A NOTE ON HILBERT SCHEMES OF NODAL CURVES

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ABSTRACT. We study the Hilbert scheme and punctual Hilbert scheme of a nodal curve, and the relative Hilbert scheme of a family of curves acquiring a node. The results are then extended to flag Hilbert schemes, parametrizing chains of subschemes. We find, notably, that if the total space X of a family X/B is smooth (over an algebraically closed field $\mathfrak{k}$), then the relative Hilbert scheme $Hilb_m(X/B)$ is smooth over $\mathfrak{k}$ and the flag Hilbert schemes are normal and locally complete intersection, but generally singular .

The Hilbert scheme parametrizes ideal sheaves or subschemes of Projective Space or more generally, or a fixed scheme X. Perhaps the simplest case is where X is a smooth curve, for then the Hilbert scheme $\mathrm{Hilb}_m(X)$ parametrizing length-m subschemes of X coincides with the symmetric product $\mathrm{Sym}^m(X)$. Our purpose in this note is to study what is in a sense the next simplest case, where X is essentially a curve with ordinary nodes, with planar equation formally equivalent to $xy = 0$. Besides $\mathrm{Hilb}_m(X)$ itself, there are (at least) 2 other types of Hilbert scheme of natural interest here: the *punctual* one $\mathrm{Hilb}^0_m(X)$, parametrizing length-m subschemes supported at a node; and the *relative* one $\mathrm{Hilb}_m(\tilde{X}/B)$, parametrizing length-$m$ subschemes in the fibres of the family $\tilde{X}/B$ that is the map $\mathbb{A}^2 \to \mathbb{A}^1$ given by $xy = t$ (both $\mathrm{Hilb}_m(X)$ and $\mathrm{Hilb}_m(\tilde{X}/B)$ may conveniently be viewed as germs, or formal schemes, along $\mathrm{Hilb}^0_m(X)$. All these will be determined below. We find that $\mathrm{Hilb}^0_m(X)$ is a connected chain of $(m-1)$ nonsingular rational curves meeting transeversely; $\mathrm{Hilb}_m(X)$ is a connected chain of $(m+1)$ nonsingular m-dimensional germs, whose first and last members are supported on points and each other member is supported on a component of $\mathrm{Hilb}^0_m(X)$, and only consecutive members intersect (and those transversely); finally, $\mathrm{Hilb}_m(\tilde{X}/B)$ is a smooth $(m+1)$-fold.

To elucidate the relations between the Hilb_m for different m, with an eye to enumerative applications based on recursion on m, we will also study some flag Hilbert schemes $\mathrm{Hilb}_{m.}$, parametrizing chains of ideals whose colengths form a given sequence $m. = (m_1 > m_2 > ...)$. For some purposes, these are easier to work with than ordinary Hilbert schemes, due to the natural maps between the $\mathrm{Hilb}_{m.}$ for different $(m.)$. For example, for $m. = (m, m-1)$, we find that the punctual Hilbert scheme $\mathrm{Hilb}^0_{m.}(X)$ is a chain of $2m-3$ copies of $\mathbb{P}^1$ that alternate between those coming from $\mathrm{Hilb}^0_m(X)$ and from $\mathrm{Hilb}^0_{m-1}(X)$. The relative Hilbert scheme $\mathrm{Hilb}_{m.}(\tilde{X}/B)$ is still smooth. Things begin to change though with $m. = (m, m-1, m-2)$. Here $\mathrm{Hilb}^0_{m.}(X)$ is still a chain of $2m-3$ components, but now only the 2 external ones on each side are $\mathbb{P}^1$'s and the rest are $\mathbb{P}^1 \times \mathbb{P}^1$. The relative

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Hilbert scheme $\mathrm{Hilb}_{m.}(\tilde{X}/B)$ is no longer smooth, but it is still normal with locally complete intersection singularities. A similar picture emerges for longer chains,and in particular for the 'full flag' case $\mathrm{fHilb}_m = \mathrm{Hilb}_{m,m-1,...,1}$.

In [R], which uses some of these results, we will develop an alternative and more 'geometric' approach to these objects. In particular, we will identify $\mathrm{fHilb}_m(\tilde{X}/B)$ with a certain space $W^m(\tilde{X}/B)$ constructed as an explicit blow-up of the relative Cartesian product $\tilde{X}^m/B$. The proof of this identification uses some of the results here. We will then apply the spaces $W^m(\tilde{X}/B)$ to some enumerative problems. In these applications, the relatively simple relationship between fHilb_m and fHilb_{m-1} is critical.

We work over a fixed algebraically closed field $\mathfrak{k}$. We denote by R localizarion of the ring

$$\mathfrak{k}[x,y]/(xy)$$

and at its maximal ideal (x,y). A typical element of R can be written in the form

$$u(a+\sum_{i\geq 1} b_i x^i + c_i y^i)$$

where u is a unit and the sum is finite. The formal completion

$$\hat{R} = \mathfrak{k}[[x,y]]/(xy) = \{a + \sum_{i\geq 1} b_i x^i + c_i y^i\}$$

(sum not necessarily finite) is isomorphic to the formal completion of the local ring at any 1-dimensional ordinary node. We seek first to determine the punctual Hilbert scheme $\mathrm{Hilb}^0_m(R)$ of colength-m ideals in R which, as is well known, is naturally isomorphic to $\mathrm{Hilb}^0_m(\hat{R})$. At this point, we do not seek to define or compute a natural scheme structure on $\mathrm{Hilb}^0_m(R)$ (so calling it the punctual Hilbert *scheme* is something of a misnomer)– this will be done later (see Corollary 8). For now we simply view $\mathrm{Hilb}^0_m(R)$ as an algebraic set endowed with a flat family of ideals that yields a bijective correspondence between the (closed, $\mathfrak{k}-$valued) points of $\mathrm{Hilb}^0_m(R)$ and the colength-m ideals of R.

Theorem 1. *(i) Every ideal $I < R$ of colength m is of one of the following, said to be of type $(c^m_i), (q^m_i)$, respectively:*

$$I^m_i(a) = (y^i + ax^{m-i}), 0 \neq a \in \mathfrak{k}, i = 1, ..., m-1;$$

$$Q^m_i = (x^{m-i+1}, y^i), i = 1, ..., m.$$

(ii) The closure C^m_i in the Hilbert scheme of the set of ideals of type (c^m_i) is isomorphic to $\mathbb{P}^1$ and consists of the ideals of types (c^m_i) or (q^m_i) or (q^m_{i+1}). In fact, we have

$$\lim_{a\to 0} I^m_i(a) = Q^m_i,$$

$$\lim_{a\to \infty} I^m_i(a) = Q^m_{i+1}.$$

(iii) The punctual Hilbert scheme $\mathrm{Hilb}^0_m(R)$, as algebraic set, is a rational chain

$$C^m_1 \cup_{Q^m_2} C^m_2 \cup \cdots \cup_{Q^m_{m-1}} C^m_{m-1}; \tag{1}$$

it has ordinary nodes at $Q_2^m, ..., Q_{m-1}^m$ *and is smooth elsewhere.*

proof. Recall that elements $z, z' \in R$ are said to be *associate* if $z = uz'$ for some unit u. Note that any nonzero nonunit $z \in R$ is associate to a uniquely determined element of the form x^α or y^β or $x^\alpha + ay^\beta, a \neq 0, \alpha, \beta > 0$, in which case we will say that z is of type $(\alpha, 0)$ or $(0, \beta)$ or (α, β), respectively. Note also that for any ideal I of colength m we have

$$x^m, y^m \in I.$$

Now given I of colength m, pick $z \in I$ of minimal type (α, β), with respect to the natural partial ordering on types. Suppose to begin with that $\alpha, \beta > 0$. Then note that

$$x^{\alpha+1}, y^{\beta+1} \in I,$$

and consequently (α, β) is unique: indeed if (α', β') is also minimal then we may assume $\alpha' > \alpha$, hence $x^{\alpha'} \in I$, hence $y^{\beta'} \in I$, contradicting minimality. Hence (α, β) is unique and it is then easy to see that the element $z' = x^\alpha + ay^\beta \in I$ is unique as well, so clearly z' generates I and I is of type (c_β^m).

Thus we may assume that any minimal element of I is of type $(\alpha, 0)$ or $(0, \beta)$. Since $x^m, y^m \in I$, I clearly contains minimal elements of type $(\alpha, 0)$ and $(0, \beta)$, and then it is easy to see that I is of type (q_β^m). This proves assertion (i). Since $I_i^m(a)$ contains y^{i+1}, x^{m-i+1}, assertion (ii) is easy. As for (iii), let $C = \bigcup_i C_i^m$ be an abstract nodal curve as in (1). It follows from (ii) that each C_i^m carries a flat family of ideals, i.e. admits a natural morphism to $\text{Hilb}_m(R)$ which is clearly injective on $\mathfrak{k}$-valued as well as $\mathfrak{k}[\epsilon]$-valued points (compare the computations in the proof of Theorem 2 below). Since these morphisms agree on the intersections $C_i^m \cap C_{i+1}^m = Q_{i+1}^m$, they yield a morphism, again clearly injective on $\mathfrak{k}$ and $\mathfrak{k}[\epsilon]$ valued points, from C to $\text{Hilb}_m(R)$, which identifies C with $\text{Hilb}_m^0(R)$ as an algebraic set. $\square$

Next we determine the structure of the full Hilbert scheme of R and $\hat{R}$:

Theorem 2. *The Hilbert scheme* $Hilb_m(R)$ *(resp.* $Hilb_m(\hat{R})$ *is a chain*

$$D_0^m \cup D_1^m \cdots D_{m-1}^m \cup D_m^m$$

where each D_i^m *is a smooth and* $m-$*dimensional germ (resp. formal scheme) supported on* C_i^m *for* $i = 1, ..., m-1$ *or* Q_i^m *for* $i = 0, m$*; for* $i = 1, ..., m-1, D_i^m$ *meets its neighbors* $D_{i\pm1}^m$ *transversely in dimension* $m-1$ *and meets no other* D_i^m*. The generic point of* D_i^m *corresponds to subscheme of* $\text{Spec}(R)$ *comprised of* $m-i$ *points on the* x*-axis and* i *points on the* y*-axis.*

proof. Clearly $\text{Hilb}_m(R)$ (resp. $\text{Hilb}_m(\hat{R})$) is a germ (resp. formal scheme) supported on $\text{Hilb}_m^0(R)$, so this is a a matter of determining the scheme structure of $\text{Hilb}_m(R)$ and $\text{Hilb}_m(\hat{R})$ at each point of $\text{Hilb}_m^0(R)$, which may be done formally by testing on Artin local algebras. Again, we shall do so at $Q_i^m, i > 1$ as the cases of $I_i^m(a)$ and Q_1^m are similar and simpler. Given S artinian local augmented, a flat S-deformation of $I = Q_i^m$ is given by an ideal

$$I_S = (f, g),$$

$$f = x^{m+1-i} + f_1(x) + f_2(y), \tag{4}$$

$$g = y^i + g_1(x) + g_2(y),$$

where f_i, g_j have coefficients in $\mathfrak{m}_S$, and such that R_S/I_S is $S-$ free of rank m, in which case it is clear by Nakayama's Lemma that

$$1, x, ..., x^{m-i}, y, ..., y^{i-1}$$

is a free basis for R_S/I_S. It is easy to see that we may assume f_1, g_1 are in fact polynomials of degree $\leq m-i$ and f_2, g_2 are of degree $< i$ and f_2, g_2 have no constant term. Let's write

$$f_1(x) = \sum_0^{m-i} a_j x^j, f_2(y) = \sum_1^{i-1} b_j y^j, \tag{5}$$

$$g_1(x) = \sum_0^{m-i} c_j x^j, g_2(y) = \sum_1^{i-1} d_j y^j. \tag{6}$$

Now obviously

$$yf - b_{i-1}g \equiv 0 \equiv xg - c_{m-i}f \mod I_S.$$

Writing these elements out in terms of $1, x, ..., x^{m-i}, y, ..., y^{i-1}$ yields relations among $1, x, ..., x^{m-i}, y, ..., y^{i-1}$. Since the latter elements form an S-free basis of R_S/I_S, those relations must be trivial. In other words, we have exact equalities rather than congruences:

$$yf - b_{i-1}g = 0 = xg - c_{m-i}f.$$

Writing out this equality term by term yields the following identities

$$b_j = b_{i-1}d_{j+1}, j = 1, ..., i-2,$$

$$b_{i-1}d_1 = a_0,$$

$$b_{i-1}c_j = 0, j = 0, ..., m-i, \tag{7}$$

$$c_j = c_{m-i}a_{j+1}, j = 0, ..., m-i-1,$$

$$c_{m-i}a_0 = 0,$$

$$c_{m-i}b_j = 0, j = 1, ..., i-1.$$

(If $i = 1$ only lines 4 and 5 of display (7) are operational.) Conversely, suppose the relations (7) are satisfied, or equivalently

$$yf - b_{i-1}g = 0 = xg - c_{m-i}f. \tag{8}$$

By Nakayama's Lemma, $1, x, ..., x^{m-i}, y, ..., y^{i-1}$ generate R_S/I_S, hence to show (4) defines a flat family it suffices to show these elements admit no nontrivial S-relations mod I_S. To this end,, suppose

$$u_{m-i}(x) + v_{i-1}(y) = A(x,y)f + B(x,y)g \tag{9}$$

where u, v, A, B are all polynomials in the indicated variables and of the indicated degees (if any) with coefficients in S and v has no constant term; in fact it clear a priori that then A, B must have coefficients in $\mathfrak{m}_S$. Then the relations (8) allow us to rewrite (9) as

$$u_{m-i}(x) + v_{i-1}(y) = A'(x)f + B'(y)g, \tag{9'}$$

with $A'(x) = \sum a'_k x^k \in \mathfrak{m}_S[[x]], B'(x) = \sum b'_k y^k \in \mathfrak{m}_S[[y]]$. Comparing coefficients of $x^{m-i+1}, ..., y^i, ...$ in (9') we get relations

$$\begin{gathered} -a'_0 = a'_1 a_{m-i} + ..., -b'_0 = b'_1 b_{i-1} + ... \\ ... \\ -a'_k = a'_{k+1} a_{m-i} + ..., -b'_k = b'_{k+1} b_{i-1} + ... \end{gathered} \tag{10}$$

Starting from the fact that $a_j, a'_k \in \mathfrak{m}_S, \forall j, k$, we infer from these first that, in fact, $a'_k \in \mathfrak{m}_S^2, \forall k$; plugging the latter fact back into the relations (10) we then infer $a'_k \in \mathfrak{m}_S^3, \forall k$, and so on. Since S is artinian it follows that $a'_k = 0, \forall k$ and likewise for b'_k. Thus

$$A' = B' = u_{m-i} = v_{i-1} = 0,$$

hence there are no nontrivial relations, as claimed.

Thus the Hilbert scheme is embedded in the space of the variables

$$a_1, ..., a_{m-i}, d_1, ..., d_{i-1}, b_{i-1}, c_{m-i},$$

i.e. $\mathbb{A}^{m+1}$, and defined by the relation

$$b_{i-1}c_{m-i} = 0. \tag{11}$$

Thus it is a union of 2 smooth $m-$dimensional components meeting transversely in a smooth $(m-1)-$dimensional subvariety. The generic point on the component where $b_{i-1} = 0$ (resp. $c_{m-i} = 0$) is clearly an ideal generated by g (resp. f), which has the properties as claimed.

In the case $I = I_i^m(a) = (y^i + ax^{m-i})$, a similar analysis shows that an S-deformation of I is given by the principal ideal

$$I_S = (y^i + \tilde{a}x^{m-i} + f_1(x) + f_2(y))$$

where

$$\tilde{a} \in S, \tilde{a} \equiv a \mod \mathfrak{m}_S,$$

$$f_1(x) = \sum_0^{m-i-1} a_j x^j, f_2(y) = \sum_1^{i-1} b_j y^j,$$

$$a_j, b_j \in \mathfrak{m}_S,$$

and via $(\tilde{a}, a_0, ..., a_{m-i-1}, b_1, ..., b_{i-1})$ we may identify Hilbert scheme locally with $\mathbb{A}^m$. $\square$

Remark 2.1. We note that in terms of the above coordinates the subset $\mathrm{Hilb}^0_m(R) \subset \mathrm{Hilb}_m(R)$ is defined by

$$a_1 = ... = a_{m-i} = d_1 = ... = d_{i-1} = 0,$$

i.e. by the conditions

$$f(x,0) = x^{m-i+1}, g(0,y) = y^i.$$

We shall see below that this, in fact, defined the 'natural' scheme structure on $\mathrm{Hilb}^0_m(R)$. $\square$

Next we consider the relative local situation, i.e. that of a germ of a (1-parameter) family of curves with smooth total space specializing to a node. Thus set

$$\tilde{R} = \mathfrak{k}[x,y]_{(x,y)}, B = \mathfrak{k}[t]_{(t)},$$

and view $\tilde{R}$ as a $B-$module via $xy = t$. As is well known, this is the versal deformation of the node singularity $xy = 0$, so any family of nodal curves is locally a pullback of this.

Theorem 3. *The relative Hilbert scheme $Hilb^m(\tilde{R}/B)$ is formally smooth, formally $(m+1)-$dimensional over $\mathfrak{k}$.*

proof. The relative Hilbert scheme parametrizes length-m schemes contained in fibres of $\mathrm{Spec}(\tilde{R}) \to \mathrm{Spec}(B)$. This means ideals $I_S < \tilde{R}_S$ of colength m containing $xy - s$ for some $s \in \mathfrak{m}_S$, such that $\tilde{R}_S/I_S$ is $S-$free. The analysis of these is virtually identical to that contained in the proof of Theorem 2, except that the relation $b_{i-1}c_{m-i} = 0$ gets replaced by

$$b_{i-1}c_{m-i} = s \tag{*}$$

and lines 3,5,6 of display (7) are replaced, respectively, by

$$b_{i-1}c_j = sa_{j+1}, j = 0, ..., m-i-1$$

$$c_{m-i}a_0 = sd_1$$

$$c_{m-i}b_j = sd_{j+1}, 1 \le j \le i-2,$$

relations which already follow from the other relations (in lines 1,2,4 of display (7)) combined with (*). Thus, the relative Hilbert scheme is the subscheme of the affine space of the variables $a_1, ..., a_{m-i}, d_1, ..., d_{i-1}, b_{i-1}, c_{m-i}, t$ defined by the relation

$$b_{i-1}c_{m-i} = t, \tag{12}$$

hence is smooth as claimed. $\square$

Remark 3.1. After this was written, the author was informed by Prof. I. Smith of some related work by himself and Prof. S. Donaldson [DS, Sm] which considers the relative Hilbert scheme of a pencil of nodal curves on a smooth surface from a rather different viewpoint, valid in the symplectic category over $\mathbb{C}$; in particular, they prove in this context an analogue of Theorem 3 (smoothness of the total space of the relative Hilbert scheme).

Construction 3.2. An explicit construction of $\mathrm{Hilb}_m(R)$, globally along $\mathrm{Hilb}^0_m(R)$, can be given as follows (also see [R2] for further developments). Let $C_1, ..., C_{m-1}$ be copies of $\mathbb{P}^1$, with homogenous coordinates u_i, v_i on the i-th copy. Let $\tilde{C} \subset C_1 \times ... \times C_{m-1} \times \mathbb{A}^1$ be the subscheme defined by

$$v_1u_2 = tu_1v_2, ..., v_{m-2}u_{m-1} = tu_{m-2}v_{m-1}.$$

Thus $\tilde{C}$ is a reduced complete intersection of divisors of type $(1, 1, 0, ..., 0)$, $(0, 1, 1, 0, ..., 0), ..., (0, ..., 0, 1, 1)$ and it is easy to check that the fibre of $\tilde{C}$ over $0 \in \mathbb{A}^1$ is

$$\tilde{C}_0 = \bigcup_i [1, 0] \times ... \times [1, 0] \times C_i \times [0, 1] \times ... \times [0, 1]$$

and that in a neighborhood of $\tilde{C}_0$, $\tilde{C}$ is smooth and $\tilde{C}_0$ is its unique singular fibre over $\mathbb{A}^1$. We may identify $\tilde{C}_0$ is an obvious way with $\mathrm{Hilb}^0_m(R)$. Next consider an affine space $\mathbb{A}^{2m}$ with coordinates $a_0, ..., a_{m-1}, d_0, ..., d_{m-1}$ and let $\tilde{H} \subset \tilde{C} \times \mathbb{A}^{2m}$ be the subscheme defined by

$$a_0u_1 = tv_1, d_0v_{m-1} = tu_{m-1}$$

$$a_1u_1 = d_{m-1}v_1, ..., a_{m-1}u_{m-1} = d_1v_{m-1}.$$

Set $L_i = p^*_{C_i}\mathcal{O}(1)$. Then consider the subscheme of $\tilde{H} \times_{\mathbb{A}^1} \mathrm{Spec}(\tilde{R})$ defined by the equations

$$F_0 := x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0 \in \Gamma(\mathcal{O}_{\tilde{H}} \otimes R)$$

$$F_1 := u_1x^{m-1} + u_1a_{m-1}x^{m-2} + ... + u_1a_2x + u_1a_1 + v_1y \in \Gamma(L_1 \otimes R)$$

...

$$F_i := u_ix^{m-i}+u_ia_{m-1}x^{m-i-1}+...+u_ia_{i+1}x+u_ia_i+v_id_{m-i+1}y+...+v_id_{m-1}y^{i-1}+v_iy^i$$

$$\in \Gamma(L_i \otimes R)$$

...

$$F_m := d_0 + d_1y_1 + ... + d_{m-1}y^{m-1} + y^m \in \Gamma(\mathcal{O}_{\tilde{H}} \otimes R).$$

In view of the above results, it is easy to check that the ideal sheaf $\mathcal{I}$ generated by $F_0, ..., F_m$ defines a subscheme of $\tilde{H} \times \mathrm{Spec}(R)$ that is flat over $\tilde{H}$ and that may serve to identify $\tilde{H}$ with $\mathrm{Hilb}(\tilde{R})/B$. Locally at a point of type c^m_i (resp. Q^m_i), $\mathcal{I}$ is generated by F_i (resp. F_{i-1}, F_i).

Now let E be the universal bundle on $\tilde{H}$, whose fibre at a point corresponding to a length-m scheme z is $\Gamma(\mathcal{O}_z)$, and let V be the trivial rank$-(2m+1)$ bundle on the symbols $1, x, ..., x^m, y, ..., y^m$. Since $1, x, ..., x^m, y, ..., y^m$ generate $\Gamma(\mathcal{O}_z)$ for all $z \in \mathrm{Hilb}_m(\tilde{R})$, we have a surjection $V \to E$. Then $F_0, ..., F_m$ together yield a map

$$F : \mathcal{O}_{\tilde{H}} \oplus \bigoplus_1^{m-1} L_i^{-1} \oplus \mathcal{O}_{\tilde{H}} \to V$$

so we get an exact sequence

$$0 \to \mathcal{O}_{\tilde{H}} \oplus \bigoplus_{1}^{m-1} L_i^{-1} \oplus \mathcal{O}_{\tilde{H}} \to V \to E \to 0.$$

In particular, it follows that $c_1(E) = L_1 + ...L_{m-1}$.

Note that $\mathbb{A}^m_{a_0,...,a_{m-1}}, \mathbb{A}^m_{d_0,...,d_{m-1}}$ may be identified, respectively, with $\mathrm{Sym}^m \mathbb{A}^1_x$, $\mathrm{Sym}^m \mathbb{A}^1_y$ and accordingly it will be convenient to use the notation $\sigma_i = a_{m-i}, \tau_i = d_{m-i}$, these being respectively the i-th elementary symmetric functions in the roots of F_0, F_m. I claim, in fact, that the projection $c : \tilde{H} \to \mathbb{A}^{2m+1}$ may be identified with the cycle map $\tilde{H} \to \mathrm{Sym}^m(\mathrm{Spec}(\tilde{R})/B)$: this is because the natural map

$$\Phi : \mathrm{Sym}^m(\mathrm{Spec}(\tilde{R})/B) \to \mathbb{A}^m_\sigma \times \mathbb{A}^m_\tau \times B$$

is an embedding. Φ is an embedding because its restrictions on the generic fibre and the special fibre over B are embeddings. Moreover, it is not hard to see that the image of Φ (which coincides with that of c) is scheme-theoretically defined by the equations

$$\sigma_i \tau_j = t\sigma_{i-1}\tau_{j-1}, \quad \forall i, j = 1, ..., m, \quad i + j > m$$

where we set $\sigma_0 = \tau_0 = 1$. Setting $S_m = \mathrm{im}(\Phi)$, we have that $\tilde{H}$ is a Weil divisor in $S_m \times \tilde{C}$. $\square$

The local analysis immediately yields some conclusions for the Hilbert scheme of a nodal curve:

Corollary 4. *Let C_0 be a curve with only k nodes as singularities and c irreducible components. Then*
(i) $Hilb_m(C_0)$ is reduced and has precisely $\binom{m+c-1}{m}$ components, the general element of each of which corresponds to a reduced subscheme of the smooth part of C_0;
(ii) let I be a point of $Hilb_m(C_0)$ having colength m_i at the i-th node of C_0; then locally at I, $Hilb_m(C_0)$ is a cartesian product of k factors, each of which is a 2-component normal crossing of dimension $m_i, i = 1, ..., k$, or a point if $m_i = 0$, times a smooth factor.
(iii) the fibre of the cycle map

$$cyc : Hilb_m(C_0) \to Sym^m(C_0)$$

(cf. [A]*) over a cycle having multiplicity m_i at the ith node is a product of 1-dimensional rational chains of length $m_i - 1$.*

proof. It is clear from the explicit analysis in the proof of Theorem 2 that any subscheme of C_0 deforms to a reduced subscheme supported on the smooth part. Such subschemes are parametrized by an open dense subset of the symmetric product $\mathrm{Sym}^m(C_0)$. This clearly yields (i) and (ii), while (iii) follows from the fact that the fibres of *cyc* are products of punctual Hilbert schemes. $\square$

Next we extend (most of) the above results to the case of *flag* Hilbert schemes (see [Se] for a general discussion of those). By definition, for any decreasing sequence of positive integers

$$m_. = (m_1 > ... > m_k),$$

the flag Hilbert scheme $\mathrm{Hilb}_{m.}(R)$ parametrizes nested chains of ideals

$$I_S^1 < \dots < I_S^k < R_S \tag{13}$$

such that R_S/I_S^j is S-free of rank $m_j, j = 1, \dots, k$ (i.e. such that each I_S^j is an $S-$point of $\mathrm{Hilb}_{m_j}(R)$); similarly for punctual and relative Hilbert schemes. Thus a flag Hilbert scheme is by definition a subscheme of a product (or, in the relative case, a fibre product) of its 'constituent' ordinary Hilbert schemes and as such comes equipped with forgetful projection morphisms to those constituents; moreover the defining equations for $\mathrm{Hilb}_{m.}(R)$ in $\prod\limits_{j=1}^{k} \mathrm{Hilb}_{m_j}(R)$ are just the conditions that the inclusions (13) hold, and these equations involve only pairs of successive factors $\mathrm{Hilb}_{m_j}(R)$, $\mathrm{Hilb}_{m_{j+1}}(R), j = 1, \dots, k-1$. In the case of 'full flags', i.e. the case

$$m_. = (m > m-1 > \dots > 1),$$

we will denote $\mathrm{Hilb}_{m.}(R)$ by $\mathrm{fHilb}_m(R)$. We begin by analyzing the case of pairs of ideals of relative colength 1:

Theorem 5. *(i)The only colength-$(m-1)$ ideal containing $I_i^m(a)$ is Q_i^{m-1}; the only colength-$(m-1)$ ideals containing Q_i^m are the $I_{i-1}^{m-1}(a)$ for $a \neq 0$ and their limits Q_i^{m-1} and Q_{i-1}^{m-1}.*
(ii) The punctual flag Hilbert scheme $Hilb^0_{m,m-1}(R)$, as algebraic set, is a chain of nonsingular rational curves of the form

$$C_1^m \cup_{(Q_2^m, Q_1^{m-1})} C_1^{m-1} \cup_{(Q_2^m, Q_2^{m-1})} C_2^m \cup \dots \cup_{(Q_{m-1}^m, Q_{m-1}^{m-1})} C_{m-1}^m;$$

it has ordinary nodes at $Q_2^m, \dots, Q_{m-1}^m$ and is smooth elsewhere. Each component C_i^m projects isomorphically to its image in $Hilb^0_m(R)$ and to a point Q_i^{m-1} in $Hilb_{m-1}(R)$, and vice-versa for C_i^{m-1}.
(iii) The flag Hilbert scheme $Hilb_{m,m-1}(R)$, as formal scheme along $Hilb^0_{m,m-1}(R)$, has normal crossing singularities and at most triple points. Each of its components is formally smooth, $m-$dimensional and of the form

$$D_{i,i'}^{m,m-1}, i = 0, \dots, m, i-1 \leq i' \leq i,$$

and projects to $D_i^m, D_{i'}^{m-1}$ respectively (cf. Theorem 2); $D_{i,i'}^{m,m-1}$ meets $D_{j,j'}^{m,m-1}$ nontrivially iff

$$|i-j| + |i'-j'| \leq 1;$$

the components of $Hilb^0_{m,m-1}(R)$ contained in $D_{i,i}^{m,m-1}$ (resp. $D_{i,i-1}^{m,m-1}$) are C_i^m and C_i^{m-1} (resp. C_{i-1}^{m-1} and C_i^m).
(iv) The relative flag Hilbert scheme $Hilb_{m,m-1}(\tilde{R}/B)$, as formal scheme along $Hilb^0_{m,m-1}(R)$, is formally smooth and $(m+1)$-dimensional over $\mathfrak{k}$. The natural map

$$Hilb_{m,m-1}(\tilde{R}/B) \to Hilb_{m-1}(\tilde{R}/B)$$

is a flat, locally complete intersection morphism of relative dimension 1.

Proof. Assertion (i) is an elementary consequence of the analysis in the proof of Theorem 1 and its proof will be omitted. Assertion (ii) and the set-theoretic portion of Assertion (iii) follow from this. To complete the proof of (iii),(iv), it remains to analyze the situation locally at each pair $(I, I') \in\mathrm{Hilb}^0_{m,m-1}(R)$. We will consider the case of $(Q^m_i, Q^{m-1}_i), 1 < i < m$, as other cases are similar or simpler. There we will focus mainly on Assertion (iv), as (iii) is essentially a special case of this (as will be indicated below). Consider then a pair $(I_S < I'_S)$ flatly deforming (Q^m_i, Q^{m-1}_i) relative to B. Then we may assume that for some $s \in \mathfrak{m}_S, I_S$ and I'_S are generated by $xy - s$ and f, g (resp. f', g') with

$$f = x^{m+1-i} + \sum_{j=0}^{m-i} a_j x^j + \sum_{j=1}^{i-1} b_j y^j,$$

$$g = y^i + \sum_{j=0}^{m-i} c_j x^j + \sum_{j=1}^{i-1} d_j y^j,$$

$$f' = x^{m-i} + \sum_{j=0}^{m-i-1} a'_j x^j + \sum_{j=1}^{i-1} b'_j y^j,$$

$$g' = y^i + \sum_{j=0}^{m-i-1} c'_j x^j + \sum_{j=1}^{i-1} d'_j y^j.$$

(For the non-relative case we take s=0.)

As we saw above, the relations as in (7), or equivalently (8), are necessary and sufficient so that I_S, I'_S are S-flat deformations of I, I' respectively. In particular, these include

$$b_{i-1}c_{m-i} = s = b'_{i-1}c'_{m-i-1}. \tag{14}$$

The other relations can be used to eliminate some of the parameters. It remains to account for the condition that $I_S < I'_S$. To this end it suffices to note that

$$1, x, ..., x^{m-i-1}, y, ..., y^{i-1}$$

form an S-free basis of R_S/I'_S, then express f, g in terms of this basis and equate the coefficients to 0. This yields the coefficient relations

$$a_0 = -(a_{m-i} - a'_{m-i-1})a'_0 + sb'_1,$$

$$a_j = a'_{j-1} - (a_{m-i} - a'_{m-i-1})a'_j, j = 1, ..., m - i - 1,$$

$$b_j = ((a_{m-i} - a'_{m-i-1})b'_j + sb'_{j+1}, j = 1, ..., i - 2,$$

$$b_{i-1} = (a_{m-i} - a'_{m-i-1})b'_{i-1},$$

$$c_j = c'_j + c_{m-i}a'_j, j = 0, ..., m - i - 1,$$

$$d_j = d'_j + c_{m-i}b'_j, j = 1, ..., i - 1.$$

These coefficient relations are equivalent to

$$f = (x + a_{m-i} - a'_{m-i-1})f', g = g' + c_{m-i}f'. \tag{14.1}$$

By formal manipulations, these relations imply that

$$c'_{m-i-1} = c_{m-i}(a_{m-i} - a'_{m-i-1}). \tag{15}$$

Therefore the 2 relations (14) are replaced by the single relation

$$(a_{m-i} - a'_{m-i-1})b'_{i-1}c_{m-i} = s. \tag{16}$$

Consequently the relative flag Hilbert scheme is smooth here, with regular parameters

$$a'_1, ..., a'_{m-i-1}, a_{m-i}, d'_1, ..., d'_{i-1}, b'_{i-1}, c_{m-i}$$

and its fibre, i.e. $\mathrm{Hilb}_{m,m-1}(R)$, is the normal-crossing triple point

$$(a_{m-i} - a'_{m-i-1})b'_{i-1}c_{m-i} = 0.$$

The 3 components are: $D^{m,m-1}_{i-1,i-1}$, defined by $a_{m-i} - a'_{m-i-1} = 0$ (which implies $b_{i-1} = c'_{m-i-1} = 0$); $D^{m,m-1}_{i,i}$, defined by $b'_{i-1} = 0$ (which implies $b_{i-1} = 0$); $D^{m,m-1}_{i,i-1}$, defined by $c_{m-i} = 0$, (which implies $c'_{m-i-1} = 0$). Finally the relation (15) exhibits $\mathrm{Hilb}_{m,m-1}(\tilde{R}/B)$ locally as a conic in an $\mathbb{A}^2$ with coordinates a_{m-i}, c_{m-i} over $\mathrm{Hilb}_{m-1}(\tilde{R}/B)$, and therefore the projection is a flat locally complete intersection morphism. $\square$

Remark 5.1. In the case of the punctual flag Hilbert scheme, we have by Remark 2.1 that the only coefficients in the above calculation not automatically 0 are $b_{i-1}, c_{m-i}, b'_{i-1}, c_{m-1-i}$, and the vanishing of $a_{m-i}, a'_{m-i-1}, d_1, d'_1$ yield the relations

$$b_{i-1} = c'_{m-i-1} = c_{m-i}b'_{i-1} = 0. \tag{17}$$

Thus $\mathrm{Hilb}^0_{m,m-1}(R)$ has two components at (Q^m_i, Q^{m-1}_i): one where $c_{m-i} = 0$, which projects to $\{Q^m_i\} \subset \mathrm{Hilb}_m(R)$ and to $C^{m-1}_{i-1} \subset \mathrm{Hilb}_{m-1}(R)$; the other where $b'_{i-1} = 0$ which projects to $C^m_i \subset \mathrm{Hilb}_m(R)$ and to $\{Q^{m-1}_i\} \subset \mathrm{Hilb}_{m-1}(R)$. Thus we recover in terms of equations the set-theoretic picture presented in Theorem 5(ii). $\square$

Remark 5.2. For future reference we note that in the analogous case of (relative) deformations of (Q^m_i, Q^{m-1}_{i-1}), f', g' take the form

$$f' = x^{m-i+1} + \sum_{j=0}^{m-i} a'_j x^j + \sum_{j=1}^{i-2} b'_j y^j, g' = y^{i-1} + \sum_{j=0}^{m-i} c'_j x^j + \sum_{j=1}^{i-2} d'_j y^j.$$

The relations (15) and $b_{i-1} = (a_{m-i} - a'_{m-i-1})b'_{i-1}$ (see just above (15)) are replaced by

$$c_{m-i} = (d_{i-1} - d'_{i-2})c'_{m-i}, b'_{i-2} = (d_{i-1} - d'_{i-2})b_{i-1}. \tag{18}$$

As we shall see, this remark is useful in studying flag Hilbert schemes with more than 2 constituents. $\square$

Remark 5.3. Note that for any $(I, I') \in \mathrm{Hilb}_{m,m-1}(R)$, the annihilator $\mathrm{Ann}(I'/I) < R$ is an ideal of colength 1. This yields a morphism

$$A_m : \mathrm{Hilb}_{m,m-1}(R) \to \mathrm{Hilb}_1(R) = \mathrm{Spec}(R).$$

For example, in the situation of (14.1), we have by (the analogue of) (8),

$$yf' = b'_{i-1}g' = b'_{i-1}g - b'_{i-1}c_{m-i}f',$$

hence $J := (x + a_{m-i} - a'_{m-i-1}, y + b'_{i-1}c_{m-i})$ annihilates $f' \mod I_S$. Hence by (14.1) again, J annihilates $I'_S \mod I_S$. Note that by (14.1), we have $b'_{i-1}c_{m-i} = d_{i-1} - d'_{i-1}$. Therefore the value of A_m on the $S-$valued point (I_S, I'_S) (which extends the closed point (Q_i^m, Q_i^{m-1})) is

$$A_m(I_S, I'_S) = (x + a_{m-i} - a'_{m-i-1}, y + d_{i-1} - d'_{i-1}).$$

Consequently, the closed (special) fibre of A_m is defined in terms of f, g, f', g' by the condition that

$$f(x, 0) = xf'(x, 0), g(0, y) = g'(0, y). \quad \square$$

Construction 5.4. An analogue of Construction 3.2 in the flag case may be given as follows. Let a_i, d_i etc be as there and let a'_i, d'_i etc. be the analogous objects for Hilb_{m-1}. Set

$$r = a_{m-1} - a'_{m-2}, s = d_{m-1} - d'_{m-2},$$

so that

$$F_0 = F'_0(x + r), F_m = F'_{m-1}(y + s)$$

and we have the relation

$$rs = t,$$

so that (r, s) give a copy of $\mathrm{Spec}(\tilde{R})/B$. Then $\mathrm{Hilb}_{m,m-1}(\tilde{R}/B)$ may be relalized as the subscheme of $\tilde{C} \times_B \mathrm{Spec}(\tilde{R}) \times_B \mathrm{Hilb}_{m-1}(\tilde{R}/B)$ defined by

$$u'_i v_i = r u_i v'_i, \quad v'_i u_{i+1} = s v_{i+1} u'_i$$

$$a'_i u_{i+1} = s d'_{m-1-i} v_{i+1}, \quad d'_{m-1-i} v_i = r a'_i u_i, i = 1, ..., m-1. \quad \square$$

In extending these results to the case of longer- a fortiori, full- flags, the same methods apply. But in the conclusions, a couple of new twists come up, already for flags of type $m. = (m, m-1, m-2)$. First, the punctual Hilbert scheme $\mathrm{Hilb}^0_{m.}(R)$ will have components of varying dimensions (in this case, 1 and 2), roughly speaking because the parameters in $\mathrm{Hilb}^0_m(R)$ and $\mathrm{Hilb}^0_{m-2}(R)$ can vary independently. Second, the relative flag Hilbert scheme $\mathrm{Hilb}_{m.}(\tilde{R}/B)$ is not the same locally at $(Q_i^m, Q_i^{m-1}, Q_i^{m-2})$ as at $(Q_i^m, Q_i^{m-1}, Q_{i-1}^{m-2})$. The following proof contains the relevant computation.

Lemma 6. *Set $m. = (m, m-1, m-2)$. Then (i) as algebraic set, $Hilb^0_{m.}(R)$ is of the form*

$$C_1^m \cup C_1^{m-1} \cup C_{2,1}^{m,m-2} \cup C_2^{m-1} \cup ... \cup C_{m-2,m-3}^{m,m-2} \cup C_{m-2}^{m-1} \cup C_{m-1}^m.$$

Each component $C_{i,i-1}^{m,m-2}$ projects isomorphically to $C_i^m \times C_{i-1}^{m-2} \subset Hilb_{m,m-2}(R)$ and to $\{Q_i^{m-1}\} \subset Hilb_{m-1}(R)$.

(ii) $Hilb_{m.}(\tilde{R}/B)$ is irreducible and is smooth except at points $(Q_i^m, Q_i^{m-1}, Q_{i-1}^{m-2})$, where is has a rank-4 quadratic hypersurface singularity with local equation

$$(a_{m-i} - a'_{m-i-1})c_{m-i} = (d'_{i-1} - d''_{i-2})c''_{m-i-1} \tag{19}$$

proof. The set-theoretic assertion (i) follows from Theorem 5. For (ii), we will analyze $\mathrm{Hilb}_{m.}(\tilde{R}/B)$ locally at points of the form $(Q_i^m, Q_i^{m-1}, Q_i^{m-2})$ or $(Q_i^m, Q_i^{m-1}, Q_{i-1}^{m-2})$ as other cases are similar or simpler. Beginning with the former case, consider a deformation $(I''_S < I'_S < I_S)$ of $(Q_i^m, Q_i^{m-1}, Q_i^{m-2})$ where I_S, I'_S are as in the proof of Theorem 5 and I''_S is analogously defined by

$$f'' = x^{m-i-1} + \sum_{j=0}^{m-i-2} a''_j x^j + \sum_{j=1}^{i-1} b''_j y^j,$$

$$g'' = y^i + \sum_{j=0}^{m-i-2} c''_j x^j + \sum_{j=1}^{i-1} d''_j y^j.$$

Then working as in the proof of Theorem 5 we find the the $(m+1)$ parameters

$$a''_1, ..., a''_{m-i-2}, d''_1, ..., d''_{i-1}, b''_{i-1}, c_{m-i}, a'_{m-i-1}, a_{m-i}$$

such that all the coefficients of all our polynomials $f, ..., g''$, as well as the parameter s, are regular expressions in these and there are no relations. This shows that $\mathrm{Hilb}_{m.}(\tilde{R}/B)$ is smooth at $(Q_i^m, Q_i^{m-1}, Q_i^{m-2})$.

In the case of $(Q_i^m, Q_i^{m-1}, Q_{i-1}^{m-2})$, we may assume

$$f'' = x^{m-i} + \sum_{j=0}^{m-i-1} a''_j x^j + \sum_{j=1}^{i-2} b''_j y^j,$$

$$g'' = y^{i-1} + \sum_{j=0}^{m-i-1} c''_j x^j + \sum_{j=1}^{i-2} d''_j y^j$$

and analogous considerations yield $(m+2)$ parameters

$$a''_1, ..., a''_{m-i-1}, d''_1, ..., d''_{i-2}, b'_{i-1}, c_{m-i}, c''_{m-i-1}, d'_{i-1}, a_{m-i}$$

such that all the coefficients of all our polynomials, as well as the parameter s, are regular expressions in these, and satisfying the relation

$$c'_{m-i-1} = (a_{m-i} - a'_{m-i-1})c_{m-i} = (d'_{i-1} - d''_{i-2})c''_{m-i-1}$$

which yields the equation (19).

Finally, irreducibility of $\mathrm{Hilb}_{m.}(\tilde{R}/B)$ follows from the fact that its natural map to $\mathrm{Hilb}_{m,m-1}(\tilde{R}/B)$ is flat with irreducible generic fibre and $\mathrm{Hilb}_{m,m-1}(\tilde{R}/B)$ is irreducible by Theorem 5(iv). $\square$

Theorem 7. *The (full) flag Hilbert scheme $fHilb_m(\tilde{R}/B)$ has locally complete intersection singularities and its natural map to B is a local complete intersection morphism. In particular $fHilb_m(\tilde{R}/B)$ is reduced and is flat over B.*

proof. The fact that $\mathrm{fHilb}_m(\tilde{R}/B)$ has locally complete intersection singularities follows by a downwards induction as in the proof of Lemma 6. A similar argument shows that each map

$$\mathrm{fHilb}_m(\tilde{R}/B) \to \mathrm{fHilb}_{m-1}(\tilde{R}/B)$$

is a locally complete intersection morphism, hence so is $\mathrm{fHilb}_m(\tilde{R}/B) \to B$. $\square$

As in Remark 5.3, we have a natural map

$$fA_m = A_m \times A_{m-1} \times ... \times A_1 : \mathrm{fHilb}_m(R) \to \mathrm{Hilb}_1(R)^m = \mathrm{Spec}(R)^m$$

which fits in the diagram

$$\begin{array}{ccc} \mathrm{fHilb}_m(R) & \to & \mathrm{Hilb}_m(R) \\ fA_m \downarrow & & \downarrow \mathrm{cyc} \\ \mathrm{Spec}(R)^m & \to & \mathrm{Sym}^m(\mathrm{Spec}(R)) \end{array}$$

Then the closed fibre of fA_m is called the mth *punctual flag-Hilbert scheme* of R, denoted $\mathrm{fHilb}^0_m(R)$, and the (scheme-theoretic) projection of $\mathrm{fHilb}^0_m(R)$ to $\mathrm{Hilb}_m(R)$, i.e. the closed fibre of the cycle map cyc, is called the mth *punctual Hilbert scheme* of R, denoted $\mathrm{Hilb}^0_m(R)$. It is clear that $\mathrm{fHilb}^0_m(R)$ and $\mathrm{Hilb}^0_m(R)$ endow the similarly denoted algebraic sets discussed previously with a scheme structure.

Corollary 8. *The full flag punctual Hilbert scheme $fHilb^0_m(R) = Hilb^0_{m,...,1}(R)$ is reduced and is the transverse union of components of the form*

$$C^{m,m-2,...,m-2j}_{i,i-1,...,i-j}, \forall i, 1 \le i \le m-1, j = \min([\frac{m}{2}], i-1, m-i-1) \ge 0,$$

which projects isomorphically to $C^m_i \times ... \times C^{m-2j}_{i-j}$, and to a point in the other factors;

$$C^{m-1,m-2,...,m-2j-1}_{i,i-1,...,i-j}, \forall i, 1 \le i \le m-2, j = \min([\frac{m-1}{2}], i-1, m-i-1-2) \ge 0,$$

which projects isomorphically to $C^{m-1}_i \times ... \times C^{m-2j-1}_{i-j}$ and to a point in the other factors. The punctual Hilbert scheme $Hilb^0_m(R)$, with the scheme structure as above is a reduced nodal curve.

proof. Except for the assertion that $\mathrm{fHilb}^0_m(R)$, hence also $\mathrm{Hilb}^0_m(R)$ are reduced, this is a straightforward extension of Lemma 6; the constraints on j come simply from the fact that the components of $\mathrm{Hilb}^0_k(R)$ are $C^k_j, j = 1, ..., k-1$. For the reducedness assertion, we argue by induction on m. We may work locally at a point of the form $(Q^m_i, Q^{m-1}_i, ...)$, as other cases are similar or simpler. Consider a deformation of the form $(I_S = (f, g) < I'_S = (f', g') < ...)$ with I_S, I'_S as in the proof of Theorem 5, that yields an S-valued point of $\mathrm{fHilb}^0_m(R)$. By induction, we may assume

$$f'(x, 0) = x^{m-i}, g'(0, y) = y^i,$$

which implies that

$$f' = x^{m-i} + b'_{i-1}y^{i-1}, g' = y^i + c'_{m-i-1}x^{m-i-1}.$$

Then by Remark 5.3 we get a similar conclusion for f, g. Thus , in terms of the coordinates $a_1, ..., a_{m-i}, d_1, ..., d_{i-1}, b_{i-1}, c_{m-i}$ as in the proof of Theorem 2, the *subscheme* $\mathrm{Hilb}^0_m(R) \subset \mathrm{Hilb}_m(R)$ is simply defined by the vanishing of $a_1, ..., d_{i-1}$, hence is reduced as claimed, and likewise for $\mathrm{fHilb}^0_m(R)$. $\square$

Remark 8.1. To 'picure' the configuration in Corollary 7, it is amusing to display the components of $\mathrm{Hilb}^0_k(R), k = 2, ..., m$ as segments arranged alternately with empty spaces in an isosceles triangle, with each segment overlying an empty space:

$$\begin{array}{c}
- \\
- \quad - \\
- \quad - \quad - \\
... \\
- \quad - ... - ... - \quad -
\end{array}$$

Then the components of $\mathrm{fHilb}^0_m(R)$ are the columnwise products of these components.

Remark 9. in [R] we construct, based on 'geometric' considerations, a space $W^m(X/B)$ together with a morphism $J_m : W^m(X/B) \to \mathrm{fHilb}_m(X/B)$, which we will prove is an isomorphism. This proof requires that we know a priori that $\mathrm{fHilb}_m(X/B)$ is *reduced*.

Remark 10. It seems likely that the above results go through without the assumption that the base field $\mathfrak{k}$ is algebraically closed, provided the fibre nodes of X/B are of 'split' type, i.e. each node p, and each of the 2 tangent directions at p, are defined over $\mathfrak{k}$. Beyond this however, it seems some further analysis is needed. For instance, if p is a node defined over $\mathfrak{k}$ of nonsplit type, i.e. with equation formally equivalent to $x^2 + y^2, \mathrm{char}(\mathfrak{k}) \neq 2$, the punctual Hilbert scheme Hilb^0_m at p apparently has just 1 or 0 components defined over $\mathfrak{k}$ depending on whether m is even or odd; the other components occur in conjugate pairs.

Correction to proof of Thm 1(i).

Recall that an element $z \in I$ is *minimal* if $z \notin \mathfrak{m}I$; a *minimal basis* of I is a collection of elements $z_1, ..., z_n \in I$ that maps to a basis of $I/\mathfrak{m}I$; by Nakayama, such a collection always generates I.

Now suppose first that I has a minimal element of the form $z = x^\alpha$. Then z alone cannot generate I so there must exist an element $z' = ax^{\alpha'} + by^{\beta'}$ independent of z mod $\mathfrak{m}I, b \neq 0$. If $\alpha' < \alpha, a \neq 0$ then z is a multiple of z', which is a contradiction. Therefore $\alpha \leq \alpha'$ hence x^α and $by^\beta = z' - ax^{\alpha'-\alpha}z$ are independent $\mod \mathfrak{m}I$. Moreover for any other $z" = a"x^{\alpha"} + b"y^{\beta"} \in I$, a similar argument shows $\alpha" \geq \alpha, \beta" \geq \beta'$, therefore $z" \in (x^\alpha, y^{\beta'})$, so $x^\alpha, y^{\beta'}$ generate I. Then a simple dimension count shows $\alpha + \beta' = m + 1$.

Now suppose I has no minimal element of the form x^α or y^β, and consider a minimal element $z = x^\alpha + by^\beta$ where α is smallest among all elements (or equivalently, among all *minimal* elements) of I. Suppose I has another minimal element $z' = x^{\alpha'} + b'y^{\beta'}$ independent of $z \mod \mathfrak{m}I$. Then $\alpha \leq \alpha'$. If $\beta' < \beta$ then $z - (b/b')y^{\beta-\beta'}z' = x^\alpha$ is minimal, contradiction. If $\beta' = \beta$ then $(z - z')/(b - b') = y^\beta$ is minimal, contradiction. If $\beta' > \beta, \alpha' = \alpha$ then $x^\alpha = z - (b/b')y^{\beta'-\beta}z'$ is minimal, contradiction. Finally, if $\alpha < \alpha', \beta < \beta'$ then z' is a multiple of z, contradiction. Thus z generates I and again a simple dimension count shows $z = x^i + ay^{m-i}$.

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