GEOMETRY ON NODAL CURVES

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ABSTRACT. Given a family X/B of nodal curves we construct canonically and compatibly with base-change, via an explicit blow-up of the Cartesian product X^r/B, a family $W^r(X/B)$ that we show isomorphic to the relative flag Hilbert scheme, parametrizing flags of subschemes of fibres of X/B, of colengths $1, ..., r$. Though $W^r(X/B)$ is singular, the important sheaves on it are locally free, which allows us to study some intersection theory on it and deduce enumerative applications, including some relative multiple point formulae, enumerating the length-r schemes contained simultaneously in some fibre of X/B and some fibre of a given map from X to a smooth variety.

One of the important facts which make geometry, in particular enumerative geometry, on a smooth curve X relatively simply is the existence of simple and quite tractable parameter spaces for subschemes of X of given length r, be it the symmetric product $\mathrm{Sym}^r(X)$, which in fact is isomorphic to the Hilbert scheme $\mathrm{Hilb}^r(X)$, or the Cartesian product X^r which parametrizes subshemes in an $(r! : 1)$ fashion, and is sufficient for many applications, especially enumerative ones. One might say that what is essential about these parameter spaces is that they are *enumerative-geometric*, i.e.

(i) *geometric*, which effectively means they admit a morphism to the Hilbert scheme, through which natural geometric loci may be defined by pullback;

(ii) *enumerative*, meaning that the relevant cohomology classes and intersection numbers that may enter into the enumeration of those natural geometric loci are computable.

Mutatis mutandis, similar considerations also hold for *families* of smooth curves. In [R1], the author studied from this viewpoint enumerative projective geometry for families of smooth curves, obtaining, *inter alia,* a general *relative multiple point formula.* This is a formula enumerating the length-r subschemes of the fibres of a given family X/B whose image under a given map

$$f : X \to Y$$

is a single reduced point.

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Incongruously, it seems these ideas and results have yet to be extended to singular curves and families of such. This paper is a step in that direction in the case of nodal curves, i.e. curves with only ordinary double points as singularities. To a family

$$\pi : X \to B$$

of nodal curves and a natural number r we shall associate canonically a family

$$\pi_r : W^r(X/B) \to B$$

which is enumerative-geometric in the above sense, and has a number of other favorable properties. $W^r(X/B)$ is functorial in B, i.e. its formation commutes with base-change; in fact, $W^r(X/B)$ is a canonical and explicit blowup of the Cartesian fibre product X^r/B (more explicitly and directly, of $W^{r-1}(X/B) \times_B X$) in a suitable sheaf of ideals. We will show that $W^r(X/B)$ admits a morphism to the relative Hilbert scheme $\mathrm{Hilb}^r(X/B)$. In fact, we will subsequently show that $W^r(X/B)$ is isomorphic to the relative *flag- Hilbert scheme*, which parametrizes chains

$$z_1 \subset z_2 ... \subset z_r$$

where each z_i is a length-i subscheme of a fibre of X/B. The relatively simple relationship of $W^r(X/B)$ and $W^{r-1}(X/B)$, in the form of the natural blow-up map

$$b_r : W^r(X/B) \to W^{r-1}(X/B) \times_B X$$

makes these spaces more amenable to enumerative computations than the Hilbert scheme itself.

The fact that $W^r(X/B)$ admits a morphism to Hilb implies that, for any vector bundle L on X, $W^r(X/B)$ carries a 'tautological' bundle (also called secant bundle) $S^r(L)$, whose fibre at a point is the restriction of L on the corresponding scheme. We will see that thanks to the simple relationship of $W^r(X/B)$ and $W^{r-1}(X/B)$, $S^r(L)$ can be analyzed conveniently with exact sequences that relate it to $S^{r-1}(L)$. This fact, together with the fact that certain 'diagonal' divisors become Cartier on $W^r(X/B)$ enables us to do some intersection theory on these spaces and apply it to enumerative questions (although the complete intersection theory of the $W^r(X/B)$ is yet to be worked out).

We note that the $W^r(X/B)$, as total spaces, are always singular for $r \geq 3$, as soon as the family X/B has singular fibres. However, when B is smooth, $W^r(X/B)$ is normal and Cohen-Macaulay (cf. Proposition 2.3). In enumerative applications, at least those developed so far, the singular nature of $W^r(X/B)$ is irrelevant; what matters is that they are enumerative-geometric in the sense described above.

The remainder of the paper is largely devoted to multiple-point formulae. In the case of a map f to a Grassmannian, one can define and enumerate a multiple-point scheme $M_r(f)$ as a more-or-less direct consequence of the existence of a tautological bundle on $W^r(X/B)$ (whose fibre at a point is the space of functions, or sections of a vector bundle, on the corresponding subscheme), and its relation to the tautological bundles on the Grassmannian. This result can be extended to the case of a target space which is 'pseudo-Grassmannian' in the sense that its diagonal admits a nice global description as zero-scheme of a vector bundle. In the case of a map to a general (smooth) variety Y, multiple-point schemes $M_r(f)$ can still be defined via

an embedding of Y in a pseudo-Grassmannian (e.g. Projective space), and then enumerated by applying a residual-intersection formula. It can, in fact, be shown that $M_r(f)$ can be defined intrinsically, independently of the pseudo-Grassmannian embedding (hence even for non-projective or non-algebraic Y), but that argument is disproportionately involved considering the paucity of applications, and is not presented here. The version we do present follows a suggestion by the referee.

Perhaps the main question raised and left open by this paper is the full computation of the subring of the cohomology (or Chow) ring of the $W^r(X/B)$ generated by the diagonal divisor classes $[\Delta^k], k = 2, ..., r$, corresponding to the loci where the k-th point coincides with one of the preceding ones. This would allow the explicit evaluation of all the multiple-point loci that we compute, beyond the few fragmentary results given here (generally for $r \leq 3$). However, see [R8, R9] for some recent progress.

The paper is organized as follows. The construction of the parameter spaces $W^r(X/B)$ is begun in §1 where the case $r \leq 3$ is considered in explicit detail. The general case is completed in §2 by an inductive construction, and the relation with the Hilbert scheme is worked out in full. In §3 we give enumerative applications, including several multiple-point formulae for maps to targets of various degrees of generality, and some concrete examples, mainly for maps to $\mathbb{P}^2$ and $\mathbb{P}^3$.

This paper is, in part, a continuation of [R6], where more particular enumerative results were obtained for $r = 2$, in effect by an *ad hoc* version of some of the methods of this paper. We will work over $\mathbb{C}$. However, we don't see any significant obstruction to extending the results over an algebraically closed field of arbitrary characteristic.

I thank the referee for his detailed, constructive suggestions which have improved the paper.

1. Parameter spaces for pairs and triples

Let
$$\pi : X \to B$$

be a flat family of curves over an irreducible variety, with all fibres nodal, i.e. each fibre of π has at worst ordinary double points as singularities. Typically, the generic fibre will be smooth, but we do not assume this. Our purpose is to construct a natural and explicit birational modification $W^r(X/B)$ of the relative cartesian product X^r/B, which will serve as our basic 'configuration space' on which to do enumerative geometry. In this section we begin by considering in explicit detail the case of $r \leq 3$. Subsequently the construction will proceed by induction.

First, we obviously set
$$W^1(X/B) = X/B.$$

Next we consider the easy but significant case $r = 2$. Note that, at a critical point p of π (i.e. a singular point of a fibre $X_b = \pi^{-1}(b)$), our family is formally equivalent to a subvariety of $\mathbb{A}^2 \times B$ given by

$$(1.1) \qquad\qquad\qquad xy = a$$

where x, y are coordinates on $\mathbb{A}^2$ and a is an element of $\hat{\mathfrak{m}}_{B,b}$, which may be also be viewed as a (formal) mapping of B to the base ($= \mathbb{A}^1$) of the versal deformation of a node, pulling back the germ at p of X/B. The analogous statements obtained

by replacing 'formal' by 'complex analytic' also hold, and it is basically a matter of taste whether one uses the formal or analytic setting. Note that X is smooth at p iff $a \notin \hat{\mathfrak{m}}_{B,b}^2$. If B is a smooth curve, a is either zero or may be taken to be of the form $a = t^k$, where t is a local parameter on B and $k \geq 1$ and $k = 1$ iff X is smooth at p. The construction we shall undertake will be manifestly local about critical points such as p, as well as compatible with base-change, therefore for many purposes it will suffice to consider the 'universal' case $xy = t$.

Now consider the fibre square X^2/B and let $D \subset X^2/B$ be the diagonal. Note that D is a Cartier divisor at all points (p, p) such that π is smooth (i.e. submersive) at p. On the other hand, at points (p, p) where p is a critical point, i.e. a fibre singularity, of π, X^2/B is given formally or analytically by

$$(1.2) \qquad\qquad x_1 y_1 = x_2 y_2 = a,$$

and D is given by the 2 equations

$$(1.3) \qquad\qquad x_1 = x_2, y_1 = y_2;$$

moreover if the total space X is smooth, then X^2/B is smooth (over $\mathbb{C}$) except at those points (p, p). We let

$$b_2 : W^2(X/B) \to X^2/B$$

denote the blowup of $D \subset X^2/B$. As D is Cartier away from the singular points (p, p) as above, the blowup coincides with the blowup locally at all those points of the ideal $(x_1 - x_2, y_1 - y_2)$, cf. (1.3). Let

$$\Delta^2 \subset W^2(X/B)$$

be the exceptional divisor, defined by

$$\mathcal{I}_{\Delta^2} = b_2^*(\mathcal{I}_D)$$

(this is not to be confused with the exceptional *locus*, i.e. the locus of all points of $W^2(X/B)$ where b_2 is not an isomorphism locally). Note that the construction of $\Delta^2 \subset W^2(X/B)$ is manifestly canonical and compatible with base-change. To analyze it, it will suffice to consider the situation locally along the exceptional locus of b_2 where we may use formal or analytic coordinates as above.

In terms of these coordinates, over a neighborhood (formal or analytic) of each (p, p), $W^2(X/B)$ is covered by 2 open affines denoted $U_{x,p}^2, U_{y,p}^2$ or just U_x^2, U_y^2 if p is understood. The coordinate ring of $U_{x,p}^2$ is generated over that of X^2/B by a symbol $[(y_2 - y_1)//(x_2 - x_1)]$ subject to the evident relation

$$(x_2 - x_1)[(y_2 - y_1)//(x_2 - x_1)] = y_2 - y_1.$$

Note that
$$x_1(x_2 - x_1)[(y_2 - y_1)//(x_2 - x_1)] = x_1(y_2 - y_1)$$
$$= y_2(x_1 - x_2),$$

hence we may, and shall, write $[(y_2-y_1)//(x_2-x_1)]$ as $-[y_2//x_1]$; similarly, the same may also be written as $-[y_1//x_2]$, therefore when the index range is understood we may write the same as $-[y//x]$. Similar comments apply over the other open $U_{y,p}^2$ where a generator $[x_1//y_2] = [x_2//y_1] = [x//y]$ is defined and of course, on the overlap $U_{x,p}^2 \cap U_{y,p}^2$ we have

$$[y_2//x_1][x_1//y_2] = 1.$$

Henceforth we shall denote $U_{x,p}^2, U_{y,p}^2$ respectively as $U([y//x]), U([x//y])$.

Thus the exceptional locus of b_2 consists of a $\mathbb{P}^1$ over each point (p,p) as above. Moreover it is easy to see that if X is smooth, then so is $W^2(X/B)$: indeed in $U([y//x])$ (resp. $U([x//y])$), a set of coordinates (or a regular system of parameters) is given by x_1, x_2, y_1 (resp. x_2, y_1, y_2). In fact, if X is a smooth surface then by (1.2), (p,p) is just a 3-fold ordinary double point and b_2 is one of its 2 small resolutions. As an aside, it is curious to note that this resolution may be obtained determinantally: i.e. via (1.2) we obtain locally a mapping

$$M : X^2/B \to M_{2\times2}^1,$$

$$M = \begin{bmatrix} x_1 & x_2 \\ y_2 & y_1 \end{bmatrix},$$

to the space of 2×2 matrices of rank ≤ 1 and $W^2(X/B)$ is just obtained by taking fibre product of X^2/B via M with what is known as the 'canonical determinantal resolution' of $M_{2\times2}^1$, defined by

$$R_{2\times2}^1 = \{(A,B) \in M_{2\times2}^1 \times \mathbb{P}^1 : BA = 0\}.$$

Next, we claim that the natural rational map of X^2/B to the Hilbert scheme $\mathrm{Hilb}_2(X/B)$, assigning a pair $p \neq q$ to the ideal $\mathcal{I}_{p,q} < \mathcal{O}_X$ lifts to a *morphism*, i.e. a regular map

$$J_2 : W^2(X/B) \to \mathrm{Hilb}_2(X/B).$$

A priori, J_2 is just a rational map, and as such it is clearly compatible with the respective natural maps of $W^2(X/B)$ and $\mathrm{Hilb}_2(X/B)$ to $\mathrm{Sym}^2(X/B)$ (see [A] for a general construction of the map from the Hilbert scheme to the Chow variety). By, respectively, the projectivity of blow-up morphisms and of Hilbert schemes, these maps are proper (even projective), hence by the GAGA principle it would suffice to check that J_2 exists as a holomorphic map (see Remark 1.1 below for a formal analogue, in arbitrary characteristic, of this argument). Clearly it suffices to check the holomorphic nature of J_2 locally along the exceptional locus. To this end it suffices finally to note that in the open subset $U_{x,p}^2$, J_2 is given explicitly by sending a point with coordinates $(x_1, x_2, [y_2//x_1])$ to the ideal

$$J_2(x_1, x_2, [y_2//x_1]) =$$

(1.4) $$(xy - a, (x - x_1)(x - x_2), y - y_1 + [y_2//x_1](x - x_1))$$

where, of course, we set

$$y_1 = x_2[y_1//x_2], y_2 = x_1[y_2//x_1], a = x_1y_1 = x_2y_2 = x_1x_2[y_2//x_1].$$

Indeed, it is elementary that the RHS of (1.4) defines an ideal of colength 2 whose cosupport contains $(x_1, y_1), (x_2, y_2)$, and therefore this RHS defines a morphism to Hilb that coincides with J_2 generically (e.g. whenever $(x_1, y_1) \neq (x_2, y_2)$). This implies our assertion. The case of $U^2_{y,p}$ is similar.

Remark 1.1. The use of holomorphic coordinates and the GAGA principle to show the regularity of J_2 can be circumvented and replaced by formal coordinates, meaningful in char. p, as follows. First, it is well known and easy to prove that any nodal curve, embedded in $\mathbb{P}^N$ by a complete linear system of sufficiently high degree on each component, yields a smooth (unobstructed) point of the Hilbert scheme of $\mathbb{P}^N$, and consequently any family of nodal curves is obtained by pullback from a family with *smooth* base. Since J_2 (as rational map), its source and target are all base-change compatible, it suffices to check regularity of J_2 in the case of a smooth base B. If B is smooth then $W^2(X/B)$ is normal by Corollary 2.2.1 below. On the other hand, formal-analytic considerations as above show that J_2 exists as a continuous, formally regular map, in the sense that for any point w in $W = W^2(X/B)$ there corresponds a well-defined point $h = J_2(w) \in H =\mathrm{Hilb}_2(X/B)$ and

$$J_2^*(\hat{\mathcal{O}}_{H,h}) \subseteq \hat{\mathcal{O}}_{W,w}$$

where $\hat{\mathcal{O}}$ denotes formal completion. Then for any $g \in \mathcal{O}_{H,h}$, $J_2^*(g)$ is a rational function on W that is also in $\hat{\mathcal{O}}_{W,w}$, and by a well-known property of normal varieties (cf. Mumford, Red Book, ch.3, §9) it follows that $J_2^*(g) \in \mathcal{O}_{H,h}$. Hence J_2 is a morphism.

Yet another approach approach to proving the regularity of J_2, suggested by an anonymous correspondent, is to note that J_2 corresponds to the subscheme $\Gamma^3 \subset W^2 \times_B X$ as in (1.5) below and the regularity of J_2 is equivalent to the flatness of Γ^3 over W^2. By the formal criterion for flatness, this flatness may be checked by passing to formal completions where we have the formal coordinates x_i, y_j as above, and the above computations as in (1.4) prove flatness. □

Next, we will study in detail the case $r = 3$. Let

$$\Gamma^3 \subset W^2 \times_B X$$

denote the pullback of the tautological subscheme of $\mathrm{Hilb}_2(X/B)$ via the map J_2. Note the scheme-theoretic equality

$$(1.5) \qquad \Gamma^3 = \Gamma^3_1 \cup \Gamma^3_2$$

where

$$\Gamma^3_i \subset W^2 \times_B X$$

is the graph of the natural projection $W^2 \to X$: indeed this follows easily from the fact that the LHS of (1.5) is flat over W^2, contains the RHS, the RHS is reduced and both sides agree generically over W^2. In particular, we see that Γ^3 is reduced. We define

$$W^3 = W^3(X/B) := B_{\Gamma^3}(W^2(X/B) \times_B X),$$

i.e. the blowup of $W^2 \times_B X$ in Γ^3, with natural map

$$b_3 : W^3 \to W^2 \times_B X.$$

Clearly W^3 is reduced. Let $\Delta^3 \subset W^3$ be the natural exceptional (Cartier) divisor supported on $b_3^{-1}(\Gamma^3)$, with ideal

$$\mathcal{I}_{\Delta^3} = b_3^*(\mathcal{I}_{\Gamma^3}),$$

and $\Delta_i^3 = b_3^{-1}(\Gamma_i^3), i = 1, 2$ its components (as Weil, in fact nonCartier divisors). From the computations below it will follow that all these divisors are reduced.

To analyze this construction, we work over $U([y//x])$. There, note that the expression $(y_3 - y_1)(x_3 - x_2)$ viewed, e.g. as a function on

$$V := U([y//x]) \times_B U,$$

is divisible by x_2: indeed setting

$$R = y_2 - [y//x]x_3 - y_3 + y_1,$$

it is easy to check that $x_2 R = (y_3 - y_1)(x_3 - x_2)$. I claim next that the ideal of Γ^3 in V is generated by $(x_3 - x_1)(x_3 - x_2)$ and R: indeed the subscheme Γ' defined by the latter projects isomorphically to the subscheme of the 'x-axis' defined by $(x_3 - x_1)(x_3 - x_2)$, hence is flat over W^2, and of course Γ^3 is also flat over W^2. Since Γ^3 and Γ' clearly coincide generically over W^2, they coincide, as claimed.

Thus we see that $b_3^{-1}(V)$ may be realized in the standard way as a subscheme of $V \times \mathbb{P}^1$ and as such is covered by the two standard opens pulled back from $\mathbb{P}^1$. One of these is the domain of regularity of the rational function

$$-[y//x^2] := \frac{R}{(x_3 - x_1)(x_3 - x_2)},$$

and it is easy to check that as rational functions,

$$[y//x^2] = \frac{y_i}{x_j x_k}$$

whenever $\{i, j, k\} = \{1, 2, 3\}$, justifying the notation; we denote this open by $U([y//x^2])$. Note, trivially, that regularity of $[y//x^2]$ implies that of $[y_i//x_j], \forall i, j$, so $U([y//x^2])$ indeed lies over $U([x//y])$. On the other standard open of $V \times \mathbb{P}^1$ the function

$$[x^2//y] = \frac{x_j x_k}{y_i}$$

is regular, however its domain of regularity does not lie entirely over $U([y//x])$ (nor, for that matter, entirely over $U([x//y])$).

Analogous comments apply to the part of W^3 over $U([x//y])$ which gives rise to symbols $[x//y^2], [y^2//x]$. Setting

$$U([y^2//x], [x^2//y]) := U([y^2//x]) \cap U([x^2//y])$$

i.e. the common regularity domain of $[y^2//x]$ and $[x^2//y]$, note by construction that the regularity domains

$$U([y//x^2]), U([y^2//x], [x^2//y]), U([x//y^2])$$

already cover $W^3(X/B)$.

I claim next that the natural rational map J_3 of $W^3(X/B)$ to the relative Hilbert scheme $\mathrm{Hilb}_3(X/B)$ is a morphism. Using again the projectivity of $W^3(X/B)$ and $\mathrm{Hilb}_3(X/B)$ over $\mathrm{Sym}^3(X/B)$ and the GAGA principle, it suffices to check that J_3 extends as a holomorphic map (the 'cycle map' $\mathrm{Hilb}_3(X/B) \to \mathrm{Sym}^3(X/B)$ is constructed in great generality by Angéniol [A], or see [Ko]); alternatively, one could argue as in Remark (1.1). The extension is a local assertion, and is moreover either obvious or a consequence of the analogous result for J_2, except at points of W^3 lying over $(p, p, p) \in X^3/B$, so it suffices to check it on opens

$$U([y//x^2]), U([y^2//x], [x^2//y]), U([x//y^2])$$

as above. Over $U([y//x^2])$, it is easy to see that the expression

$$(y - y_1)(x - x_2)(x - x_3),$$

considered as a function on $W^3 \times_B X$, i.e. taken modulo $xy - x_1 y_1$, is divisible by $x_2 x_3$: explicitly, setting

$$R = ([y_2//x_3] - [y//x^2])x^2 + ([y_1//x_3] + [y_1//x_2])x + y - (y_1 + y_2 + y_3)$$

(recall that $[y_2//x_3] = x_1[y//x^2]$ etc.), we have

$$x_2 x_3 R \equiv (y - y_1)(x - x_2)(x - x_3) \mod xy - x_1 y_1.$$

Then on $U([y//x^2])$, J_3 takes a point with coordinates x_i, y_j to the ideal

$$J_3 = ((x - x_1)(x - x_2)(x - x_3), R, xy - x_1 y_1).$$

Since the latter ideal evidently has colength 3 over W^3, this makes J_3 a morphism over $U([y//x^2])$. The case of $U([x//y^2])$ is similar. Over $U([y^2//x], [x^2/y])$, it is elementary to check as above that, always modulo $xy - x_1 y_1$, $(y - y_1)(y - y_2)(x - x_3)$ is divisible by x_3 and $(x - x_1)(x - x_2)(y - y_3)$ is divisible by y_3 and the ideal

$$J_3 = (\frac{(y - y_1)(y - y_2)(x - x_3)}{x_3}, \frac{(x - x_1)(x - x_2)(y - y_3)}{y_3}, xy - x_1 y_1)$$

has colength 3 over W^3 and yields the map to Hilb (cf. [R7]).

2. Parameter spaces for r-tuples

In this section we will give the construction of our parameter spaces $W^r(X/B)$ for general r, proceeding by induction. It is convenient to summarize the construction and its main properties as follows.

Theorem 2.1. *Define*

$$(2.1) \qquad\qquad b_r : W^r(X/B) \to W^{r-1}(X/B) \times_B X$$

inductively as the blowup of the canonical subscheme

$$(2.2) \qquad\qquad \Gamma^r \subset W^{r-1}(X/B) \times_B X$$

corresponding to the morphism

$$J_{r-1} : W^{r-1}(X/B) \to Hilb_{r-1}(X/B),$$

let $\Delta^r = b_r^{-1}(\Gamma^r)$ be the exceptional divisor and

(2.3) $$w_r : W^r(X/B) \to X^r/B$$

be the natural map. Then

(i) $W^r(X/B)$ *is reduced, and is irreducible if X is; Γ^r is reduced and has $r-1$ irreducible components each isomorphic to W^{r-1}; as functor of the family X/B, $W^r(X/B)$ commutes with base-change;*

(ii) *the natural rational map*

$$J_r : W^r(X/B) \to Hilb_r(X/B)$$

is a morphism;

(iii) *for each node p of X/B, an analytic or formal neighborhood N of $w_r^{-1}(p, ..., p)$ in $W^r = W^r(X/B)$ carries meromorphic or formal-rational functions*

(2.4) $$[y^i//x^{r-i}] = [y^I//x^J], [x^i//y^{r-i}] = [x^I//y^J]$$

where $I \subset [1, r]$ is any index-set of cardinality i and complement J and the domains of regularity

$$U([y//x^{r-1}]), ..., U([y^i//x^{r-i}], [x^{r-i+1}//y^{i-1}]), ..., U([x//y^{r-1}])$$

form a covering of N;

(iv) *setting*

(2.5) $$P_i^r = \prod_{j=1}^{i}(y - y_j) \prod_{j=i+1}^{r} (x - x_j) \in \mathcal{O}_{W^r}[x, y]/(xy - x_1 y_1),$$

over $U([y^i//x^{r-i}])$, P_i^r is divisible by $x_{i+1} \cdots x_r$ and over $U([x^{r-i}//y^i])$, P_i^r is divisible by $y_1 \cdots y_i$; over $U([y^i//x^{r-i}], [x^{r-i+1}//y^{i-1}])$, the map J_r is given by

(2.6) $$J_r = (xy - x_1 y_1, \frac{P_i^r}{x_{i+1} \cdots x_r}, \frac{P_{i-1}^r}{y_1 \cdots y_{i-1}});$$

(v) W^{r+1} *is covered by open sets over each of which either*

(a)$[y^i//x^{r+1-i}]$ *is regular and $\frac{P_{i-1}^r(x_{r+1}, y_{r+1})}{y_1 \cdots y_{i-1}}$ is an equation for Δ^{r+1} (called an 'x−based equation'); or*

(b)$[x^{r-i+1}//y^i])$ *is regular and $\frac{P_i^r(x_{r+1}, y_{r+1})}{x_{i+1} \cdots x_r}$ is an equation for Δ^{r+1} (called a 'y−based equation').*

proof. For $r \le 3$ all of this has already been proven, so we may assume it holds for $r - 1$. First (i) is clear from the fact that W^r is an iterated blowup of the cartesian product X^r/B, while Γ^r coincides with the union of the graphs (over B) of the

coordinate projections $p_i : W^{r-1} \to X$ (proof as in the $r = 3$ case). For the rest, we may as before work over

$$U([y^i//x^{r-1-i}], [x^{r-i}//y^{i-1}]) \subset W^{r-1}$$

so in a suitable open set in $W^{r-1} \times_B X$, the ideal of Γ^r is generated by

$$\frac{P_i^{r-1}}{x_{i+1} \cdots x_{r-1-i}}, \frac{P_{i-1}^{r-1}}{y_1 \cdots y_{i-1}}$$

(where we plug in (x_r, y_r) for (x, y)). Thus the blowup (in the part under consideration) is covered by two standard opens. In the first we have the regular function

$$\frac{P_i^{r-1}}{x_{i+1} \cdots x_{r-1-i}} / \frac{P_{i-1}^{r-1}}{y_1 \cdots y_{i-1}} = \frac{y_r - y_i}{x_r - x_i} \frac{y_1 \cdots y_{i-1}}{x_i \cdots x_{r-1-i}}$$

and it is easy to see as before that this coincides as rational function with

$$-\frac{y_1 \cdots y_i}{x_{i+1} \cdots x_r}$$

and, for that matter, with any $-y^I/x^J$ as in (iii) so we may denote it by

$$-[y^i//x^{r-i}].$$

It is also easy to see as before that this standard open coincides with the regularity domain of this function so we denote it by $U([y^i//x^{r-i}])$. Similarly we get a rational function $[x^{r-i}//y^i]$.

Now we can prove (iii). Given $z \in W^r$, such that $w_r(z)$ is close to $(p, ..., p)$, we may assume z projects to

$$z' \in U([y^i//x^{r-1-i}], [x^{r-i}//y^{i-1}]) \subset W^{r-1},$$

and in particular

$$\frac{y_1 \cdots y_i y_r}{x_{i+1} \cdots x_{r-1}} = [y^{i+1}//x^{r-1-i}],$$

$$\frac{x_i \cdots x_{r-1} x_r}{y_1 \cdots y_{i-1}} = [x^{r-i+1}//y^{i-1}]$$

is regular at z. As we have seen, either

$$[y^i//x^{r-i}] \quad \text{or} \quad [x^{r-i}//y^i]$$

are regular at z. Therefore either

$$z \in U([y^i//x^{r-i}], [x^{r-i+1}//y^{i-1}])$$

or

$$z \in U([x^{r-i}//y^i], [y^{i+1}//x^{r-1-i}]).$$

Thus (iii) is proved.

We will now prove (iv), which of course implies (ii) (our purpose in stating (ii) separately was rhetorical). To begin with, note that the question is local (on W^r, *a fortiori* on X/B) so we may assume X/B is the versal family $xy = t$ over $B = \mathbb{A}^1$ (actually we just need that B is integral). Now note

Claim. *(a) Over $U([y^i//x^{r-i}])$, P_i^r is divisible by $x_{i+1} \cdots x_r$;*
(b) over $U([x^{r-i}//y^i])$, P_i^r is divisible by $y_1 \cdots y_i$.

proof. We prove(a) as the proof of (b) is similar. Now P_i^r (a polynomial in x, y subject to the relation $xy = x_1 y_1$) is a sum of terms of the form

$$M_{i-a}(y) M_{r-i-b}(x) y^a x^b$$

where the M's are monomials in distinct variables $y_1, ..., y_i, x_{i+1}, ..., x_r$ of the indicated degrees. If $a \leq b$, use the relations

$$xy = x_j y_j, \forall j$$

to rewrite this term as

$$x^{b-a} y_1 \cdots y_i M_{r-i-b+a}(x),$$

which is clearly divisible as claimed. If $b \leq a$ this term can be rewritten as

$$y^{a-b} M_{i-a+b}(y) x_{i+1} \cdots x_r,$$

which is even more clearly divisible as claimed. This proves the claim.

Note that the above calculation shows $P_i^r / x_{i+1} \cdots x_r$ and $P_{i-1}^r / y_1 \cdots y_{i-1}$ can be written respectively as

$$y^i + f_1(x) + f_2(y),$$
$$x^{r+1-i} + g_1(x) + g_2(y)$$

where f_1, f_2 have degree $\leq r - i$ and g_1, g_2 have degree $\leq i - 1$ and all have regular functions as coefficients. By the proof of [R7], Theorem 2, we see that for J_r as defined to yield a morphism

$$J_r : W^r(X/B) \to \mathrm{Hilb}^r(X/B)$$

is equivalent to certain identities (4) among the coefficients of the f_i and g_j. Since J_r clearly coincides *generically* with the evident rational map, these identities hold generically, hence they hold period, so J_r is indeed a lifting of the evident rational map as a morphism to Hilb. This completes the proof of (iv). Finally, in light of the fact that

$$\frac{P_{i-1}^r(x_{r+1}, y_{r+1})}{y_1 \cdots y_{i-1}} / \frac{P_i^r(x_{r+1}, y_{r+1})}{x_{i+1} \cdots x_r} = [x^{r-i+1}//y^i],$$

(v) follows directly from (iv) and the definition of blowup (the two opens in question are the standard affine opens of $\mathbb{P}^1$ over $U([y^i//x^{r-i}], [x^{r-i+1}//y^{i-1}])$. This completes the proof of Theorem 2.1.

A posteriori, we can identify $W^r(X/B)$ with the flag-Hilbert scheme studied in [R7]: recall that the flag Hilbert scheme $\mathrm{fHilb}_m(X/B)$ parametrizes $\mathcal{O}_B-$ chains of ideals

$$I_m < ... < I_1 < \mathcal{O}_X$$

where $\mathcal{O}_X/I_j$ is $\mathcal{O}_B-$locally free of length j. Note that the ideal
$$\mathfrak{a}_j = \mathrm{A}nn(I_{j-1}/I_j)$$
is $\mathcal{O}_B-$locally free of colength 1, giving rise to a B-map fHilb$_m(X/B) \to X$. Putting these together, we get a map
$$\mathrm{fHilb}_m(X/B) \to X^m/B$$
Note that the various maps J_j together yield a morphism
$$\eta_r : W^r(X/B) \to \mathrm{fHilb}_r(X/B),$$
which evidently commutes with the natural maps of its source and target to X^r/B. We showed in [R7] that any length-m subscheme of a fibre of X/B, supported at a fibre node with local coordinates as above is either $Q_i^m = (x^{m-1+1}, y^i)$ or $I_i^m(a) = (y^i + ax^{m-i}), a \in \mathbb{C}^*$ and a is part of a local coordinate system on Hilb$_m(X/B)$.

Proposition 2.2. *The map η_r is an isomorphism.*

proof. The proof that follows uses the detailed, local-coordinate description of Hilb given in [R7]. Clearly, both $W^r(X/B)$ and fHilb$_r(X/B)$ are proper, even projective, over the fibre power X^r/B, and fHilb$_r(X/B)$ is reduced by [R7], Theorem 8. Therefore η_r is scheme-theoretically surjective. To show η_r is an isomorphism, it suffices by an obvious induction to prove that its fibre over a flag supported at a point p that is a relative node reduces (scheme-theoretically) to a point. By induction, we may assume η_{r-1} is an isomorphism and in particular fHilb$_r(X/B)$ is a subscheme of $W^{r-1} \times_B \mathrm{Hilb}_r(X/B)$. It then suffices to prove that a fibre F of the natural map
$$\zeta_r : W^r \to W^{r-1} \times_B \mathrm{Hilb}_r(X/B)$$
is a point. Note that, because fHilb$_r(X/B)$ admits an rth projection map to X, F is contained in a fibre of the natural map
$$b_r : W^r \to W^{r-1} \times_B X.$$
Let $w \in W^r$ be 'supported' at a fibre node p (i.e. $w_r(w) = (p, ..., p)$ or equivalently, $J_r(w)$ is supported at p) and write
$$\zeta_r(w) = (w', z).$$
Note also that by construction, if
$$w' \in U([y^i//x^{r-1-i}], [x^{r-i}//y^{i-1}])$$
then a fibre of the map b_r already is coordinatized locally by either
$$[y^i//x^{r-i}] \text{ or } [x^{r-i}//y^i].$$
Assume first that $w \in U([y^i//x^{r-i}])$. Then by [R7], Theorem 2 and the computations in the proof of Theorem 2.1(iv) it is clear that z, the point in Hilb$_r$ corresponding to w is either
$$I_i^r(a), \text{ with } a = [y^i//x^{r-i}]$$
if $a \neq 0$ (note that a is the coefficient of x^{r-i} in $P_i^r/x_{i+1}\cdots x_r$, and it has already been noted that on Hilb$_r$, a is part of a coordinate system), or Q_i^r if
$$[y^i//x^{r-i}] = [x^{r-i+1}//y^{i-1}] = 0.$$
Analogous conclusions hold if $w \in U([x^{r-i}//y^i])$. In either case, the computations show that $[y^i//x^{r-i}]$ or $[x^{r-i}//y^i]$ can be read off from $z \in$ Hilb$_r$, consequently ζ_r is scheme-theoretically injective, hence so is η_r. $\square$

Corollary 2.2.1. *If X/B is flat and B is integral, $W^r(X/B) \to B$ is a flat, locally complete intersection morphism. In particular, if B is Cohen-Macaulay or a locally complete intersection, so is $W^r(X/B)$.*

proof. It suffices to prove this locally on X. But locally, any X/B is induced by base-change from the versal family $X_0/B_0 \subset \mathbb{A}^2 \times \mathbb{A}^1/\mathbb{A}^1$ given by $xy = t$. For this family, the analogous assertions for the flag Hilbert scheme were proven in [R7], Theorem 8, and by base change this implies the general case.

Unfortunately, W^r is never smooth if $r \geq 3$, whenever X/B has a singular fibre, but we still have

Proposition 2.3. *If B is smooth and X/B is flat then $W^r(X/B)$ is normal and Cohen-Macaulay.*

proof. We have essentially seen this already in [R7] in the case of the flag Hilbert scheme, but will give another proof for W^r. We first show inductively that W^r is R_1. For $r = 1$ this is clear (indeed $W^1 = X$ is smooth or has at most rational double points). Inductively, if W^{r-1} is R_1 then clearly so is $W^{r-1} \times_B X$. Moreover the blowup map

$$b_r : W^r \to W^{r-1} \times_B X$$

has at most $\mathbb{P}^1$ fibres and those only over a codimension-3 locus. Hence W^r is R_1.

Now one can see either by applying Corollary 2.2.1 above or imitating the foregoing proof that W^r is CM, hence S2. Hence W^r is normal. $\square$

3. Enumerative results

Now let

$$\Gamma^{r+1} \subset W^r \times_B X$$

be the 'tautological divisor' (=pullback of universal divisor over the Hilbert scheme via J_r). For any sheaf L on X, set

$$(3.1) \qquad\qquad S^r(L) = p_{W^r *}(p_X^*(L) \otimes \mathcal{O}_{\Gamma^{r+1}}).$$

If L is locally free, $S^r(L)$ is clearly locally free of rank $r.\mathrm{rk}(L)$, and we call it the $r-$th *secant bundle* associated to L. It was introduced in the smooth case by Schwarzenberger [Sc]. Because Γ^{r+1} is the pullback of the universal divisor on $\mathrm{Hilb}_r(X/B) \times_B X$, $S^r(L)$ is the pullback of the analogous bundle on $\mathrm{Hilb}_r(X/B)$. However the recursive structure of W^r makes $S^r(L)$ easier to compute with, as we proceed to show.

So start with, note trivially that by flatness of Γ^{r+1}/W^r, S^r is an exact functor from coherent (resp. locally free coherent) sheaves on X to coherent (resp. locally free coherent) sheaves on W^r. Also, if V is a vector space and $\phi : V \otimes \mathcal{O}_X \to L$ is a map, ϕ lifts naturally to a map $V \otimes \mathcal{O}_{W^r \times_B X} \to p_X^*(L)$ whence a map

$$\phi^r : V \otimes \mathcal{O}_{W^r} \to S^r(L)$$

and clearly ϕ^r is surjective is ϕ is. Next, note that $W^r(X/B)$ is not symmetric with respect to permuting the factors, but still there are projections 'to the first s factors', for all $s \leq r$:

$$\gamma^{r,s} : W^r \to W^s.$$

We also set

$$\gamma^r = \gamma^{r,r-1}.$$

Also, denote by

$$\Delta^r \subset W^r$$

be the exceptional divisor of b_r, i.e. $b_r^*(\Gamma^r)$, which is by definition of blowup a Cartier divisor. As in Theorem 2.1, we have for any r that Γ^r splits up as

$$\Gamma^r = \bigcup_{i=1}^{r-1} \Gamma_i^r$$

with each Γ_i^r, being the graph of the $i-$th coordinate projection to X, is isomorphic to W^{r-1} and in particular is reduced always and irreducible if X is. Similarly Δ^r splits

$$\Delta^r = \bigcup_{i-1}^{r-1} \Delta_i^r$$

with each Δ_i^r reduced and birational to Γ_i^r (and in general a non-Cartier divisor on W^r).

Proposition 3.1. *Let $p_i : W^r(X/B) \to X$ denote the ith natural projection for $1 \le i \le r$. There is an exact sequence of vector bundles on $W^r(X/B)$*

$$(3.2) \qquad 0 \to p_r^*(L) \otimes \mathcal{O}(-\Delta^r) \to S^r(L) \to \gamma^{r*}(S^{r-1}(L)) \to 0$$

proof. There is clearly a natural surjection

$$S^r(L) \to \gamma^{r*}(S^{r-1}(L))$$

whose kernel K is locally free of rank $\mathrm{rk}(L)$ and moreover admits a generically injective map

$$k : K \to p_r^*(L).$$

Clearly, k vanishes at each generic point of Δ^r, hence factors through $p_r^*(L) \otimes \mathcal{O}(-\Delta^r)$. It is easy to see that the factored map

$$K \to p_r^*(L) \otimes \mathcal{O}(-\Delta^r)$$

is an isomorphism in codimension 1 between locally free sheaves of the same rank, hence an isomorphism since W^r is pure-dimensional. $\square$

Thus $S^r(L)$ has a natural filtration with quotients

$$(\gamma^{r,s})^*(p_s^*(L) \otimes \mathcal{O}(-\Delta^s)),$$

and consequently we have

Corollary 3.2. *The total Chern class $c(S^r(L))$ satisfies*

$$c(S^r(L)) = \prod_{i=1}^{r} c(L_i(-(\gamma^{r,i})^* \Delta^i))$$

where L_i denotes (the class of) $p_i^*(L)$ and $\Delta^1 = 0$. In particular, if L is invertible, we have

$$(3.3) \qquad c(S^r(L)) = \prod_{i=1}^{r} (1 + L_i - (\gamma^{r,i})^* \Delta^i) \quad \square$$

After this was written, the author became aware of the work of Lehn [L], which computes the Chern classes of the analogue of the $S^r(L)$ over Hilbert schemes of smooth surfaces, in terms of Nakajima's theory of the cohomology of these schemes. See [R8] for further discussion the the connection and for a verification that Corollary 3.2 is consistent with Lehn's formula.

In a nutshell, it is the possibility of results like the foregoing Corollary that make the spaces W^r amenable to enumerative computations. As a first application, we use this result to give a *multiple-point formula* for maps to a Grassmannian. Let X/B be as above and

$$f : X \to G$$

be a map to a Grassmannian $G = G(k, m + k) = G(k, V)$, with tautological sub- and quotient bundles S_G, Q_G respectively. Set

$$S_X = f^* S_G, Q_X = f^* Q_G.$$

Note that on $G \times W^r(X/B)$ we have a natural map

$$\psi : p_G^*(S_G) \to p_W^* S^r(Q_X)$$

which is the composite of the obvious inclusion $p_G^*(S_G) \subset V \otimes \mathcal{O}$ with the natural surjection $\phi^r : V \otimes \mathcal{O} \to p_W^* S^r(Q_X)$ deduced from the tautological quotient $\phi : V \otimes \mathcal{O} \to Q_X$. Put another way, ψ is obtained from the diagram

$$(3.4) \qquad \begin{array}{c} G \times \Gamma^{r+1} \quad \overset{1_G \times p_X}{\longrightarrow} \quad G \times X \quad \overset{1_G \times f}{\longrightarrow} \quad G \times G \\ 1_G \times p_{W^r} \downarrow \\ G \times W^r \end{array}$$

by pulling back the equation of the diagonal in $G \times G$ to $G \times \Gamma^{r+1}$ and pushing down to $G \times W^r$. Thus, ψ vanishes at a point (w, g) iff all the elements of the subspace $g \subset V$ vanish in $Q_X \otimes \mathcal{O}_{J_r(w)}$, in other words, iff f maps the scheme $J_r(w)$ corresponding to w to the reduced point scheme g. We call the latter locus $M_r^+(f)$; by definition, it is a subscheme of $G \times W^r$. We may replace $M_r^+(f)$ by a subscheme of W^r, as follows. Define a bundle $S^{r,1}(L)$, for any L, by the exact sequence

$$0 \to S^{r,1}(L) \to S^r(L) \overset{\alpha}{\to} (\gamma^{r,1})^*(L) \to 0.$$

Then clearly the zero-scheme of the composite map

$$\alpha \circ \psi : p_G^*(S_G) \to p_W^*(\gamma^{r,1})^*(Q_X)$$

is just the graph R of the composite

$$f \circ p_1 : W^r \to G.$$

Thus, $M_r^+(f)$ is a subscheme of R, and as such, it is the zero-scheme of a map $p_G^*(S_G) \to p_W^* S^{r,1}(Q)$. Identifying R with W^r by projection, we see that $M_r^+(f)$ projects isomorphically to its image $M_r(f)$ in W^r and $M_r(f)$ is the zero-scheme of an analogous map

$$p_1^*(S_X) \to S^{r,1}(Q_X).$$

Then we have shown the following.

Theorem 3.3. *In the above situation, $M_r^+(f)$ and $M_r(f)$ have a natural scheme structure as zero-scheme of vector bundle maps. When $M_r^+(f)$ and $M_r(f)$ have their expected codimension, i.e.*

$$codim(M_r^+(f), W^r \times G) = rkm,$$

or equivalently

$$codim(M_r(f), W^r) = (r-1)km,$$

then the cohomology (or rational equivalence) class of $M_r(f)$ on W is

$$(3.5) \qquad [M_r(f)] = c_{(r-1)km}(p_1^*(S_X^*) \otimes S^{r,1}(Q_X)).$$

In case $G = \mathbb{P}^m = G(m, m+1)$, the formula (3.4) simplifies somewhat. Let us write

$$L_i = (f \circ p_i)^*(\mathcal{O}_G(1)), M_1 = (f \circ p_1)^*(S_X).$$

Then we get

$$[M_r(f)] = \prod_{i=2}^{r} c_m(M_1^* \otimes (L_i - (\gamma^{r,i})^*(\Delta^i)))$$

$$(3.6) \qquad = \prod_{i=2}^{r} (\sum_{j=0}^{m} L_1^j (L_i - (\gamma^{r,i})^*(\Delta^i))^{m-j}).$$

Consider now the case $m = 2, r = 3$. Then the RHS of (3.5) yields

$$(3.7) \qquad (L_1^2 + L_1(L_2 - \Delta^2) + (L_2 - \Delta^2)^2)(L_1^2 + L_1(L_3 - \Delta^3) + (L_3 - \Delta^3)^2).$$

We want to compute the image of this on $X = W^1$. To this end we must first compute the image on the second factor on W^2 via $\gamma^{3,2}$. This computation follows formally from the following.

Lemma 3.4. *Set*

$$f_0^i = \text{ fibre of } W^i(X/B) \text{ over } 0 \in B,$$

$$d = \deg(f_0^1) = f_0^1.L, b = \pi_* L^2.$$

Then we have

(i)
$$\gamma_*^{32}(L_1^2) = 0$$

(ii)
$$\gamma_*^{32}(L_1 L_3) = dL_1$$

(iii)
$$\gamma_*^{32}(L_1 \Delta^3) = 2L_1$$

(iv)
$$\gamma_*^{3,2}(L_3^2) = \pi_2^*(b)$$

(v)
$$\gamma_*^{3,2}(L_3 \Delta^3) = L_1 + L_2$$

(vi)
$$\gamma_*^{3,2}((\Delta^3)^2) = -K_1 - K_2 + 2\Delta^2$$

where $K_i = p_i^*(K_{X/B})$

proof. Assertions (i), (ii) and (iv) are obvious. Assertions (iii) and (v) are immediate from the fact that Δ^3 has 2 components mapping birationally to W^2. To prove (vi) we may work off the (codimension-2) exceptional locus of the natural birational map

$$W^2 \to X^2/B$$

and its inverse image in W^3; on this open set, Δ^3 consists of 2 components Δ_1^3, Δ_2^3, each a pullback of the diagonal via the p_{13}, p_{23} projections, which meet in a locus projecting isomorphically to $\Delta^2 \subset W^2$.

Remark 3.4.1. Actually analogues of formulae (i-vi) hold for any $\gamma^{r.r-1}, r \geq 3$ in place of $\gamma^{3,2}$, where the analogue of (vi) is

$$(\gamma^{r,r-1})_*((\Delta^r)^2) = -\sum_{i=1}^{r-1} K_i + 2\sum_{i=2}^{r-1}(\gamma^{r-1,i})^*(\Delta^i)$$

where we set $\gamma^{r-1,r-1} =$identity. All these formulae are but the tip of a sizable iceberg, that it explored more deeply in [R9].

Given formulae 3.4(i-vi), an elementary formal calculation yields

$$\gamma_*^{3,2}((L_1^2 + L_1(L_3 - \Delta^3) + (L_3 - \Delta^3)^2))$$

$$= (d-4)L_1 - 2L_2 - K_1 - K_2 + 2\Delta^2 + \pi_2^*(b)$$

Therefore by the projection formula the image of (2.12) via $\gamma^{3,2}$ is

$$(L_1^2 + L_1(L_2 - \Delta^2) + (L_2 - \Delta^2)^2).$$

(3.8) $((d-4)L_1 - 2L_2 - K_1 - K_2 + 2\Delta^2 + \pi_2^*(b))$

In the computation of the product in (3.8) and its image in X, the main point is the following. Observe that the normal bundle to Δ^2 in W^2 is just $-b_2^* K_1 + E$ where

$$b_2 : W^2 \to X^2/B$$

is the natural blowup map and E its exceptional locus, which is a divisor on Δ^2. Indeed the restriction of b_2 over the diagonal

$$\Delta_X = X \subset X^2/B$$

is just the blowup of the critical locus of π, which we denote by σ (cf. [R6]). Therefore

$$\gamma_*^{2,1}(\Delta^2)^2 = -K,$$
$$\gamma_*^{2,1}(\Delta^2)^3 = \gamma_*^{2,1}(K-E)^2 = K^2 - \sigma.$$

From these facts the computation of (3.8) and its image in X are routine.

Now by construction $M_3(f)$ parametrizes filtered length-3 schemes contained in fibres of f, and the filtration induces an ordering on the support. Therefore, the image on X of $M_3(f)$ is geometrically twice the locus of points contained in a relative triple point of f, while the image on B of the same is 6=3! times the locus of fibres containing a triple point. Thus writing out the product yields the following result. As a matter of terminology, we will say that a locus Z has 'virtual class z' if whenever Z has its expected dimension, then its cohomology or rational equivalence class is given by z.

Theorem 3.5. *Let $\pi : X \to B$ be a family of connected nodal curves of arithmetic genus g and $f : X \to \mathbb{P}^2$ a morphism. Then the virtual class on X of the locus of points contained in a relative triple point of f is*

(3.9) $N_{3,X}(f) = \dfrac{1}{2}((3d^2 - 18d + 24 + 6g - 6)L^2 + (18 - 3d)KL + 4K^2 - 2\sigma);$

the locus in B of fibres containing a relative triple point of f has virtual class

$$N_{3,B}(f) = \frac{1}{3}\pi_*(N_{3,X}(f)).$$

Here $L = f^*\mathcal{O}(1), K = \omega_{X/B}, d = \deg(f(\pi^{-1}(pt.))), \sigma = $ class of critical locus of f.

Let us finally specialize to the case where X/B is the normalization of the pencil (i.e. 1-parameter family) of rational curves in $\mathbb{P}^2$ through $3d - 2$ assigned generic points. In this case all the ingredients of (3.9) have been computed recursively before, e.g. in [R2-R5]; the needed results from these papers are summarized in [RA]. We have $L^2 = N_d$, the number of rational curves of degree d through $3d - 1$ generic points,

$$K = -2s_1 - m_d f_0^1 + R_1$$

(cf. [RA],(11)) where s_1 is a section of X/B contracted by f to a point (viz. one of the base points of the pencil), R_1 is the sum of all fibre components disjoint from s_1, and $m_d = -s_1^2$ is given by [RA],(5). Hence

$$L.K = -dm_d + L.R_1, K^2 = -\sigma$$

(cf. [RA], (14),(15)) and σ coincides with the number of reducible fibres of X/B, also recursively computed. We conclude

Corollary 3.6. *The number of rational curves of degree d in $\mathbb{P}^2$ having a triple point and going through $3d - 2$ generic points is (cf.[R6])*

$$(3.10) \qquad N_{d,3} = \frac{1}{2}((d^2 - 6d + 10)N_d + (d - 6)(dm_d - L.R_1) - 2\sigma)$$

Example 3.6.1. $N_{4,3} = 60$, a number first computed by Zeuthen and rederived with modern methods by Kleiman and Piene [KP] (I am grateful to Steve Kleiman for this reference). See [KP,R6] for some similar examples.

When f is a map to $\mathbb{P}^m$, one is interested classically not only in the relative multiple-point loci of f but also in its relative *multisecant loci*, that is the locus of length-r subschemes of fibres whose $f-$image is contained in a linear $\mathbb{P}^k$. This locus can be enumerated by the above results, as the $r-$fold locus of the natural projection

$$I_X \to \mathbb{G}(k, m) := G$$

where I_X is the incidence variety, i.e.

$$I_X = \{(x, L) : f(x) \in L\} \subset X \times G.$$

But it is simpler and more direct to enumerate this locus as follows. Set

$$(3.11) \qquad \mathrm{Sec}_k^r(f) = \{(z, L) : f(\mathrm{sch}(z)) \subset L \ \text{as schemes}\} \subset W^r(X/B) \times G$$

where $\mathrm{sch}(z)$ is the subscheme of X corresponding to z. Clearly $\mathrm{Sec}_k^r(f)$ is just the zero- scheme of the natural map

$$p_2^*(Q_G^*) \to S^r(L)$$

where Q_G is the tautological quotient bundle (of rank $m - k$) on G and $L = f^*(\mathcal{O}(1))$. Thus we conclude

Theorem 3.7. *For a family of nodal curves X/B and a morphism*

$$f : X \to \mathbb{P}^m,$$

the virtual locus on $W^r(X/B) \times G$ of relatively $r-$secant k-planes to X/B in $\mathbb{P}^m$ is given by

$$(3.12) \qquad [Sec_k^r(f)] = c_{r(m-k)}(p_1^*(S^r(L)) \otimes p_2^*(Q_G)).$$

If the RHS of (3.12) is nonzero (resp. not representable by an effective cycle), then the locus of relative $r-$secant $k-$planes is nonempty (resp. of dimension larger than the expected, viz. $\dim(B) + r + (k + r + 2)(m - k)$). $\square$

Note that the projection of $\mathrm{Sec}_k^r(f)$ to $W^r(X/B)$ coincides with the locus where the natural map

$$H^0(\mathcal{O}_{\mathbb{P}^m}(1)) \otimes \mathcal{O}_{W^r(X/B)} \to S^r(L)$$

has rank at most $k + 1$, and consequently can be enumerated directly via Porteous' formula [F].

Now Theorem 3.3 above 'repackages' the multiple-point locus $M_r^+(f)$ in a certain way, and it is possible to repackage differently; the alternative repackaging is useful, notably, for recursion, and also allows some more general, 'pseudo Grassmannian' target spaces. We proceed to define these.

Let us say that a smooth m-dimensional variety G is *pseudo- Grassmannian* (with bundle E and section ψ, if these need be specified) if E is a rank-m vector bundle on $G \times G$ and the diagonal

$$\Delta_G \subset G \times G$$

is the zero-scheme of the section ψ of E.

Examples. (i) Clearly a Grassmannian has this property, with

$$E = p_1^* S^* \otimes p_2^* Q$$

where S and Q are respectively the tautological sub- and quotient bundles.

(ii) Trivially, any curve is pseudo-Grassmannian

(iii) Generally, a product of pseudo-Grassmannians is pseudo-Grassmannian, therefore any product of curves and Grassmannians is pseudo-Grassmannian.

Now suppose we have a morphism $f : X \to G$ to a pseudo-Grassmannian with bundle E and section ψ. Then we get a diagram as in (3.4), and pulling back and pushing forward E, ψ we get a bundle E^r with section ψ^r on $G \times W^r$, and we define $M_r^+(f)$ as the zero scheme of ψ^r. As in the proof of Proposition 3.1, we have an exact sequence

$$0 \to (p_G \times (f \circ p_r))^*(E)(-\Delta^r) \to E^r \to (\gamma^{r,r-1})^* E^{r-1} \to 0.$$

Consequently, $M_r^+(f)$ is a subscheme of $\gamma^{r,r-1*}(M_{r-1}^+(f))$ and as such is a zero scheme of $(p_G \times (f \circ p_r))^*(E)(-\Delta^r)$. As in the Grassmannian case, the fact that $M_r^+(f)$ is contained in $(\gamma^{r,1})^* M_1^+(f)$, where $M_1^+(f)$ is just the graph of f, shows that $M_r^+(f)$ is isomorphic to its image $M_r(f)$ on W^r and that as subscheme of $M_{r-1}(f)$, $M_r(f)$ is a zero scheme of $((f \circ p_1) \times (f \circ p_r))^*(E)(-\Delta^r)$.

Theorem 3.3 bis. *In the situation of Theorem 3.3, assume only that G is pseudo-Grassmannian with bundle E. Then*
(i) $M_r(f)$ is a zero-scheme on W^r of

$$(3.13) \qquad \bigoplus_{i=2}^r ((f \circ p_1) \times (f \circ p_i))^*(E)(-(\gamma^{r,i})^* \Delta^i)$$

(ii) if B is irreducible, $M_r(f)$ is locally defined by $(r-1)m$ equations on $W^r(X/B)$ hence is purely at least $(\dim(B) + r - (r-1)m)-$ dimensional and if equality holds then

$$(3.14) \qquad [M_r(f)] = \prod_{i=2}^r ((f \circ p_1) \times (f \circ p_i))^*(c_m(E(-(\gamma^{r,i})^* \Delta^i))).$$

proof. (i) has been proved above. In (ii), the assertion about the number of equations is clear from the definition. As for the assertion about the cohomology class,

working by induction on r, it is clear from the Fulton-MacPherson residual- intersection formula [F] provided both M_r and M_{r-1} have their expected dimensions. In the general case, let $C_1, ..., C_k$ be the irreducible components of M_{r-1}. By Fulton's theory, there is a cycle U_i of dimension $\dim(B) + r - 1 - (r-2)m$ on each C_i such that

$$\sum [U_i] = m_{r-1}.$$

Since M_r is locally defined by m equations over $(\gamma^r)^{-1}(M_{r-1})$ but still has its expected dimension, it follows that the contribution of each oversize component C_i to M_r is empty, and in particular

$$(\gamma^r)^*(U_i).\mu_r(f) = 0.$$

So these oversize components contribute nothing to either M_r or m_r, so (3.14) still holds. $\square$

Though in most classical application the target is in fact pseudo-Grassmannian, it is worth noting that Theorem 3.3 bis can be extended to mappings with target an arbitrary smooth m-dimensional variety Y embedded in a pseudo-Grassmannian G with bundle E. Note that any projective variety admits such an embedding (e.g. with G a projective space). Thus, let

$$f : X \to Y$$

be a mapping to a smooth variety and fix an embedding

$$Y \subset G$$

in a pseudo-Grassmannian. Of course, the diagonal $\Delta_Y \subset Y \times Y$ is a zero-scheme, albeit not of the expected dimension, of $E \otimes \mathcal{O}_{Y \times Y}$. Let $f' : X \to G$ be the induced map. We set

$$M_r(f) = M_r(f').$$

In fact, we will show that $M_r(f)$ depends only on f and not the embedding $Y \subset G$. As noted above, as subscheme of $(\gamma^{r,r-1})^{-1}(M_{r-1}(f))$, $M_r(f)$ is a zero scheme of a section of $((f' \circ p_1) \times (f' \circ p_r))^*(E)(-\Delta^r)$ induced by a section ψ of E whose zero scheme is Δ_G. The same section yields a section of $((f' \circ p_1) \times (f' \circ p_r))^*(E)$, whose zero-scheme consists of Δ^r plus

$$((f \circ p_1) \times (f \circ p_r))^*(\Delta_G \cap Y \times Y) = ((f \circ p_1) \times (f \circ p_r))^*(\Delta_Y)$$

Put another way, $M_r(f)$ is the residual scheme to Δ^r in $((f \circ p_1) \times (f \circ p_r))^*(\Delta_Y)$. This first shows that $M_r(f)$ is independent of the choice of embedding $Y \subset G$. Next, it allows us to compute $M_r(f)$ using residual-intersection theory. To that end, set for $k \geq 2$

$$(3.15) \qquad \mu_k(f) = ((f \circ p_1) \times (f \circ p_k))^*(\Delta_Y) - \Delta^k \{ \frac{(f \circ p_1)^*(c(T_Y))}{1 + \Delta^k} \}_{m-1}$$

Also let

$$(3.16) \qquad m_r(f) = \prod_{k=2}^{r} (\gamma^{r,k})^*(\mu_k(f))$$

where

$$\gamma^{r,k} : W^r(X/B) \to W^k(X/B)$$

is the natural map. The residual intersection formula of [F], §9.2 yields that whenever $M_r(f)$ has the expected codimension, viz. m, in $(\gamma^{r,r-1})^{-1}(M_{r-1}(f))$, then its class is given by $\mu_r(f)$. Thus:

Theorem 3.3 ter. *Let X/B be a family of nodal curves and*

$$f : X \to Y$$

a morphism to a smooth m-dimensional variety embeddable in a pseudo-Grassmannian G. Then

(i) there is a natural scheme structure $M_r(f)$ on the locus in $W^r(X/B)$ of points whose associated scheme is mapped by f to a reduced point;

(ii) if B is irreducible, $M_r(f)$ is locally defined by $(r-1)m$ equations on $W^r(X/B)$ hence is purely at least $(\dim(B)+r-(r-1)m)-$ dimensional and if equality holds then

(3.17) $$[M_r(f)] = m_r(f).$$

proof. Part (i) has been proved above and the proof of (ii) is essentially identical to that of Theorem 3.3 bis, (ii). Note that the local defining equations for $M_r(f)$ in $(\gamma^{r,r-1})^{-1}(M_{r-1}(f))$ arise by pulling back equations for Δ_Y (or what is the same, for Δ_G), and factoring out the equation of Δ^r.   □

Remark. An earlier version of this paper gave a more general version of Theorem 3.3 ter, not assuming any embedding of Y, and with a more complicated proof. The foregoing argument is due to the referee. The general idea of defining a multiple-point locus recursively as a residual scheme inside a diagonal pullback is old folklore. Having the secant bundles is what makes it work.

Corollary 3.8. *In the situation of Theorem 3.3 ter, if $m_r(f) \neq 0$ then $M_r(f)$ is nonempty.*

Consider the case $r = 2, m = 3$. Thus we have a family of nodal curves mapping to Y and are enumerating the relative multiple points of their images in Y (at least if we assume that a general fibre of X/B is smooth and embeds in Y and that every fibre maps in with degree 1). Then it is easy to see that (3.13) or (3.15) specializes to (writing $f_i = f \circ p_i$):

(3.18) $$[M_2(f)] = (f_1 \times f_2)^*(\Delta_Y) - ((\Delta^2)^3 - (\Delta^2)^2 f_1^* K_Y + \Delta^2 f_1^* c_2(Y)).$$

By the calculations in [R6], we have as in the proof of Corollary 3.6,

$$p_{1*}(\Delta^2)^3 = K^2 - \sigma,$$

$$p_{1*}((\Delta^2)^2 f_1^* K_Y) = -K.f^*(K_Y), \; p_{1*}(\Delta^2 f_1^* c_2(Y)) = f^*(c_2(Y))$$

where as before $K = \omega_{X/B}$ and σ is the critical locus of π. Thus we obtain

Corollary 3.9. *For a family of nodal curves X/B mapping via f to a smooth pseudo-Grassmannian 3-fold Y, the virtual locus on X of relative double points of f is*

(3.19) $$[N_{2,X}(f)] = p_{1*}(f_1 \times f_2)^*(\Delta_Y) - (K^2 - \sigma + K.f^* K_Y + f^* c_2(Y))$$

The expression $p_{1*}(f_1 \times f_2)^*(\Delta_Y)$ (which is a 0-cycle or just a number if B is 1-dimensional) may be evaluated in various ways. For example, working in singular

cohomology over $\mathbb{C}$, let (α_i) be a homogeneous basis for the total cohomology $H^*(B)$ and let (α_i^*) be the dual basis. Then the class of the diagonal Δ_B in $B \times B$ is given by

$$[\Delta_B] = \sum \alpha_i \otimes \alpha_i^*$$

setting

$$\beta_i = f_*(\pi^*(\alpha_i)), \beta_i^* = f_*(\pi^*(\alpha_i^*))$$

we have

$$(3.20) \qquad f_*(p_{1*}(f_1 \times f_2)^*(\Delta_Y)) = \sum \beta_i . \beta_i^*$$

(which coincides with $p_{1*}(f_1 \times f_2)^*(\Delta_Y)$ or $(f_1 \times f_2)^*(\Delta_Y)$ when they are of top degree, i.e. numbers. Note also that when B is a curve, we have

$$[\Delta_B] \equiv [B] \otimes [pt] + [pt] \otimes B \mod H^1 \otimes H^1$$

so if $H^3(Y) = 0$ then the first term in (3.19) reduces to $2f_*([f_0]).f_*([X])$. Finally note that one customarily denotes

$$\pi_*(K^2) = \kappa, \pi_*(\sigma) = \delta.$$

Thus we have

Corollary 3.10. *In the above situation, suppose*
(i) a generic fibre of X/B is smooth and embedded via f;
(ii)$\dim(B) = 1$;
(iii) $H^3(Y) = 0$.
Then the virtual number of double of relative double points of f is given by

$$(3.21) \qquad n_2(f) = \frac{1}{2}(2f_*([f_0]).f_*([X]) - \kappa + \delta - K.f^*K_Y - f^*c_2(Y)).$$

In particular, if $n_2(f) \neq 0$ then f does not embed all fibres of X/B and if $n_2(f) < 0$ then f has degree > 1 on some fibre.

We note that if $Y = \mathbb{P}^3$, (3.13) simplifies to

$$(3.21\text{bis}) \qquad n_2(f) = \frac{1}{2}(2bd - 6b - 4L.K - K^2)$$

where as usual $L = f^*\mathcal{O}(1), d = L.\pi^{-1}(pt.), b = L^2$. This formula was first given in [CR] where it was derived from a general double-point formula referred to Fulton's book [F] (in fact, the book does not appear to contain such a formula explicitly in this generality, though it should be possible to derive one from the case treated there, due to the fact that any double point is automatically curvilinear). As shown in [CR], Corollary 3.10 has the following geometric consequence

Corollary 3.11. *There is no nontrivial family of nonsingular rational curves of degree $d \geq 3$ in $\mathbb{P}^3$ parametrized by an irreducible projective variety of positive dimension.*

proof. We reproduce the short argument from [CR]. If the assertion fails, there is a family as above with X/B a $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ where E is a rank-2 vector bundle over a smooth curve B and f restricted on each fibre is an embedding in $\mathbb{P}^3$, so that $n_2(f) = 0$, and moreover $f(X)$ is 2-dimensional, so that $b = L^2 > 0$. With no loss of generality, one may assume $c_1(E) = 0$ or -1. Set $D = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. If $c_1(E) = 0$, we have

$$K = -2D, K^2 = 0,$$

and we can write

$$L = dD + xF, F = \pi^{-1}(pt.), x \in \mathbb{Z}.$$

Since $b = L^2 > 0$, we have $x > 0$, hence

$$L.K = -2x < 0.$$

Since $d \geq 3$, (3.21bis) yields a contradiction. If $c_1(E) = -1$, we can write, with similar notations

$$K = -2D + F, D^2 = 1$$

so

$$K^2 = 0, L.K = -d - 2x < 0,$$

so again (3.18bis) yields a contradiction. $\square$

As another special case of Corollary 3.10, we recover a result from [R6]. We use the notation developed in [RA]; in particular $N_d^{\mathrm{red}}(a.)$ denotes the number of *reducible* rational curves of degree d in $\mathbb{P}^3$ satisfying the incidence conditions indicated by $(a.)$ and $m_1 = -s_1^2$ where s_1 is the section of X/B corresponding to an incident linear subspace of codimension a_1. Both these numbers are recursively computable. See the [RA] for more details.

Corollary 3.12. *With the notations of* [RA], *the number of singular rational curves of degree d through a generic points and $4d - 2a - 1$ generic lines in $\mathbb{P}^3$ is*

$$(3.22) \quad (d-2)N_d(3^a 2^{4d-2a}) + N_d^{\mathrm{red}}(3^a 2^{4d-2a-1}) - 2dm_1(3^a 2^{4d-2a-1} + 2L.R_1$$

if $a > 0$ and

$$(d-2)N_d(3^a 2^{4d-2a}) + N_d^{\mathrm{red}}(3^a 2^{4d-2a-1})$$

$$(3.23) \quad -2dm_{a+1}(3^a 2^{4d-2a-1}) + 2L.R_{a+1} - 4N_d(3^{a+1} 2^{4d-2a-2})$$

if $4d - 2a - 1 > 0$.

proof. We use Corollary 3.10 for the family $X/B(3^a 2^{4d-2a-1})$. The the RHS of (3.21bis) yields, with $(a.) = (3^a 2^{4d-2a-1})$ that, in the the above notation,

$$n_2(f) = \frac{1}{2}(2dN_d(3^a 2^{4d-2a}) - K^2 + N_d^{\mathrm{red}}(a.) + 4LK - 4N_d(3^a 2^{4d-2a})).$$

Using the formula [RA],(14) for $i = 1$ and noting that $N_d(4, ...) = 0$ yields (3.22), while the same formula with $i = a + 1$ yields (3.23).

Remark 3.13. Theorem 3.3, as well as the other multiple-point results in this paper, admit straightforward generalizations to the relative case, where Y is replaced by a smooth morphism

$$\rho : Y \to B$$

and f is a B-morphism, i.e. the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \searrow & & \swarrow \rho \\ & B. & \end{array}$$

Note that the 'absolute' case discussed above becomes a special case of the relative case by replacing Y by $Y \times B \to B$. In the relative case The factors $\mu_k(f)$ are replaced by

$$(3.24) \qquad \mu_k(f/B) = (f_1 \times_B f_k)^*(\Delta_{Y/B}) - \Delta^k \{ \frac{f_1^*(c(T_{Y/B}))}{1 + \Delta^k} \}_{m-1}$$

where $\Delta_{Y/B}$ is the diagonal in $Y \times_B Y$ and $T_{Y/B}$ is the relative or vertical tangent bundle of $Y \to B$ (which coincides with the normal bundle of $\Delta_{Y/B}$ in $Y \times_B Y$) and m is the relative dimension of Y/B. With (3.24) in place of (3.15), the analogue of Theorem 3.2 and its consequences hold. The analogous generalizations of Theorem 2.6 and its consequences, which concern maps to projective and Grassmannian bundles, also hold. The proofs are the same, because for a $B-$map f, multiple-point loci involve only the 'vertical' coordinates of Y over B.

Concluding remark 3.14. Hopefully, the enumerative results of this section provide sufficient motivation for wanting to determine completely the multiplicative structure of the subring of the Chow ring of W^r generated by $\Delta^2, ..., \Delta^r$ together with the Chow ring of X. As remarked above, this indeed is the subject of [R9].

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