\partial}$-) coboundary, we conclude similarly that

$$u_2 \equiv \frac{1}{2} u_0 \times \phi \times \phi \mod m_n^4.$$

Hence, considering the 'horizontal', i.e. bracket-induced coboundary of u_2 it follows that, in fact,

$$\sum \bar{\partial} v_i \times \phi_i \equiv \bar{\partial} u \times \phi \mod m_n^4$$

as well. As above, it follows that

$$u_1 \equiv u_0 \times \phi \mod m_n^4.$$

Continuing in this way we prove (4.8).

Step5: putting things togther

Now we may easily complete the proof as in Sect.3. First, taking the 'tautological' element $[\epsilon(\phi_n)]$ for $R_n = R_n^u$ corresponding to the identity on $\mathbb{H}^0(J_n)$, we obtain a corresponding deformation X_n^u/R_n^u. Next, given any X_n/R_n, with corresponding $\phi_n^0, \alpha_n^0 = \alpha_n(X_n/R_n)$ (= homomorphism $R_n^u \to R_n$ corresponding to the morphic class $[\epsilon(\phi_n^0)]$), it is clear by construction that ϕ_n^0 coincides with the ϕ associated to the deformation $(\alpha_n^0)^*(X_n^u/R_n^u)$, hence

$$\alpha_n^0 = \alpha_n((\alpha_n^0)^*(X_n^u/R_n^u).$$

Since a deformation is determined by its ϕ_n, hence by its α_n it follows that

$$X_n/R_n \simeq \alpha_n^*(X_n^u/R_n^u).$$

Thus X_n^u/R_n^u is n-universal. Finally it is clear by construction that for different n these are mutually compatible so the limit $\hat{X}_n^u/\hat{R}_n^u$ exists and is formally universal, completing the (Dolbeault) proof of Theorem 0.1. □

Note that in case $\phi = 0$ the above argument becomes simpler and does not require $H^0(T) = 0$, hence it works with our compact X replaced, e.g. by an acyclic open subset U. In particular, since the restriction map

$$\mathbb{H}^0(X < n >, J_n(g^\cdot)) \to \mathbb{H}^0(U < n >, J_n(g^\cdot))$$

vanishes if U is acyclic (provided $H^0(X,T) = 0$), it follows that for any ϕ, the restriction of $\epsilon(\phi)$ on $U < n >$ can be represented by 0, hence the deformation corresponding to ϕ is trivial on U (though, of course, $T(U) \neq 0$). In other words, any flat deformation is automatically locally trivial (of course this is well known anyway).

We now consider a translation of the above proof into the Čech language, and compatibility of the two proofs. To this end, we replace $g^\cdot$ by the Čech complex $\check{C}^\cdot(T)$ which, together with the Čech differential δ and the natural bracket [,] forms a DGLA. By analogy with ϕ_n, we set

$$u. = u_n = \delta(s.) = \exp(-s.)\delta(\exp(s.)) = -\delta(\exp(-s.))\exp(s.).$$

As C is globally defined (albeit nonholomorphic), it commutes with δ hence by (4.3)

$$u_{n\alpha} = (\psi_\alpha^n)^{-1}\delta\psi_\alpha^n$$

so actually $u_n \in \check{C}^1(T)$, i.e. $\bar{\partial}(u_n) = 0$. In particular,

$$\bar{\partial}(\exp(-s.)\delta(\exp(s.)) = -\exp(-s.)\bar{\partial}\delta(\exp(s.).$$

On the other hand $\delta(\phi.) = 0$ yields

$$\delta(\exp(-s.))\bar{\partial}(\exp(s.) = -\exp((-s.)\delta\bar{\partial}(\exp(s.).$$

As δ and $\bar{\partial}$ commute it follows that

$$\delta(\exp(-s.)\bar{\partial}((\exp(s.) = \bar{\partial}((\exp(-s.))\delta(\exp(s.).$$

Hence

$$u.\phi. = \exp(-s.)\delta(\exp(s.)\exp(-s.)\bar{\partial}(\exp(s.) = -\delta(\exp(-s.)\bar{\partial}(\exp(s.))$$

$$= -\bar{\partial}(\exp(-s.))\delta(\exp(s.)) = \phi.u.,$$

i.e. $u., \phi.$ commute:

$$[u., \phi.] = 0 \qquad (4.8)$$

Now as above we have formally that

$$\delta(u.) = \delta(\exp(-s.))\delta(\exp(s.) = \frac{-1}{2}[u., u.],$$

and therefore $v_n = \epsilon(u_n) \in \check{C}^0(J_n(T))$ is a morphic hypercocycle,which may be used to define the required Kodaira-Spencer homomorphism $\alpha_n(X_n/R_n) : R_n^u \to R_n$. That the cohomology class of v_n is independent of choices may be proved in the same way as in the Dolbeault proof (it is also a consequence of the latter).

The interpretation of u_n is analogous to that of ϕ: i.e. the operator

$$\delta + u_n : \check{C}^{\cdot}(\mathcal{O}) \otimes R_n \to \check{C}^{\cdot+1}(\mathcal{O}) \otimes R_n$$

corresponds to the coboundary operator on $\check{C}^{\cdot}(\mathcal{O}_n)$ under the local trivialisation (ψ_α^n) above. Thus to reverse this construction we may proceed analogously as in the Dolbeault case. Firstly we represent a morphic element $v_n \in \mathbb{H}^0(J_n) \otimes m_n$ in the form

$$v_n = \epsilon(u_n), u_n \in \check{C}^1(T) \otimes m_n \tag{4.9}$$

where $u. = u_n$ satisfies the Čech integrability equation

$$\delta(u.) = \frac{-1}{2}[u., u.]. \tag{4.10}$$

Now thanks to (4.9), the deformed coboundary operator

$$\delta' = \delta + u_n : \check{C}^{\cdot}(\mathcal{O}) \otimes R_n \to \check{C}^{\cdot+1}(\mathcal{O}) \otimes R_n$$

satisfies $(\delta')^2 = 0$, thus making $(\check{C}^{\cdot}(\mathcal{O}) \otimes R_n, \delta')$ as well as its sheafy version $(\check{\mathcal{C}}^{\cdot}(\mathcal{O}) \otimes R_n, \delta')$ into complexes where the latter is exact in positive degrees. Hence as before,

$$\mathcal{O}_n = \ker(\delta', \check{\mathcal{C}}^0(\mathcal{O}) \otimes R_n)$$

is a sheaf of flat R_n-algebras yielding a flat deformation

$$\Phi_n(v_n) = X_n/R_n = Specan(\mathcal{O}_n).$$

Now the proof can be completed as in the Dolbeault case.

It is worth noting that the Čech construction yields the same deformation as the Dolbeault one: this follows easily from (4.8). Also, it is clear from the construction that either u_n determines the deformation X_n/R_n up to isomorphism. Since we know from the Dolbeault case that the associated deformation depends only on the cohomology class of $\epsilon(\phi)$, it follows that X_n/R_n depends only on the Čech cohomology class of v_n (or, the latter can also be proved independently in an analogous way).

Remark See [R3] for an 'interpretation' of the construction of X_n^u.

5. Generalizations

Let g be a sheaf of $\mathbb{C}$- Lie algebras on X with $H^0(g) = 0$, E a g-module. Replacing g by its unique quotient acting faithfully on E, we may assume E is faithful. We also assume X, g, E are reasonably tame so cohomology can be computed by Čech complexes. We further assume g and E admit compatible soft resolutions $g^{\cdot}, E^{\cdot}$ where $g^{\cdot}$ is a DGLA acting on $E^{\cdot}$. Typically, E will have some additional structure and g will coincide with the full Lie algebra of infinitesimal automorphisms of the structure: e.g. when E is a ring g may be the algebra of internal derivations. For any artin local $\mathbb{C}$-algebra (R, m)), we have a Lie group sheaf $Aut_R^\circ(E \otimes R)$ of

R-linear automorphisms of $E \otimes R$ which act as the identity on $E = (E \otimes R) \otimes_R \mathbb{C}$, and we assume given a Lie subgroup sheaf

$$G_R \subset Aut_R^\circ(E \otimes R)$$

with Lie algebra $g \otimes m$, which coincides - by definition if you will -with the subgroup of structure-preserving automorphisms in $Aut_R^\circ(E \otimes R)$. For the argument in Step 2 of Sect. 4, we require that $\Gamma(g^\cdot)$ should carry a topology making it a separated topological vector space and inducing a separated topology on cohomology (i.e. coboundaries are closed). Then the above constructions, being essentially formal in nature, carry over to this setting essentially verbatim, yielding n-universal deformations E_n^u/R_n^u, $n \geq 1$, and a formally universal deformation $\hat{E}^u/\hat{R}^u$.

Examples (cf. [R5])

5.*a*. E is a simple locally free finite-rank $\mathcal{O}_X$-module and g is the algebra of all traceless $\mathcal{O}_X$-linear endomorphisms of E. The deformation obtained is the usual universal deformation of E as $\mathcal{O}_X$-module.

Subexamples

5.a_1. $\mathcal{O}_X$ is the ring of locally constant functions on the topological space X assumed 'nice', e.g. a manifold. In this case E is a local system (i.e. a π_1 representation), and we obtain its universal deformation as such.

5.a_2. $\mathcal{O}_X$ is the sheaf of holomorphic functions on a complex manifold (or regular functions on a proper $\mathbb{C}$-scheme). In this case E is a (holomorphic) vector bundle and we obtain its universal formal deformation as such.

5.*b*. Let $Y \subset X$ be an embedding of compact complex manifolds, $g = T_{X/Y}$ the algebra of vector fields on X tangent to Y along Y, which may be identified with the algebra of infinitesimal automorphisms(i.e. internal derivations) of $\mathcal{O}_X$ preserving the subsheaf $\mathcal{I}_y$. Assuming $H^0(T_{X/Y}) = 0$, we obtain the universal deformation of the pair (X, Y).

The case of general holomorphic map $f : Y \to X$ may be treated in a similar way using the algebra T_f (cf. [R2]); in fact it is sufficient for many purposes to replace f by the embedding of its graph in $Y \times X$(cf.[R4]). On the other hand the case of deformations of $Y \to X$ with X fixed requires the DGLA formalism and is considered in [R3].

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