JACOBI COHOMOLOGY, LOCAL GEOMETRY OF MODULI SPACES AND HITCHIN CONNECTIONS

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Introduction

The main purpose of this paper is to develop some cohomological tools for the study of the local geometry of moduli and parameter spaces in complex Algebraic Geometry. The main ingredient will be the language of Lie algebras, in particular differential graded Lie algebras, their representations, and certain complexes associated to these that we generally call *Jacobi* complexes. Why the presence of Lie algebras ? We understand since Felix Klein that geometry, in one way or another, is conveniently expressed in terms of symmetry groups, so it is reasonable to expect a similar thing to be true of deformations or variations of a geometric object. Now a geometric structure on a topological space X may be described by gluing data on a collection of 'standard' or 'trivial' pieces (e.g. polydiscs in the case of a manifold, or or free modules in the case of a vector bundle), and a deformation of this structure may be obtained by varying the gluing data. Now, infinitesimal variations of gluing data can be described in terms of Lie algebras (e.g. of vector fields or linear endomorphisms). Consequently, infinitesimal deformations of geometric structures can be systematically expressed in terms of a sheaf of Lie algebras on X. Thus, such sheaves will play a fundamental role in our work.

One of the main tools we develop here is a direct cohomological construction, in terms of the moduli problem, of the vector fields and differential operators on moduli spaces, together with their action on functions, as well as on 'modular' modules, i.e. those associated to the moduli problem, including formulae for composition and Lie bracket (commutator); in particular, we obtain a canonical formula for the Lie algebra of vector fields on a moduli space together with its natural representation on (formal) functions, as well as extensions to the case of differential operators acting on modular vector bundles . This material is developed is §§ 1-4 beginning with the fundamental construction of modular modules given in Thm. 1.1 (which clarifies and streamlines some constructions from [10]). In §5 we present a general, formal study of connections on moduli spaces, which proceeds by identifying a class of Lie algebras that we call *connection algebras*, which are universal, in a suitable sense, with respect to the presence of an integrable connection over their deformation spaces.

As an application of these methods we will study the relation between the geometry and deformations of a given complex manifold X and that of a moduli space $\mathcal{M}_X$ of vector bundles on X. Since $\mathcal{M}_X$ is a functor of X, it seems intuitively plausible that an automorphism of X should act on $\mathcal{M}_X$, and likewise for infinitesimal automorphisms. This intuitive idea obviously needs some precising, because on the one hand the Lie algebra T_X of holomorphic vector fields on X will typically admit no global sections, and on the other hand as sheaves, T_X and $T_{\mathcal{M}_X}$ live on different spaces. In fact, we will show that there is a Lie homomorphism Σ_X from the differential graded Lie algebra associated to T_X to that of $T_{\mathcal{M}_X}$. This is useful because a Lie homomorphism induces a map on the associated deformation spaces, so Σ_X can be used to relate deformations of X to those of $\mathcal{M}_X$.

The latter result will be further refined in case X has dimension 1, i.e. is a compact Riemann surface, by showing that the map Σ_X factors through a Lie homomorphism to a certain Lie-theoretic object the we call *Lie atom*, associated to $\mathcal{M}_X$ (more specifically, it is a 'heat atom', having to do with Heat deformations of $\mathcal{M}_X$) . The new notion of Lie atom, barely touched on here, seems to deserve further study and receives some in [Rla]. As an essentially immediate consequence of the factorization we will deduce the so-called Hitchin or Knizhnik- Zamolodchikov (KZH) flat connection over the moduli of curves. This is a holomorphic connection on the projective bundle associated to the vector bundle $\mathfrak{V}$ with fibre $H^0(SU_X(r,L),G)$, where $SU_X(r,L)$ is the moduli space of (S-equivalence classes of) semistable bundles of rank r and determinant L on X, and G is a line-bundle on $SU_X(r,L)$ (which is necessarily, by results of Drezet-Narasimhan [4], a power of the modular theta bundle, and a fractional power of the canonical bundle). That the projectivization of $\mathfrak{V}$ should admit a flat connection was conjectured by physicists based on ideas from Conformal Field Theory, and subsequently treated by a number of mathematicians including Beilinson-Kazhdan, Hitchin, Faltings, Ueno and Witten (cf. [2] [3] [6] [Fa][8] [16] [17] [19] and references therein). Our approach is quite close to Hitchin's as regards the construction of the connection; the ideas here go back to some degree to Welters [20]. The novelty is that we are able to extend the Welters-Hitchin construction, which is essentially 'plain' first-order deformation theory, to the Lie theoretic context, via a connection algebra. This yields a transparent proof that the connection thus obtained is automatically flat, a second-order conclusion– modulo showing that the relevant maps are Lie homomorphisms. Thus we get a new and essentially 'algebraic' proof of the flatness of the connection, replacing some arguments by Hitchin [6] which appeal to infinite-dimensional symplectic geometry. This serves to illustrate the deformation-theoretic usefulness of a philosophy familair in Lie theory, that the Lie algebra framework allows automatic passage from first order to arbitrary order. In the interest of perspective, we must add that our main focus in this paper is not to give the umpteenth construction of the KZH connection, but rather to develop general methods and use KZH as a benchmark.

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Remark. After this was written the author was made aware of two other papers on the construction of Hitchin connections, by V. Ginzburg (Resolution of diagonals and moduli spaces,in The moduli space of curves (Texel Island, 1994), 231–266, Progr. Math., 129, Birkhuser Boston, Boston, MA, 1995.) and S. Barannikov

(Quantum periods-I (alg-geom/0006193)).

1. Moduli modules revisited

Our purpose here is to revisit and modify some basic deformation-theoretic constructions from [9], using a slightly different viewpoint that seems more convenient for applications, such as the construction of Lie brackets on moduli. See [9] for definitions and explanations. Let $(\mathfrak{g}, E)$ be an admissible pair with $H^0(\mathfrak{g}) = 0$ on a Hausdorff space X; thus $\mathfrak{g}$ is a sheaf of $\mathbb{C}$-Lie algebras, E is a $\mathfrak{g}$-module, and both satisfy some reasonable 'nice cohomology' assumptions. In [9] we constructed the universal deformation ring $\hat{R}(\mathfrak{g}) = \varprojlim R_m(\mathfrak{g}), R_m(\mathfrak{g}) = \mathbb{C} \oplus \mathbb{H}^0(J_m(\mathfrak{g}))^*$ where $J_m(\mathfrak{g})$ is the Jacobi complex of $\mathfrak{g}$, as well as the flat $R_m(\mathfrak{g})$-module $E_m = M_m(\mathfrak{g}, E)$ that is the universal $\mathfrak{g}$-deformation of E, and whose cohomology groups may be called the 'moduli modules' associated to E. As in [9] we let $(\mathfrak{g}^\cdot, \delta)$ be a soft dgla resolution of $\mathfrak{g}$ and $(E^\cdot, \partial)$ be a soft resolution of E that is a graded $\mathfrak{g}^\cdot$-module. Then the standard (Jacobi) complex $J_m(\mathfrak{g}^\cdot) =: \mathfrak{J}^\cdot_m$ has terms which may be decomposed as

$$\lambda^i(\mathfrak{g}^\cdot) = \bigoplus_j \mathfrak{g}_j^{-i}$$

where each $\mathfrak{g}_j^{-i}$ has total degree $j - i$ and is a sum of terms of the form

$$(\lambda^{a_1}\mathfrak{g}^{b_1} \boxtimes \cdots) \boxtimes (\sigma^{c_1}\mathfrak{g}^{d_1} \boxtimes \cdots)$$

with b_k even, d_k odd and

$$\sum a_k + \sum c_k = i, \sum a_k b_k + \sum c_k d_k = j.$$

Thus $\mathbb{H}^\cdot(J_m(\mathfrak{g}))$ is $H^\cdot$ of a complex $\Gamma J^\cdot_m$ with

$$\Gamma J^r_m = \bigoplus_{i=1}^m \Gamma(\mathfrak{g}^{-i}_{r+i}).$$

It is convenient to augment ΓJ^r_m by adding the term $\mathfrak{g}^0_0 = \mathbb{C}$ in degree 0.

Note that ΓJ^r_m depends only on the differential graded Lie algebra $g^\cdot = \Gamma(\mathfrak{g}^\cdot)$; moreover the quasi-isomorphism class of ΓJ^r_m depends only on the dgla quasi-isomorphism class of $\mathfrak{g}^\cdot$. Also, it is worth noting at this point that the differential δ on $\mathfrak{J}^\cdot_m$ and $\Gamma J^\cdot_m$ is a 'graded coderivation', in the sense of commutativity of the following diagram

$$\begin{array}{ccc} \mathfrak{J}^\cdot_m & \stackrel{\delta}{\to} & \mathfrak{J}^\cdot_m \\ \downarrow & & \downarrow \\ \sigma^2\mathfrak{J}^\cdot_{m-1} & \xrightarrow{\sigma^2(\delta)} & \sigma^2\mathfrak{J}^\cdot_{m-1} \end{array}$$

where the vertical arrows are the comultiplication maps and $\sigma^2(\delta)$ is induced by δ by functoriality, i.e. is the map given by extending δ 'as a derivation', given by the rule

$$ab \mapsto \delta(a)b \pm a\delta(b).$$

Thus $\mathfrak{J}^\cdot_m$ and $\Gamma J^\cdot_m$ possess 'differential graded OS structures' (see [Rcid, Sec. 1]; OS structure essentially means filtered comultiplicative structure) . It follows that the same is true not only of their H^0 but of the entire cohomology

$$H^\cdot(\Gamma J^\cdot_m) = \bigoplus H^i(\Gamma J^\cdot_m) = \bigoplus \mathbb{H}^i(J_m(\mathfrak{g})).$$

Now to get an algebra out of this one may either dualize the cohomology, as was done in [9]; alternatively, one may dualize the complex and then take cohomology. Thus let $\Gamma^* J^\cdot_m$ be the complex dual to $\Gamma J^\cdot_m$, with

$$\Gamma^* J^r_m = (\Gamma J^{-r}_m)^*$$

(vector space dual) and dual differentials. Thus $\Gamma^* J^r_m = \bigoplus\limits_{i=1}^m \Gamma(\mathfrak{g}^{-i}_{-r+i})^*$, with

$$\Gamma(\mathfrak{g}_j^{-i})^* = \bigoplus [\bigwedge^{a_1}\Gamma(\mathfrak{g}^{b_1})\otimes\cdots]\otimes[\mathrm{sym}^{c_1}\Gamma(\mathfrak{g}^{d_1})\otimes\cdots],$$

with sum over all nonnegative with b_k even, d_k odd and

$$\sum a_k + \sum c_k = i, \sum a_k b_k + \sum c_k d_k = j.$$

Clearly

$$H^i(\Gamma^* J^\cdot_m) = H^{-i}(\Gamma J^\cdot_m)^*.$$

Then $\mathbb{C}\oplus\Gamma^* J^\cdot_m$ is in fact a differential, graded- commutative associative algebra (graded commutative means two homogeneous elements commute unless they are both odd, in which case they anticommute). Indeed, since the OS or comultiplicative structure on J was derived from exterior comultiplication on $\lambda^\cdot(\mathfrak{g}^\cdot)$, clearly the multiplication on $\mathbb{C}\oplus\Gamma^* J^\cdot_m$ is induced by graded exterior multiplication, hence is obtained by tensoring together the various exterior products on the $\bigwedge^{a_k}\Gamma(\mathfrak{g}^{b_k})$, b_k even and symmetric products on the $\mathrm{sym}^{c_k}\Gamma(\mathfrak{g}^{d_k})$, d_k odd. Thus the total cohomology

$$\tilde{R}_m(\mathfrak{g}) := H^\cdot(\Gamma^* J^\cdot_m) = (H^\cdot(\Gamma J^\cdot_m)^*)$$

is a local graded Artin algebra, and in particular the degree-0 piece

$$R_m(\mathfrak{g}) = H^0(\Gamma^* J^\cdot_m) = (H^0(\Gamma J^\cdot_m)^*)$$

is the mth universal deformation ring mentioned above.

Next, we revisit the construction of the m-universal deformation $M_m(\mathfrak{g}, E)$ of a given $\mathfrak{g}-$module E. The proof of ([10], Thm 3.1)shows that the MOS structure $V^m(E)$- which in turn determines $M_m(\mathfrak{g}, E)$ purely formally- is a cohomology sheaf $\mathcal{H}^0$ of a (multiple) complex $(K^\cdot_m, d^\cdot) = (K^\cdot_m(\mathfrak{g}, E), d^\cdot)$ whose part in total degree r is

$$K^r_m = \bigoplus_{i,j,j\geq -m}\Gamma(\mathfrak{g}^j_{r-j-i})\otimes E^i = \bigoplus_i \Gamma J^{r-i}_m \otimes E^i$$

(note there is a misprint in the corresponding formula in ([10], p.430, l.-5). 'Transposing' this, we define a complex

$$L^\cdot = {}^t K^\cdot_m(\mathfrak{g}, E)$$

of sheaves with

$$L^r = \bigoplus_{i,j}\Gamma(\mathfrak{g}^{-j}_{i+j-r})^*\otimes E^i = \bigoplus_i \Gamma^* J^{r-i}_m \otimes E^i$$

with differentials the 'transpose' of those of $K^\cdot$, where the transpose of a map

$$d : A\otimes E \to B\otimes E'$$

with A, B finite-dimensional vector spaces, is the map

$${}^t d : B^*\otimes E \to A^*\otimes E'$$

defined by the rule

$$<{}^t d(b^*\otimes e), a> = < b^*, d(a\otimes e)>$$

in which $<,>$ refers to the natural pairings

$$A^*\otimes E' \times A \to E', B^*\times B\otimes E' \to E'.$$

Since ΓJ_m has finite-dimensional cohomology we may 'approximate' it by a finite-dimensional subcomplex quasi- isomorphic to it (this remark will be used frequently in the sequel). Since the comultiplicative structure on $K^\cdot$ as defined in [10] coincides with the evident structure induced by comultiplication on the $\mathfrak{g}$ factor, clearly $L^\cdot$ has a natural structure of a sheaf of differential graded $\mathbb{C}\oplus\Gamma^* J^\cdot_m$-modules, and consequently the total cohomology $\mathcal{H}^\cdot(L^\cdot)$ is a sheaf of graded $\tilde{R}_m(\mathfrak{g})$-modules and in particular $\mathcal{H}^0(L^\cdot)$ is a sheaf of $R_m(\mathfrak{g})$-modules. Note also that for any $\mathfrak{g}-$modules E_1, E_2, the multiplicative structure on $\Gamma^* J^\cdot_m$ gives rise to a natural pairing

$$(1.1)\qquad {}^t K^\cdot_m(\mathfrak{g}, E_1)\times {}^t K^\cdot_m(\mathfrak{g}, E_2) \to {}^t K^\cdot_m(\mathfrak{g}, E_1\otimes E_2).$$

Note also that $\Gamma L^\cdot$ depends only on $\Gamma E^\cdot$ (as differential graded $g^\cdot$)$-$ module), and we may similarly associate a complex, still denoted ${}^t K^\cdot_m(g^\cdot, F^\cdot)$, to any differential graded $g^{\cdot}-$module $F^{\cdot}$. The following result streamlines and extends some of the foundational results of [10].

Theorem 1.1. *We have natural isomorphisms*

$$E_m \simeq M_m(\mathfrak{g}, E) \simeq \mathcal{H}^0({}^tK^{\cdot}_m(\mathfrak{g}, E)).$$

proof. The first isomorphism is just [10], Thm 3.1, but it is worth observing that the proof given there involves an implicit spectral sequence argument, and may be shortened by making this argument explicit. Thus, write $K^{\cdot}$ as a double complex

$$K^{i,j} = \bigoplus_k \Gamma(\mathfrak{g}^k_{j-k}) \otimes E^i = \Gamma J^j_m \otimes E^i.$$

Now because $H^0(\mathfrak{g}) = 0$, the map $\delta : \Gamma(\mathfrak{g}^0) \to \Gamma(\mathfrak{g}^1)$ is injective, so choosing a complement to its image we obtain a quasi-isomorphic complex in strictly positive degrees, and it will be convenient to replace all resulting complexes by ones formed with this modified complex. This in particular ensures that $K^{i,j} = 0$ for $j < 0$. Note that the vertical differentials $K^{i,j} \to K^{i,j+1}$ are of the form $\delta \otimes id$ where δ is a differential of $\Gamma J^{\cdot}_m$. Consequently, the first spectral sequence of the double complex has an E_1 term

$$\begin{aligned} E_1^{p,q} &= (\mathbb{C} \oplus H^0(\Gamma J^{\cdot}_m)) \otimes E^p, q = 0 \\ &= H^q(\Gamma J^{\cdot}_m) \otimes E^p, q > 0. \end{aligned}$$

Our assumption that $H^0(\mathfrak{g}) = 0$ easily implies that $H^q(\Gamma J^{\cdot}_m) = 0$ for $q < 0$, so this is a first-quadrant spectral sequence and consequently

$$H^0(K^{\cdot}) = \ker(E_1^{0,0} \to E_1^{0,1}) = \ker(V_0^m \otimes E^0 \to V_0^m \otimes E^1),$$

and identifying the map and applying a suitable linear functor gives the result.

For the second isomorphism we argue analogously, considering the double complex

$$L^{i,j} = \bigoplus_k \Gamma(\mathfrak{g}^k_{-j-k})^* \otimes E^i = \Gamma^* J^j_m \otimes E^i.$$

As above, this vanishes for $j > 0$. We get a spectral sequence whose E_1 term may be identified as

$$E_1^{p,q} = \tilde{R}_m(\mathfrak{g})^q \otimes E^p$$

where $\tilde{R}_m(\mathfrak{g})^q$ denotes the part in degree q i.e. $H^q(\Gamma^* J^{\cdot}_m)$ for $q > 0$ and $\mathbb{C} \oplus H^0(\Gamma^* J^{\cdot}_m)$ for $q = 0$. Thanks again to our hypothesis that $H^0(\mathfrak{g}) = 0$, this vanishes for all $q > 0$, so in this case we have a fourth-quadrant spectral sequence. Now note that if we view $(\bigoplus_q E_1^{p,q})$ as a complex indexed by p only, the differentials are $\tilde{R}_m(\mathfrak{g})$-linear, so as such this is a finite complex of free $\tilde{R}_m(\mathfrak{g})$-modules. Now we use the following fairly standard fact.

Lemma 1.2. *Let $A^{\cdot}$ be a complex of flat modules (not necessarily of finite type) over a local Artin algebra S with residue field k. Suppose $A^{\cdot} \otimes_S k$ is exact in positive degrees. Then so is $A^{\cdot}$*

proof. Use induction on the length of S. Let $I < S$ be a nonzero 'socle' ideal, with $I.\mathfrak{m}_S = 0$. By flatness we have a short exact sequence of complexes

$$0 \to IA^{\cdot} \to A^{\cdot} \to A^{\cdot} \otimes (S/I) \to 0.$$

As $IA^{\cdot}$ is isomorphic to a direct sum of copies of $A^{\cdot} \otimes_S k$, it is exact in positive degrees; by induction, the same is true of $A^{\cdot} \otimes (S/I) \to 0$. By the long cohomology sequence, it follows that the middle is also exact in positive degrees, proving the

Lemma. $\square$ (The Lemma is perhaps more familiar in the case where the A^i are of finite type (hence free) and zero for $i >> 0$, and S is not necessarily Artinian.)

For our complex above tensoring with the residue field just yields the original complex $E^\cdot$ which is of course exact in positive degrees, so Lemma 1.2 applies. We conclude that $E_1^{p,q}$ is exact at all terms with $p > 0$ and in particular we have

$$H^0(L^\cdot) = \ker(E_1^{0,0} \to E_1^{0,1}) = \ker(R_m(\mathfrak{g}) \otimes E^0 \to R_m(\mathfrak{g}) \otimes E^1),$$

and again the map may be easily identified as the one yielding E_m. $\square$

We will now formalize some constructions which occurred in the foregoing proof. For a double complex $K^{\cdot\cdot}$ we will denote by $K^{\cdot\cdot}_{\lceil j}$ its jth lower truncation, which is the double complex defined by

$$K^{h,i}_{\lceil j} = \begin{cases} 0, & i > j, \\ \ker(K^{h,j} \to K^{h,j+1}), & i = j \\ K^{h,i}, & i < j. \end{cases}$$

Similarly, we will denote by $K^{\cdot\cdot}_{\lfloor j}$ the jth upper truncation, which is the double complex defined by

$$K^{h,i}_{\lfloor j} = \begin{cases} 0, & i < j, \\ K^{h,j}/K^{h,j-1}, & i = j \\ K^{h,i}, & i > j. \end{cases}$$

Motivated by the foregoing proof, we set

$$\tilde{L}_m(\mathfrak{g}, E) = {}^t(K^\cdot_m(\mathfrak{g}, E))_{\lceil 0} \tag{1.2}$$

which, as we have seen, is a double complex in nonpositive vertical degrees (indeed in the fourth quadrant) quasi-isomorphic to ${}^t(K^\cdot_m(\mathfrak{g}, E))$ itself. Also set

$$L_m(\mathfrak{g}, E) = \tilde{L}_m(\mathfrak{g}, E)_{\lfloor 0} \tag{1.3}$$

which is thus to be considered as a simple (horizontal) complex.

Corollary 1.3. *Assumptions as in 1.1, we have*

$$(i)\quad \mathcal{H}^i(L^\cdot_m(\mathfrak{g}, E)) = \begin{cases} 0 & i \neq 0 \\ M_m(\mathfrak{g}, E) & i = 0 \end{cases}$$

$$(ii)\quad H^i(X, M_m(\mathfrak{g}, E)) = H^i(\Gamma(L^\cdot_m(\mathfrak{g}, E))) \quad \square$$

Note that the above constructions make sense for any differential graded Lie algebra $g^\cdot$ and differential graded $g^\cdot$-module $G^\cdot$ and yield (double) complexes

$$K_m(g^\cdot, G^\cdot): \quad K^{i,j} = J^j_m(g^\cdot) \otimes G^i$$
$${}^tK_m(g^\cdot, G^\cdot): \quad {}^tK^{i,j} = {}^* J^j_m(g^\cdot) \otimes G^i.$$

Likewise for $\tilde{L}_m, L_m$. Moreover, clearly the quasi-isomorphism classes of these complexes depends only on that of $G^\cdot$ as $g^\cdot$-module. We collect some of the simple properties of these constructions in the following

Lemma 1.4. *In the above situation, assume $H^{\leq 0}(g^\cdot) = 0$. Then*

(i) If $G^\cdot$ is acyclic in negative degrees, then so is $K_m(g^\cdot, G^\cdot)$(i.e. it is acyclic in negative total degrees).

(ii) If $G^\cdot$ is acyclic in positive degrees, then so is ${}^tK_m(g^\cdot, G^\cdot)$.

(iii) If $G^\cdot$ is acyclic in negative degrees, then $\tilde{L}_m(g^\cdot, G^\cdot)$ is a 4th- quadrant bicomplex and $L_m(g^\cdot, G^\cdot)$ is acyclic in negative degrees.

(iv) The complex ${}^*G^\cdot$ dual to $G^\cdot$ is also a $g^\cdot$-module, and we have

$${}^*(K_m(g^\cdot, G^\cdot)) = {}^t K_m(g^\cdot, {}^* G^\cdot). \quad \square$$

Corollary 1.5. *In the situation of Corollary 1.3, assume moreover that for some i we have*
$$H^j(E) = 0, \forall j \neq i.$$
*Then we have (where * denotes vector space dual)*

$$\text{(i)} \qquad H^i(K_m(\mathfrak{g}, E))^* = H^{-i}({}^tK_m(\Gamma(\mathfrak{g}^\cdot), \Gamma(E^\cdot)^*) = H^{-i}(L_m(\Gamma(\mathfrak{g}^\cdot), \Gamma(E^\cdot)^*)),$$

$$\text{(ii)} \qquad H^i(M_m(\mathfrak{g}, E))^* = H^{-i}(K_m(\Gamma(\mathfrak{g}^\cdot), \Gamma(E^\cdot)^*)).$$

proof. (i) The first equality is just the fact that cohomology commutes with dualizing. The second follows by applying Lemma 1.4 to $\Gamma(E^\cdot)[-i]$ which is acyclic except in degree 0 and consequently $H^{-i}({}^tK_m(\Gamma(\mathfrak{g}^\cdot), \Gamma(E^\cdot)$ only involves $H^0(\Gamma J_m)$. (ii) follows from Corollary 1.3. □

A sheaf (or complex) satisfying the condition of Corollary 1.5 will be said to be *equicyclic of degree i* or *i-equicyclic.* Next, note that a $g^\cdot-$bilinear pairing of differential graded $g^\cdot-$modules
$$G_1 \times G_2 \to G_3$$
gives rise to a pairing of double complexes

$$\tilde{L}_m(G_1) \times \tilde{L}_m(G_2) \to \tilde{L}_m(G_3) \tag{1.4}$$

hence also

$$L_m(G_1) \times L_m(G_2) \to L_m(G_3) \tag{1.5}$$

Clearly these pairings are compatible with the ${}^*J_m(g^\cdot)$-module structure on these complexes, so we get an $R_m(g^\cdot)$- linear pairing
$$H^i(L_m(G_1)) \times H^j(L_m(G_2)) \to H^{i+j}(L_m(G_3)).$$
In particular, for any differential graded $g^\cdot$-module $G^\cdot$, using the natural $g^\cdot$-linear pairing
$$G^\cdot \times^* G^\cdot \to \mathbb{C}$$
(where $\mathbb{C}$ is endowed with the trivial $g^\cdot$-action), we obtain an $R_m(g^\cdot)-$linear pairing

$$H^i(L_m(G^\cdot)) \times H^{-i}(L_m({}^*G^\cdot)) \to H^0(L_m(\mathbb{C})) = R_m(g^\cdot). \tag{1.5}$$

Now suppose moreover that $G^\cdot$ is $i-$equicyclic. Then clearly $H^i(L_m(G^\cdot))$ is a free $R_m(g^\cdot)-$module (as obstructions lie in $H^{i+1}(G^\cdot) = 0$) and similarly for $H^{-i}(L_m({}^*G^\cdot))$. Since the pairing (1.5) yields the natural perfect pairing of $H^i(G^\cdot)$ and $H^{-i}({}^*G^\cdot)$ modulo the maximal ideal of $R_m(g^\cdot)$, it is likewise perfect. Thus

Corollary 1.6. *In the above situation, if $G^\cdot$ is $i-$equicyclic, then $H^i(L_m(G^\cdot))$ and $H^{-i}(L_m({}^*G^\cdot))$ are free $R_m(g^\cdot)-$modules naturally dual to each other.*

Corollary 1.7. *In the situation of Corollary 1.5, $H^i(K_m(\mathfrak{g}, E))$ is the $R_m(\mathfrak{g})$-module dual to the free module $H^{-i}(K_m(\Gamma(\mathfrak{g}^\cdot), \Gamma^*(E^\cdot)))$.*

proof. This follows from Corollary 1.6 and the fact that $H^i(K_m(\mathfrak{g}, E))$ and $\mathrm{Hom}(H^{-i}(L_m(\Gamma^*(E))), \mathbb{C})$ coincide as R_m-modules. □

Lemma 1.8. *For any $g^\cdot-$module $G^\cdot$,*
(i) $K_m(G)$ and $L_m(G)$ are $g-$modules;
(ii) there is a natural inclusion $L_{m+k}(G)\to L_k(L_m(G))$.

proof. (i) The point is that the natural $g-$ action on the components of $K(G)$ and $L(G)$ (suppressing m for convenience) commutes with the differentials. Firstly for K and for its first differential $K^{-1}(G)=g\otimes G\to K^0(G)=G$, this commutativity is verified by the fact that

$$<v,<w,a>>=<[v,w],a>+<w,<v,a>>,\forall v,w\in g,a\in G,$$

which means that the following diagram commutes

$$\begin{array}{ccc} g\times g\otimes G & \stackrel{\mathrm{id}\times\delta}{\to} & g\times G \\ \downarrow & & \downarrow \\ g\otimes G & \stackrel{\delta}{\to} & g \end{array}$$

(vertical arrows given by the action; NB g acts both on g (adjoint action) and G).

Next, the case of an arbitrary differential of K follows by noting the inclusion $K^{-i}(G)\subset K^{-1}(\bigwedge^{i-1} g\otimes G),\forall i$, which makes the following diagram commute

$$\begin{array}{ccc} K^{-i}(G) & \stackrel{\delta}{\to} & K^{-i+1}(G) \\ \downarrow & & \| \\ K^{-1}(\bigwedge^{i-1} g\otimes G) & \stackrel{\delta}{\to} & K^{0}(\bigwedge^{i-1} g\otimes G). \end{array}$$

For the case of the L complex, again it suffices to prove commutativity of the action with the first differential, i.e. commutativity of the diagram

$$\begin{array}{ccc} g\times G & \stackrel{id\times{}^t\mathrm{br}}{\to} & g\times {}^*g\otimes G \\ \downarrow & & \downarrow \\ G & \stackrel{{}^t\mathrm{br}}{\to} & {}^*g\otimes G, \end{array}$$

which amounts to

$$ {}^t\mathrm{br}(<v,a>)=<v,{}^t\mathrm{br}(a)>,\forall a\in G,v\in g. \tag{1.6}$$

This is verified by the following computation. Pick $y\in g$ and write

$${}^t\mathrm{br}(<v,a>)=\sum w_i^*\otimes b_i.$$

Then (assuming a,v,y are all even)

$$\begin{gathered} <y.{}^t\mathrm{br}(<v,a>)>=<y,<v,a>>=<[y,v],a>+<v,<y,a>> \\ =<[y,v],a>+<v,<y.{}^t\mathrm{br}(a)> \\ =<[y,v],a>+<[v,y].{}^t\mathrm{br}(a)>+<y.<v,{}^t\mathrm{br}(a)>> \\ =<[y,v],a>+<[v,y],a>+<y.<v,{}^t\mathrm{br}(a)>>=<y.<v,{}^t\mathrm{br}(a)>>. \end{gathered}$$

Similar computations can be done for other parities. Thus (1.6) holds, as claimed.

(ii) Note that $L_k(L_m(G))$ is naturally a double complex with vertical differentials those coming from $L_m(G)$. Then each term of the associated total complex is a sum of copies of $\bigwedge^i({}^*g)\otimes G$ and naturally contains $\bigwedge^i({}^*g)\otimes G$ itself, embedded diagonally. It is easy to check that this yields a map of complexes $L_{m+k}(G)\to L_k(L_m(G))$. $\square$

Remark. The latter inclusion is analogous, and closely related to, the natural map on jet or principal parts modules

$$P^{m+k}(M)\to P^k(P^m(M))$$

for any module M (over a commutative ring).

2. Tangent Algebra

The purpose of this section is to construct the tangent (or derivation) *Lie algebra* of vector fields on a moduli space $\mathcal{M}$, together with its natural representation on the (formal) functions on $\mathcal{M}$. More specifically, we will say that a locally $\mathbb{C}-$ringed topological space $\mathcal{M}$ is a *locally fine moduli space* (of $\mathcal{O}_X$ or $(\mathcal{O}_X, \tilde{\mathfrak{g}})$-modules, if this is not understood), if there exists an $\mathcal{O}_X$- Lie algebra $\tilde{\mathfrak{g}}$ and a sheaf $\mathcal{E}$ of $\mathcal{O}_X$ and $\tilde{\mathfrak{g}}$-modules on $\mathcal{M} \times X$, with $\tilde{\mathfrak{g}}$ acting $\mathcal{O}_X-$linearly, such that for each point $[E] \in \mathcal{M}$, we may identify

$$E = \mathcal{E}|[E] \times X := \mathcal{E} \otimes \mathbb{C}(E)$$

($\mathbb{C}(E) =$ residue field of $\mathcal{M}$ at $[E]$), and the formal completion $\hat{\mathcal{E}}$ of $\mathcal{E}$ along $[E] \times X$ is isomorphic to the universal formal $\mathfrak{g}_E$-deformation of E, where $\mathfrak{g}_E = \tilde{\mathfrak{g}} \otimes \mathbb{C}(E)$, as constructed in [9] and above, so that for each m, or at least a cofinal set of m's, we have (compatible) isomorphisms

$$\mathcal{E} \otimes (\mathcal{O}_{\mathcal{M}}/\mathfrak{m}_{[E]}^{m+1}) \simeq M_m(\mathfrak{g}_E, E), \quad \mathcal{O}_{\mathcal{M}}/\mathfrak{m}_{[E]}^{m+1} \simeq R_m(\mathfrak{g}_E). \tag{2.2}$$

We do *not* assume points of $\mathcal{M}$ correspond bijectively with 'equivalence' classes of objects $[E]$ (which we don't even define)– when a fine moduli space $\mathfrak{M}$ does exist, our assumptions imply that the natural classifying map $\mathcal{M} \to \mathfrak{M}$ is étale. Of course by definition the above properties essentially depend on $\tilde{\mathfrak{g}}$ only and not on the particular $\tilde{\mathfrak{g}}$ -module $\mathcal{E}$. Then the tangent sheaf

$$T_{\mathcal{M}} \simeq R^1 p_{1*}(\tilde{\mathfrak{g}}) \tag{2.3}$$

(isomorphism as $\mathcal{O}_{\mathcal{M}}$-modules). Now fix a point $[E] \in \mathcal{M}$ and set $\mathfrak{g} = \mathfrak{g}_E$. Viewing $\tilde{\mathfrak{g}}$ as a module over itself via the adjoint representation, we get an isomorphism of the jet or principal part space

$$P^m(T_{\mathcal{M}}) \otimes \mathbb{C}(E) \simeq M_m(\mathfrak{g}, \mathfrak{g}). \tag{2.4}$$

Now the Lie bracket on $T_{\mathcal{M}}$ is a first-order differential operator in each argument, hence yields an $\mathcal{O}_{\mathcal{M}}$-linear 'bracket' pairing

$$B_m : P^m(T_{\mathcal{M}}) \times P^m(T_{\mathcal{M}}) \to P^{m-1}(T_{\mathcal{M}}). \tag{2.5}$$

Likewise, the action of $T_{\mathcal{M}}$ on $\mathcal{O}_{\mathcal{M}}$ yields an 'action' pairing

$$A_m : P^m(T_{\mathcal{M}}) \times \mathcal{O}_{\mathcal{M}}/\mathfrak{m}_{[E]}^{m+1} \to \mathcal{O}_{\mathcal{M}}/\mathfrak{m}_{[E]}^{m} \tag{2.6}$$

The problem is to identify the pairings (2.5), (2.6) in terms of the identifications (2.2) and (2.4). We shall proceed to define some pairings on complexes that will yield this. For a down-to-earth cocycle version of this, the reader may wish to consult Elaboration 2.2 below.

First, it was already observed above that the dgla $\Gamma(\mathfrak{g}^\cdot)$ is quasi-isomorphic to a sub-dgla of itself concentrated in strictly positive degrees. More canonically, we may set

$$\Gamma^i = \{ \begin{array}{cc} 0, & i \leq 0 \\ \mathfrak{g}^1/\delta(\mathfrak{g}^0), & i = 1 \\ \Gamma(\mathfrak{g}^i)/[\delta\Gamma(\mathfrak{g}^0), \Gamma(\mathfrak{g}^{i-1})], & i > 1. \end{array}$$

Then $\Gamma^{\cdot}$ is a canonical quasi-isomorphic quotient dgla of $\Gamma(\mathfrak{g}^{\cdot})$ concentrated in positive degrees. Although a given $\mathfrak{g}-$module E may not give rise to a $\Gamma^{\cdot}-$module, still for the purposes of this section we may as well replace $\Gamma(\mathfrak{g}^{\cdot})$ by $\Gamma^{\cdot}$ and assume it is concentrated in positive degrees. Let us also set, for convenience

$$g^{\cdot} = \Gamma(\mathfrak{g}^{\cdot}).$$

We now begin constructing the action pairing. Note that the complex ${}^*g^{\cdot} = \Gamma(\mathfrak{g}^{\cdot})^*$ is naturally a graded module over the dgla $g^{\cdot}$ known as the *coadjoint representation*, via the rule

$$<< a, b^* >, b >=< b^*, [a, b] >, a, b, \in \Gamma(\mathfrak{g}), b^* \in \Gamma^*(\mathfrak{g}).$$

Hence we get a complex which we will write as ${}^tK_m(\mathfrak{g}, \mathfrak{g}^*)$ or ${}^tK_m(g^{\cdot}, {}^*g^{\cdot})$. To abbreviate, we will also write $\Gamma({}^tK_m(\mathfrak{g}, E))_+$ as $\tilde{L}_m(E)$ and ${}^tK_m(\mathfrak{g}, \mathfrak{g}^*)_+$ as $\tilde{L}_m(\mathfrak{g}^*)$, and we will view them as double complexes in nonpositive vertical degrees (in the latter case, nonpositive horizontal degrees as well). One can check easily that the duality pairing $g^{\cdot} \otimes^* g^{\cdot} \to \mathbb{C}$, viewed as a map between $\mathfrak{g}^{\cdot}$-modules (where $\mathbb{C}$ has the trivial action), is $g^{\cdot}$-linear, hence gives rise to a pairing of double complexes (preserving total bidegree)

$$\tilde{L}_m(\mathfrak{g}) \times \tilde{L}_m(\mathfrak{g}^*) \to \tilde{L}_m(\mathfrak{g} \otimes \mathfrak{g}^*) \to \tilde{L}_m(\mathbb{C}) = \Gamma^* J_m \tag{2.7}$$

(where the RHS is viewed as a double complex concentrated in bidegrees $(0, \cdot \leq 0)$). Next, note the natural map $\Gamma^* J_{m+1} \to \tilde{L}_m(\mathfrak{g}^*)[-1]$, where the shift is taken vertically. This map comes about by writing symbolically

$$\begin{array}{rcccccccl} \Gamma^* J_{m+1}: & \mathbb{C} & \xrightarrow{0} & {}^*g & \to & \bigwedge^2 {}^*g & \to & \dots & , \\ \tilde{L}_m(\mathfrak{g}^*)[-1]: & 0 & \to & {}^*g & \to & {}^*g \otimes^* g & \to & \dots & \end{array}$$

and mapping $\mathbb{C}$ to 0 and $\bigwedge^i {}^*g \to \bigwedge^{i-1} {}^*g \otimes^* g$ in the standard way. According to our conventions, this map only preserves total degree; it induces

$$\tilde{L}_{m+1}(\mathbb{C}) = \Gamma^* J_{m+1} \to \tilde{L}_m(\mathfrak{g}^*)[-1].$$

Combining the latter with the pairing (2.7), we get a pairing

$$< . >: \tilde{L}_m(\mathfrak{g}) \times \tilde{L}_{m+1}(\mathbb{C}) \to \tilde{L}_m(\mathbb{C})[-1].$$

Now this map takes an element of bi-bidegree $((a_1, a_2), (0, b))$ to a sum of elements of bidegrees $(a_1 + b_1 = 0, a_2 + b_2 + 1)$, where $b_1 + b_2 = b$ and (b_1, b_2) is the bidegree of an element in $L_m(\mathfrak{g}^*)$. Since $a_2, b_2 \leq 0$, it follows that $a_2 + b_2 + 1 < 1$ if either $a_2 < 0$ or $b < 0$. Therefore there is an induced map

$$< . >: L_m(\mathfrak{g}) \times L_{m+1}(\mathbb{C}) \to L_m(\mathbb{C})[-1], \tag{2.8}$$

whence a pairing on cohomology

$$H^1(L_m(\mathfrak{g})) \times H^0(L_{m+1}(\mathbb{C})) \to H^0(L_m(\mathbb{C})).$$

Note that $H^0(L_m(\mathbb{C})) = R_m(\mathfrak{g})$. We set

$$\Theta_m(\mathfrak{g}) = H^1(L_m(\mathfrak{g})).$$

By Corollary 1.3, this group coincides with $H^1(M_m(\mathfrak{g}, \mathfrak{g}))$, i.e. the m-th principal part of the tangent sheaf to moduli. Thus we have defined a pairing (action pairing)

$$< . >: \Theta_m(\mathfrak{g}) \times R_{m+1}(\mathfrak{g}) \to R_m(\mathfrak{g}). \tag{2.9}$$

Now it is easy to see that the pairing

$$L_m(\mathbb{C}) \times L_m(\mathfrak{g}) \to L_m(\mathfrak{g})$$

(which comes from the 'product of L's maps to L of product' rule (1.4)) induces $R_m(\mathfrak{g}) \times \Theta_m(\mathfrak{g}) \to \Theta_m(\mathfrak{g})$ which endows $\Theta_m(\mathfrak{g})$ with an $R_m(\mathfrak{g})$-module structure.

Next, we undertake to define a pairing on $\Theta_m(\mathfrak{g})$ that will yield the Lie bracket. For this consider the complex $\tilde{L}_m(\mathrm{sym}^2 g^\cdot)$ which may be written in the form

$$\mathrm{sym}^2 g^\cdot \to (\Gamma^* J_m^{(1)})_{[-1} \otimes \mathrm{sym}^2 \mathfrak{g}^\cdot \to (\Gamma^* J_m^{(2)})_{[-2} \otimes \mathrm{sym}^2 g^\cdot \to \cdots$$

where ${}^*J_m^{(i)}$ denotes the sum of the terms in ${}^*J_m^\cdot$ of *tensor* degree i (i.e. products of i factors) and $({}^*J_m^{(i)})_{[-i}$ is its truncation in (total) degrees $\geq -i$. Now the duality pairing $g^\cdot \times^* g^\cdot \to \mathbb{C}$ extends to 'contraction' maps (analogous to interior multiplication)

$$(\Gamma^* J_m^{(r)})_{[-r} \otimes \mathrm{sym}^2 \mathfrak{g}^\cdot \to (\Gamma^* J_{m-1}^{r-1})_{[-r+1} \otimes g^\cdot.$$

Thanks to the alternating nature of the bracket on $g^\cdot$, it is easy to check that these maps together yield a map of double complexes

$$\tilde{L}_m(\mathrm{sym}^2 g^\cdot) \to \tilde{L}_{m-1}(g^\cdot)[-1].$$

Now recall the map $\mathrm{sym}^2 \tilde{L}_m(g^\cdot) \to \tilde{L}_m(\mathrm{sym}^2 g^\cdot)$ as in (1.4). Composing, we get a map of double complexes

$$b_m : \mathrm{sym}^2 \tilde{L}_m(g^\cdot) \to \tilde{L}_{m-1}(g^\cdot)[-1],$$

which induces a map on the respective truncations, whence a map on cohomology

$$H^2((\mathrm{sym}^2 \tilde{L}_m(\mathfrak{g}))_+) \to H^1(\tilde{L}_{m-1}(\mathfrak{g})_+) = H^1(L_{m-1}(\mathfrak{g})).$$

Note that $(\mathrm{sym}^2 \tilde{L}_m(\mathfrak{g}))_+ = \mathrm{sym}^2(\tilde{L}_m(\mathfrak{g})_+) = \mathrm{sym}^2 L_m(\mathfrak{g})$ because these are complexes concentrated in nonpositive vertical degrees. Then define the bracket pairing

$$B_m : \bigwedge^2 \Theta_m(\mathfrak{g}) \to \Theta_{m-1}(\mathfrak{g})$$

as the induced map

$$\bigwedge^2 H^1(L_m(\mathfrak{g})) \to H^2(\mathrm{sym}^2(L_m(\mathfrak{g}))) \to H^1(L_{m-1}(\mathfrak{g})).$$

Theorem 2.1.

(i) The above pairings (2.9) yield a compatible sequence of natural homomorphisms

$$A_m : \Theta_m(\mathfrak{g}) \to \mathrm{Der}(R_{m+1}(\mathfrak{g}), R_m(\mathfrak{g})).$$

A_1 is always an isomorphism.

(ii) Via A_m, the commutator of derivations is given by

$$[A_{m+1}(u), A_{m+1}(v)] = A_m(B_m(u \wedge v)), \forall u, v \in \Theta_{m+1}(\mathfrak{g}).$$

(iii) The induced pairing $\hat{B} = \varprojlim B_m$ on $\hat{\Theta}(\mathfrak{g}) = \varprojlim \Theta_m(\mathfrak{g})$ turns it into a Lie algebra. If $\mathfrak{g}$ has unobstructed deformations, then the induced map

$$\hat{A} = \varprojlim A_m : \hat{\Theta}(\mathfrak{g}) \to \mathrm{Der}(\hat{R}(\mathfrak{g}))$$

is a Lie isomorphism.

proof. (i) We first show Θ_m acts on R_{m+1} (we drop the $\mathfrak{g}$ for convenience) as derivations, i.e. that

$$< A_m(u), fg >= g < A_m(u), f > +f < A_m(u), g >, \quad \forall f, g \in R_{m+1}, u \in \Theta_m.$$

This results from the commutative diagram

$$(2.10) \qquad \begin{array}{ccc} L_m(\mathfrak{g}) \times L_{m+1}(\mathbb{C}) \times L_{m+1}(\mathbb{C}) & \longrightarrow & L_m(\mathbb{C}) \times L_m(\mathbb{C})[-1] \\ \mathrm{id} \times \mu_{m+1} \downarrow & & \mu_m \downarrow \\ L_m(\mathfrak{g}) \times L_{m+1}(\mathbb{C}) & \xrightarrow{<.>} & L_m(\mathbb{C})[-1] \end{array}$$

where $\mu_m : L_m(\mathbb{C}) \times L_m(\mathbb{C}) \to L_m(\mathbb{C})$ is the multiplication mapping as in (1.4), which yields the multiplication in R_m, and which is simply induced by (graded) exterior multiplication in $\dot{\bigwedge}({}^*g)$, $< . >$ is the pairing (2.9), and the top horizontal arrow in induced by $< . >$ via the derivation rule, i.e. $u \times f \times g \mapsto < u, f > \times g + < u, g > \times f$. Commutativity of this diagram is immediate from the definitions.

Next we show A_m is R_m-linear. This again follows from the (easily checked) commutativity of a suitable diagram, namely

$$\begin{array}{ccc} L_m(\mathbb{C})\times L_m(\mathfrak{g})\times L_{m+1}(\mathbb{C}) & \stackrel{\mathrm{id}\times<.>}{\longrightarrow} & L_m(\mathbb{C})\times L_m(\mathbb{C})[-1] \\ \mu'\times\mathrm{id}\downarrow & & \downarrow\mu_m \\ L_m(\mathfrak{g})\times L_{m+1}(\mathbb{C}) & \stackrel{<.>}{\longrightarrow} & L_m(\mathbb{C})[-1] \end{array} \tag{2.11}$$

where μ' is the multiplication mapping $L_m(\mathbb{C})\times L_m(\mathfrak{g})\to L_m(\mathfrak{g})$ which induces the R_m-module structure on Θ_m.

Note that A_1 is just a map
$$H^1(\mathfrak{g})\to\mathfrak{m}^*_{R_1(\mathfrak{g})}=H^1(\mathfrak{g}),$$
and it is immediate from the definitions that this is just the identity.

(ii) To be precise, what is being asserted here is that for all $u,v\in\Theta_{m+1}$, if u',v' are the induced elements in Θ_m, then
$$A_m(u')\circ A_{m+1}(v)-A_m(v')\circ A_{m+1}(u)=A_m(B_m(u\wedge v)).$$
This, in turn, results from the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{sym}^2L_m(\mathfrak{g})\times L_{m+1}(\mathbb{C}) & \stackrel{b_m\times\bar{\mathrm{id}}}{\longrightarrow} & L_{m-1}(\mathfrak{g})\times L_m(\mathbb{C})[-1] \\ \mathrm{id}\times<.>\downarrow & & \downarrow<.> \\ L_{m-1}(\mathfrak{g})\times L_m(\mathbb{C})[-1] & \stackrel{<.>}{\longrightarrow} & L_{m-1}(\mathbb{C})[-2] \end{array}$$

where the left vertical arrow is induced by $<.>$ again via the derivation rule, i.e. $uv\times w\mapsto u\times<v.w>+v\times<u.w>$.

(iii) The fact that $\hat{\Theta}(\mathfrak{g})$ is a Lie algebra amounts to the Jacobi identity. To verify this, note that b_m induces via the derivation rule a map
$$\mathrm{sym}^3L_m(\mathfrak{g})\to L_m(\mathfrak{g}^\cdot)\otimes L_{m-1}(\mathfrak{g})[-1]\to\mathrm{sym}^2L_{m-1}(\mathfrak{g})[-1].$$
Then the Jacobi identity amounts to the vanishing of the composite of this map and
$$b_{m-1}:\mathrm{sym}^2L_{m-1}(\mathfrak{g})[-1]\to L_{m-2}(\mathfrak{g})[-2].$$
This may be verified easily.

Finally in the unobstructed case, clearly both $\hat{\Theta}(\mathfrak{g})$ and $\mathrm{Der}(\hat{R}(\mathfrak{g}))$ are free $\hat{R}(\mathfrak{g})$-modules, and since A_1 is an isomorphism it follows that so is $\hat{A}$. □

Elaboration 2.2. In term of cocycles, we may describe $\Theta_1(\mathfrak{g})$ as follows. Set $V=H^1(\mathfrak{g})$ which we view as a subspace of $\Gamma(\mathfrak{g}^1)$.Then
$$\Theta_1(\mathfrak{g})=\{(a,\sum b_i\otimes c_i^*)\in V\oplus\mathfrak{g}^1\otimes V^*|{}^tbr(a)=\sum\delta(b_i)\otimes c_i^*\}$$
where ${}^tbr$ is the adjoint of the bracket, defined by

$$\begin{array}{rcl} {}^tbr(a) & = & \sum b_i\otimes c_i^* \\ [a,x] & = & \sum<c_i^*.x>b_i\quad\forall x\in V. \end{array}$$

Thus the condition defining $\Theta_1(\mathfrak{g})$ is
$$[a,x]=\sum<c_i^*.x>\delta(b_i)\quad\forall x\in V.$$
Now the bracket
$$[.,.]:\bigwedge^2\Theta_1(\mathfrak{g})\to\Theta_0(\mathfrak{g})=V$$
is given by
$$[(a,\sum b_i\otimes c_i),(a',\sum b_i'\otimes c_i'^*)]=\sum<c_i'^*.a>b_i'-\sum<c_i^*.a'>b_i.$$
Note that neither sum is $\delta-$ closed, but the difference is because
$$\sum<c_i'^*.a>\delta(b_i')-\sum<c_i^*.a'>b_i=[a',a]-[a,a']=0$$
(recall that the bracket is symmetric on $\mathfrak{g}^1$).

3. Differential operators

We shall require an extension of the results the last section from the case of derivations on $R_m(\mathfrak{g})$ itself to that of differential operators on 'modular' $R_m(\mathfrak{g})$-modules (those that come from $\mathfrak{g}-$ modules).To this end we will construct, given a dgla $g^{\cdot}$ and $g^{\cdot}-$modules G_1, G_2, complexes $LD^m_k(G_1, G_2)$ whose cohomology will act as mth order differential operators from $H^{\cdot}(M_{m+k}(G_2)$ to $H^{\cdot}(M_k(G_1))$ and will coincide with the module of all such operators, i.e.
$D^m(H^{\cdot}(M_{m+k}(G_2), H^{\cdot}(M_k(G_1))$, under favorable circumstances ('no obstructions'). This will apply in particular to an admissible Lie pair $(\mathfrak{g}, E)$ on X with suitable (dgla, dg-module) resolution $(\mathfrak{g}^{\cdot}, E^{\cdot})$, by taking as usual $g^{\cdot} = \Gamma(\mathfrak{g}^{\cdot})$, $G^{\cdot} = \Gamma(E^{\cdot})$. See Elaboration 3.8 below for a concrete cocycle (partial) version.

To begin with, set, for any $g^{\cdot}$-modules G_1, G_2,

$$K_m(G_1, G_2) = K_m(g^{\cdot}, G_{1triv} \otimes^* G_2),$$

where $G_{1triv} \otimes^* G_2$ is $G_1 \otimes^* G_2$ as a complex but with $g^{\cdot}$ acting through the ${}^*G_2$ factor only. Note that as a complex, we may identify $K_m(G_1, G_2) = G_1 \otimes K(g^{\cdot},{}^* G_2)$. A fundamental observation is the following

Lemma 3.1. *Let G_1, G_2 be $g^{\cdot}-$modules. Then the duality pairings between $g^{\cdot}$ and ${}^*g^{\cdot}$ and G_2 and ${}^*G_2$ extends to a pairing $K_m(G_1, G_2) \times L_m(G_2) \to G_1$.*

proof. There is clearly no loss of generality in assuming $G_1 = \mathbb{C}$ with trivial $g-$action. Write these complexes schematically as

$$K := K_m(g^{\cdot},{}^* G_2) \quad \cdots \bigwedge^2 g \otimes^* G_2 \to g \otimes^* G_2 \to^* G_2,$$
$$L := L_m(g^{\cdot}, G_2) \quad G_2 \to^* g \otimes G_2 \to \bigwedge^2 ({}^*g) \otimes G_2 \cdots .$$

Then we have

$$(K \otimes L)_0 = \bigoplus_{i=0}^{m} \bigwedge^i g \otimes \bigwedge^i ({}^*g) \otimes^* G_2 \otimes G_2.$$

We map this to $\mathbb{C}$ in the obvious way by contracting together all the g and ${}^*g$ factors and likewise for ${}^*G_2$ and G_2. What has to be proved is that this yields a map of complexes $K \otimes L \to \mathbb{C}$, i.e. that the composite

$$(K \otimes L)_{-1} \overset{\delta_{K \otimes L}}{\to} (K \otimes L)_0 \to \mathbb{C}$$

vanishes, in other words that for each $i = 1, ..., m$ the composite

$$\bigwedge^i g \otimes \bigwedge^{i-1} ({}^*g) \otimes^* G_2 \otimes G_2 \overset{\delta_K \otimes \mathrm{id} \oplus \mathrm{id} \otimes \delta_L}{\to}$$
$$\bigwedge^{i-1} g \otimes \bigwedge^{i-1} ({}^*g) \otimes^* G_2 \otimes G_2 \bigoplus \bigwedge^i g \otimes \bigwedge^i ({}^*g) \otimes^* G_2 \otimes G_2 \to^* G_2 \otimes G_2 \to \mathbb{C}$$

is zero. Now it is easy to see from the definitions (compare the proof of Lemma 1.8) that it suffices to check this for $i = 1$. So pick an element

$$v \times a^* \times a \in g \times^* G_2 \times G_2.$$

Its image under the first map has two components, the first of which is $< v, a^* > \times a$ where $< \cdot, \cdot >$ denotes the action, while the second component has the form

$$v \times a^* \times \sum w_j^* \otimes b_j$$

where the sum denotes the cobracket of a, defined by

$$\sum < w_j^*.y > b_j =< y, a >, \forall y \in g,$$

where $< \cdot . \cdot >$ denotes the duality pairing. Clearly the image of this second component in $\mathbb{C}$ (i.e. its trace) is just $< a^*. < v, a >>$. However by definition of the dual action we have

$$< a^*. < v, a >>= - << v, a^* > .a > .$$

Thus the image of $v \times a^* \times a$ in $\mathbb{C}$ is zero, as claimed. $\square$

Next, recall by Lemma 1.8 that $K_m({}^*G_2)$ is a $g^\cdot$-module, hence so is $K_m(G_1, G_2) = G_1 \otimes K_m({}^*G_2)$. It gives rise to a complex

$$L_k(g^\cdot, K_m(G_1, G_2)) =: LD_k^m(g^\cdot, G_1, G_2).$$

When $g^\cdot$ is understood, we may denote the latter by $LD_k^m(G_1, G_2)$, and when $G_1 = G_2 = G$ the same may also be denoted by $LD_k^m(G)$ From (1.4) and Lemma 3.1 we deduce a pairing

$$LD_k^m(g^\cdot, G_1, G_2)) \times L_m(g^\cdot, L_k(g^\cdot, G_2)) \to L_k(G_1),$$

hence by Lemma 1.8(ii) we get a pairing

$$LD_k^m(g^\cdot, G_1, G_2)) \times L_{m+k}(g^\cdot, G_2) \to L_k(G_1) \tag{3.1}$$

Our next goal is to show that, via this pairing, we may, at least under favorable circumstances, identify $H^{j-i}(LD_k^m(g^\cdot, G_1, G_2)$ with the k-jet of the $m-$th order differential operators on the $R_{m+k}(g^\cdot)$-module corresponding to $H^i(G_2)$ with values in $H^j(G_1)$, provided these are the unique nonvanishing respective cohomology groups. In fact, it will be convenient to prove the stronger result saying that this assertion essentially holds already 'on the level of complexes'. To explain what that means, note that via the pairing (1.4), $L_m(\mathbb{C})$- and likewise $L_m(A)$ for any $\mathbb{C}$-algebra A-forms a 'ring complex', i.e. a ring object in the category of complexes; this ring structure is the one that induces the ring structure on $R_m(g) = \mathbb{H}^0(L_m(\mathbb{C}))$. Moreover, for any g-module G, $L_m(G)$ is an $L_m(\mathbb{C})-$module. There is an evident notion of $L_m(\mathbb{C})-$linear map of $L_m(\mathbb{C})$-modules, and any $g-$linear map $G_1 \to G_2$ induces such a map $L_m(G_1) \to L_m(G_2)$. Likewise, the natural pairing

$$L_m(G) \times L_m({}^*G) \to L_m(\mathbb{C})$$

is $L_m(\mathbb{C})-$linear. Given this, the notion of differential operators of any order (over $L_m(\mathbb{C})$) can be defined inductively: given complexes D, M, N of $L_m(\mathbb{C})-$modules and a pairing

$$a : D \times M \to N,$$

a is said to be of differential order $\leq m$ in the M factor if the composite map

$$L_m(\mathbb{C}) \times D \times M \to N,$$
$$(v, d, m) \mapsto a((vd), m) - a(d, (vm))$$

is of differential order $\leq m - 1$ in the M factor.

Lemma 3.2. *The pairing (3.1) is of differential order $\leq m$ in $L_{m+k}(G_2)$.*

proof. By induction on m, naturally. For $m = 0$ the result is clear (and was already noted above). For the induction step, it suffices to show that the map

$$LD_k^m(G_1, G_2) \times L_{m+k}(\mathbb{C}) \to LD_k^m(G_1, G_2)$$

given by (premultiplication)-(postmultiplication) factors through $LD_k^{m-1}(G_1, G_2)$. As for the premultiplication map, it is induced by the $L_m(\mathbb{C})-$module structure on $K_m({}^*G_2)$, i.e. the natural map (cf. Lemma 1.8(i))

$$L_m(\mathbb{C}) \times K_m({}^*G_2) \to K_m({}^*G_2).$$

Tensoring by G_1, applying L_k and using Lemma 1.8(ii) we get a map

$$L_{m+k}(\mathbb{C}) \times LD_k^m(G_1, G_2) \to LD_k^m(G_1, G_2)$$

that is the premultiplication map. This map clearly factors through $L_m(\mathbb{C}) \times LD_k^m(G_1, G_2)$. It is essentially obtained by contracting together some g and ${}^*g$ factors and exterior-multiplying others; in particular the induced map on any term involving $\bigwedge\limits^m(g)$ going to a similar term cannot involve any contraction, hence is simply given by exterior- multiplying the factor from $L_m(\mathbb{C})$ by the one from $LD_k^m(G_1, G_2)$. It is easy to see that similar comments apply to the postmultiplication map. Thus the two induced map (from pre and post) between terms involving $\overset{m}{\bigwedge}(g)$ agree, and consequently the difference (pre)-(post) goes into $LD_k^{m-1}(G_1, G_2)$, which proves the Lemma. $\square$

Remark 3.2.1. As was observed in the course of the proof, $LD_k^m(G_1, G_2)$ has the structure of $L_{m+k}(\mathbb{C})-$ bimodule, corresponding to the pre-post-multiplications. This is analogous, and closely related to the bimodule structure on the space of differential operators $D^m(M_1, M_2)$ between a pair of modules.

Next we will construct a pairing that will yield the *composition* of differential operators.

Lemma 3.3. *For any $g-$modules $G_1, G_2, G3$ and natural numbers m, k, j, n with $k \geq j - m \geq 0$,there is a natural pairing of $g-$ modules*

$$LD_k^m(G_1, G_2) \times LD_j^n(G_2, G_3) \to LD_{j-m}^{m+n}(G_1, G_3). \tag{3.2}$$

proof. There is clearly no loss of generality in assuming $k = j - m$. Then using Lemma 1.8 we are easily reduced to the case $j = m$ where the point is to construct a $g-$linear pairing

$$G_1 \otimes K_m({}^*G_2) \times L_m(G_2 \otimes K_n({}^*G_3)) \to G_1 \otimes K_{m+n}({}^*G_3). \tag{3.3}$$

There is obviously no loss of generality in assuming $G_1 = \mathbb{C}$. Then the LHS is a direct sum of terms of the form

$$\overset{i}{\bigwedge} g \otimes^* G_2 \times \overset{j}{\bigwedge}({}^*g) \otimes \bigwedge^k g \otimes G_2 \otimes G_3$$

which has degree $i + k - j$. We map this term to zero if $i + j - k < 0$, and otherwise to

$$\overset{i-j+k}{\bigwedge} g \otimes^* G_3 = K_{m+n}^{i-j+k}({}^*G_3)$$

in the standard way, by contracting away all the ${}^*g$ factors against the $g's$, as well as ${}^*G_2$ against G_2. If we can prove this is a map of complexes, then $g-$linearity comes for free, due to the $g-$linearity of contraction.

Now to prove we have a map of complexes one may reduce as in the proof of Lemma 1.8 to the case $i = k = 1, j = 0$ and commutativity of the following diagram

(3.4)

$$\begin{array}{ccc}
g \otimes^* G_2 \otimes g \otimes G_2 \otimes^* G_3 & \to & [g \otimes^* G_2 \times G_2 \otimes^* G_3]^{\oplus 2} \oplus g \otimes^* G_2 \otimes^* g \otimes g \otimes G_2 \otimes^* G_3 \\
\downarrow & & \downarrow \\
\overset{2}{\bigwedge} g \otimes^* G_3 & \to & g \otimes^* G_3
\end{array}$$

where the top map is of the form
(g-action on ${}^*G_2$, g-action on ${}^*G_3$, g-coaction on $g \otimes G_2 \otimes^* G_3$). Given an element

$$v_1 \times a^* \times v_2 \times a \times b^* \in g \otimes^* G_2 \otimes g \otimes G_2 \otimes^* G_3,$$

its image going counterclockwise is clearly

$$< (v_1 \wedge v_2), < a.a^* > b^* >=$$

$$< a.a^* > (v_1 \times < v_2, b^* > -v_2 \times < v_1, b^* > -[v_1, v_2] \times b^*). \tag{3.5}$$

On the other hand, the image of this element under the top map is

$$(< v_1, a^* > \times v_2 \times a \times b^*, v_1 \times a^* \times < v_2, b^* > \times a, v_1 \times a^* \times^t \mathrm{br}(v_2 \times a \times b^*)).$$

Now from the definition of ^{t}br, the fact that it acts as a derivation, plus the definition of the dual action, it is elementary to verify that the image of the latter element under the right vertical map coincides with (3.5), so the diagram commutes. $\square$

Lemma 3.4. *Via the action pairing (3.1), the 'composition' pairing (3.2) corresponds to composition of operators.*

proof. Our assertion means that

$$<< d_1, d_2 >, a >=< d_1, < d_2, a >>,$$
$$\forall d_1 \in LD_k^m(G_1, G_2), d_2 \in LD_j^n(G_2, G_3), a \in L_r(G_3),$$

assuming $r \geq j - m \geq 0$ (and abusing $< >$ to denote the various pairings involved), which amounts to commutativity of the obvious diagram

$$\begin{array}{ccc} LD_k^m(G_1, G_2) \times LD_j^n(G_2, G_3) \times L_r(G_3) & \to & LD_{j-m}^{m+n}(G_1, G_3) \times L_r(G_3) \\ \downarrow & & \downarrow \\ LD_k^m(G_1, G_2) \times L_j(G_2) & \to & L_{j-m}(G_1). \end{array} \tag{3.6}$$

Now all the maps involved are essentially given by exterior multiplication and contraction, so commutativity of (3.6) follows from the associativity of exterior multiplication. $\square$

In particular, taking $G_1 = G_2 = G_3 = G$ we get a (composition) pairing, whenever $k \geq m$,

$$LD_k^m(G) \times LD_k^n(G) \to LD_{k-m}^{m+n}(G).$$

It is easy to see by sign considerations as in the proof of Lemma 3.2 that the 'commutator' associated to this pairing takes values in $LD_{k-\max(m,n)}^{m+n-1}(G)$. In particular, we get a skew-symmetric pairing

$$B_k : \bigwedge^2 LD_k^1(G) \to LD_{k-1}^1(G).$$

Lemma 3.5. *Under $B_\infty = \lim\limits_{\leftarrow} B_k$, $LD_\infty^1(G) = \lim\limits_{\leftarrow} LD_k^1(G)$ is a Lie algebra object in the category of complexes, and admits a natural representation on $L_\infty(G) = \lim\limits_{\leftarrow} L_k(G)$.*

proof. Most of this has been proved above. The only remaining point is the Jacobi identity for B_∞, which can be proved as in the case of the trivial module $G = \mathbb{C}$ (cf. Theorem 2.1). $\square$

The pairings discussed above naturally induce pairings on cohomology groups. This leads to the following Theorem 3.6 . First some notation and terminology. For any $g-$module $G, k \leq \infty$, set

$$H^i(G, k) = H^i(L_k(G))$$

As we have seen, if (g, G) comes from sheaves $(\mathfrak{g}, E)$ on X then this coincides with the sheaf cohomology $H^i(X, M_k(\mathfrak{g}, E))$, i.e. the $k-$universal $g-$deformation of $H^i(X, E)$. We will say that G is *strongly* $i-$*unobstructed* if for all $v \in g^1$ that is $\delta-$closed (i.e. $\delta(v) = 0$) and all $a \in G^i$ (closed or not), we have that $< v, a >$ is exact; we will say that g itself is *strongly unobstructed* if it is strongly 1-unobstructed in the adjoint representation. It is easy to see that if g is strongly unobstructed then $R_\infty(g)$ is regular (i.e. smooth) and that if G is strongly i-unobstructed then $H^i(G, \infty)$ is $R_\infty(g)$ -free. Also, it is obvious that if G is i-equicyclic then it is strongly i-unobstructed.

Theorem 3.6. *Let G_1, G_2, G_3 be modules over the dgla g with $H^{\leq 0}(g) = 0$. Then*

(i) there is a natural pairing, for any $0 \leq k \leq n - m$

$$H^{j-i}(LD_k^m(G_1, G_2)) \times H^i(G_2, n) \to H^j(G_1, k)$$

which induces a map

$$A : H^{j-i}(LD_k^m(G_1, G_2)) \to D_{R_n(g)}^m(H^i(G_2, n), H^j(G_1, k);$$

(ii) there is a natural pairing, for any $0 \le j-m \le k$,
$$C : H^i(LD^m_k(G_1,G_2)) \times H^j(LD^n_j(G_2,G_3)) \to H^{i+j}(LD^{m+n}_{j-m}(G_1,G_3),$$
via which A *corresponds to composition of operators; in particular there are natural Lie algebra structures on* $H^0(LD^1_\infty(G))$ *and* $H^0(LD^\infty_\infty(G))$ *with representations on* $H^i(G,\infty)$ *for all* i*;*

(iii) if g *is strongly unobstructed and* G_1 *and* G_2 *are equicyclic of degrees* i,j *respectively, then the map*

$$A_\infty : H^{i-j}(LD^m_\infty(G_1,G_2)) \to D^m_{R_\infty(g)}(H^j(G_2,\infty), H^i(G_1,\infty)) \tag{3.7}$$

is an isomorphism for all m.

proof. Items (i) and (ii) follow directly from the results above. We prove (iii). Clearly the target of A_∞ , with respect to its left (postmultiplication) structure, is a free module with fibre
$$H^i(G_1) \otimes D^m_0(H^j(G_2),\mathbb{C}) = H^i(G_1) \otimes H^j(G_2,m)^*.$$
As for the source, note that $K_m({}^*G_2)$ is strongly $(-j)$-unobstructed and has no cohomology in degree $< -j$. Consequently, $G_1 \otimes K_m({}^*G_2)$ is strongly $(i-j)$-unobstructed and
$$H^{i-j}(G_1 \otimes K_m({}^*G_2)) = H^i(G_1) \otimes H^{-j}(K_m({}^*G_2)) = H^i(G_1) \otimes H^j(G_2,m)^*.$$
By definition, the latter is precisely the fibre of $H^{i-j}(LD^m_\infty(G_1,G_2))$ with respect to its postmultiplication module structure (which structure we now know is free, thanks to unobstructedness). Thus the source and target of A_∞ have isomorphic fibres; moreover it is easy to see, for instance by considering the other (right or premultiplication) structure that A_∞ induces an isomorphism. But clearly a linear map of free modules over a local ring inducing an isomorphism on fibres is itself an isomorphism, proving our assertion. $\square$

Corollary 3.7. *If* G *is an* $i-$*equicyclic module and* g *is strongly unobstructed then the Lie algebra* $H^0(LD^1_\infty(G))$ *is canonically isomorphic to* $D^1_{R_\infty(G)}(H^i(G,\infty))$ $\square$

In particular, in the geometric situation with $(\mathfrak{g}, E)$ an admissible pair, $\mathfrak{g}$ unobstructed and E i-equicyclic, we get a canonical recipe for the Lie algebra which is the formal completion of $D^1_{\mathcal{M}}(\mathcal{H})$ where $\mathcal{H}$ is the sheaf on the moduli space $\mathcal{M}$ associated to the unique nonvanishing cohomology group $H^i(E)$.

Elaboration 3.8. Let us write down the bracket pairing B_1 in terms of cocycles. This comes about by considering the diagram

$$\begin{array}{ccc} g\otimes\ G\otimes^* G & \xrightarrow{b} & G\otimes^* G \\ {}^t b \downarrow & & {}^t b\downarrow \\ {}^*g\otimes g\otimes\ G\otimes^* G & \xrightarrow{b} & {}^*g\otimes\ G\otimes^* G \end{array} \tag{3.8}$$

where the maps b are induced by the action of g on ${}^*G$, while the maps ${}^tb$ are induced by the transpose of the g action on G. A cocycle for $LD^1_1(G)$ is a 4-tuple $(\phi,\psi,\phi',\psi')$of cochains of the four complexes in (3.8) such that
$$\partial(\phi)=0, b(\phi)=\partial(\psi),^t b(\phi)=\partial(\phi'), b(\phi')+^t b(\psi)=\partial(\psi').$$
The pairing
$$B_1 : \bigwedge^2(H^0(LD^1_1(G))) \to H^0(LD^1_0(G)$$
is given by

$$B_1((\phi_0,\psi_0,\phi_0',\psi_0')\wedge(\phi_1,\psi_1,\phi_1',\psi_1'))=(\phi_2,\psi_2)$$

where $\psi_2=[\psi_0,\psi_1]+<\phi_0,\psi_1'>-<\phi_1,\psi_0'>$, $\phi_2=<\phi_0,\phi_1'>-<\phi_1,\phi_0'>$ (compare Elaboration 2.2). Here $[\ ]$ is the usual commutator on $G\otimes^* G$ while $<\ >$ is the pairing induced by $[\ ]$ and the duality between g and $^*g$. To check that this is indeed a cocycle, we compute:

$$\partial(\psi_2)=[\partial(\psi_0),\psi_1]-[\psi_0,\partial(\psi_1)]-<\phi_0,\partial(\psi_1')>+<\phi_1,\partial(\psi_0')>$$
$$=[b(\phi_0),\psi_1]-[\psi_0,b(\phi_1)]-<\phi_0,b(\phi_1')+^t b(\psi_1)>+<\phi_1,b(\phi_0')+^t b(\psi_0)>$$
$$=[b(\phi_0),\psi_1]-[\psi_0,b(\phi_1)]-<\phi_0,b(\phi_1')>+<\phi_1,b(\phi_0')>-[b(\phi_0),\psi_1]+[\psi_0,b(\phi_1)]$$
$$=-<\phi_0,b(\phi_1')>+<\phi_1,b(\phi_0')>=b(<\phi_0,\phi_1'>-<\phi_1,\phi_0'>)=b(\phi_2).$$

Analogous formulae may be given for the bracket 'action' of $LD_k^1(G)$ on $LD_k^m(G)$. These actions being compatible for different m, there is an induced action on $LD_k^m(G)/L_k^{m-1}(G)=L_k(\bigwedge^m g\otimes G\otimes^* G)[m]$. In particular, we get a pairing

$$LD_1^1(G)\times L_1(\bigwedge^2 g\otimes G\otimes^* G)[2]\to(\bigwedge^2 g\otimes G\otimes^* G)[2]$$

Now note the natural map

$$L_1(\bigwedge^2 g\otimes G\otimes^* G)[1]\to LD_1^1(G)$$

which is induced by the map $\bigwedge^2 g\otimes G\otimes^* G[1]\to K_1(g,G\otimes^* G)$ that is part of the complex $K_2(g,G\otimes^* G)$. Hence we get a pairing

$$L_1(\bigwedge^2 g\otimes G\otimes^* G)[1]\times L_1(\bigwedge^2 g\otimes G\otimes^* G)[2]\to(\bigwedge^2 g\otimes G\otimes^* G)[2]$$

i.e. a (symmetric) bracket pairing

$$\mathrm{Sym}^2(L_1(\bigwedge^2 g\otimes G\otimes^* G)[1])\to\bigwedge^2 g\otimes G\otimes^* G[1].$$

This pairing has the following interpretation. Suppose $\mathcal{M}$ is a locally fine moduli space with Lie algebra $\tilde{\mathfrak{g}}$ on $X\times\mathcal{M}$ as above and $\mathcal{H}$ is the locally free $\mathcal{O}_{\mathcal{M}}-$sheaf $R^ip_{\mathcal{M}*}(\mathcal{H})$ for a suitable $\mathfrak{g}-$module E on $X\times\mathcal{M}$ (assuming this is the only nonvanishing derived image). Then as in Example 7.2.C we get a heat atom $(\mathfrak{D}^1_{\mathcal{M}}(\mathcal{H}),\mathfrak{D}^2_{\mathcal{M}}(\mathcal{H}))$ on $\mathcal{M}$, hence a Lie bracket on the (shifted) quotient $\mathrm{sym}^2(T_{\mathcal{M}})\otimes\mathcal{H}^*\otimes\mathcal{H}[-1]$. This bracket can be identified 'fibrewise' with the map induced by the pairing (3.1).

4. Connection Algebra

Our purpose in this section is to construct, for a given representation $(\mathfrak{g},E)$, a canonical 'thickening' $\mathfrak{k}(\mathfrak{g},E)$ of $\mathfrak{g}$ which is another Lie algbera which acts on E, such that the sections of E extend canonically over the universal deformation associated to $\mathfrak{k}(\mathfrak{g},E)$. Our construction refines and generalizes one in first-order deformation theory due to Welters [W] and Hitchin [Hit, Thm 1.20]. They noted that given a line bundle L on a compact complex manifold X, together with a holomorphic section $s\in H^0(L)$, 1-parameter deformations of the triple (X,L,s) are parametrized by $\mathbb{H}^1$ of the complex

$$\mathfrak{D}^1(L)\xrightarrow{\cdot s}L.$$

Consequently, a family, in a suitable sense, of such $\mathbb{H}^1$ cohomology classes yields a *connection* ∇ on the 'bundle of $H^0(L)$'s (that is, it yields the covariant derivative $\nabla\cdot s$). Our construction extends that of Welters-Hitchin from first-order to arbitrary m-th order deformations. Applied in their original context with m at least 2, it shows that the connection ∇ is automatically *flat*, a fact which could not be seen by first-order considerations alone.

Now let $(\mathfrak{g},E)$ be an admissible pair, with soft resolution $(\mathfrak{g}^\cdot,E^\cdot,\partial)$. Then

$\Gamma^*(E^\cdot)\otimes E^\cdot$ is a complex (via tensor product of complexes) and a $\mathfrak{g}^\cdot$-module (acting on the $E^\cdot$ factor only), which makes it a differential graded $\mathfrak{g}^\cdot-$module. There is a tautological map

$$\mathfrak{g}^\cdot \xrightarrow{\delta} \Gamma^*(E^\cdot)\otimes E^\cdot \tag{4.1}$$

which is easily seen to be a derivation. Thus, (the mapping cone of) (4.1) yields a differential graded Lie algebra , which we denote $\mathfrak{k}(\mathfrak{g},E)$. Note that $\mathfrak{k}(\mathfrak{g},E)$ is itself a differential graded $\mathfrak{g}^\cdot-$module, and that we have a natural dgla homomorphism

$$\mathfrak{k}(\mathfrak{g},E)\to\mathfrak{g}^\cdot$$

Note also that if $H^{\leq 0}(\mathfrak{g})=0$, then we have that $H^{\leq 0}(\mathfrak{k}(\mathfrak{g},E))=0$ if and only if E, that is, $\Gamma E^\cdot$, is i-equicyclic for some i; if this holds, we have an exact sequence

$$0\to H^i(E)\otimes H^i(E)^*\to H^1(\mathfrak{k}(\mathfrak{g},E))\to H^1(\mathfrak{g}).$$

Similar constructions can be make purely algebraically. Thus let $(g^\cdot,G^\cdot)$ be a dg Lie representation. We consider ${}^*G^\cdot\otimes G^\cdot$ as another dg representation of $g^\cdot$ (with differential as tensor product of complexes and $g^\cdot$-action on the $G^\cdot$ factor only), and note the graded derivation

$$g^\cdot \xrightarrow{\delta} {}^*G^\cdot\otimes G^\cdot. \tag{4.2}$$

Then (4.2) forms a dgla which we denote by $k(g^\cdot,G^\cdot)$, and in which g^i has degree i and ${}^*G^i\otimes G^j$ has degree $i+j+1$. Thus

$$\Gamma\mathfrak{k}(\mathfrak{g},E)=k(\Gamma\mathfrak{g}^\cdot,\Gamma E^\cdot).$$

Obviously, $k(g^\cdot,G^\cdot)$ is a $g^\cdot$-module; indeed the canonical 'identity' element

$$I\in {}^*G\otimes G$$

yields an inclusion of $g^\cdot-$modules $k(g^\cdot,G^\cdot)\subset LD^1_0(G)$ (cf. §6). Note that the $g^\cdot$-action on ${}^*G^\cdot\otimes G^\cdot$ evidently extends to an action of $k^\cdot=k(g^\cdot,G^\cdot)$, by letting ${}^*G^\cdot\otimes G^\cdot$ act trivially on itself. Consequently we get for each $m\geq 1$ a complex $L_m(k(g^\cdot,G^\cdot),\ {}^*G^\cdot\otimes G^\cdot)$ which we write schematically as a double complex (with components which are themselves multiple complexes) in the form

(4.3)

$$\begin{array}{ccccccc}
 & & & & \mathrm{sym}^2({}^*G^\cdot\otimes G^\cdot)\otimes\ {}^*G^\cdot\otimes G^\cdot & \to & \dots\\
 & & & & \downarrow & & \\
 & & {}^*G^\cdot\otimes G^\cdot\otimes\ {}^*G^\cdot\otimes G^\cdot & \to & {}^*g^\cdot\otimes\ {}^*G^\cdot\otimes G^\cdot\otimes\ {}^*G^\cdot\otimes G^\cdot & \to & \dots\\
 & & {}^*\delta\otimes\mathrm{id}\downarrow & & \downarrow & & \\
{}^*G^\cdot\otimes G^\cdot & \to & {}^*g^\cdot\otimes\ {}^*G^\cdot\otimes G^\cdot & \to & \bigwedge^2({}^*g^\cdot)\otimes\ {}^*G^\cdot\otimes G^\cdot & \to & \dots
\end{array}$$

Thus the $i-$th column in (4.3) is the complex $\bigwedge^i({}^*k(g^\cdot,G^\cdot))\otimes\ {}^*G^\cdot\otimes G^\cdot$.

Lemma 4.1. *The identity element $I\in\ {}^*G^\cdot\otimes G^\cdot$ lifts canonically to a compatible sequence of elements*

$$I_m\in H^0(L_m(k(g^\cdot,G^\cdot),\ {}^*G^\cdot\otimes G^\cdot)),m\geq 1.$$

proof. Let I_m be the cochain consisting of the elements $\mathrm{sym}^i I\otimes I$ in position (i,i) in the above complex, for all $0\leq i\leq m$. It is trivial to check that this is a cocycle. □

Theorem 4.2. *In the situation of Theorem 1.1, assume moreover that E is equicyclic of degree i. Then we have a canonical isomorphism (or 'trivialization')*

$$H^i(M_m(\mathfrak{g},E))\otimes_{R_m(\mathfrak{g})} R_m(\mathfrak{k}(\mathfrak{g},E))\simeq H^i(E)\otimes_{\mathbb{C}} R_m(\mathfrak{k}(\mathfrak{g},E)) \tag{4.4}$$

Moreover, $R_m(\mathfrak{k}(\mathfrak{g},E))$ is universal with respect to this property, i.e. given a deformation E^τ parametrized by S and an S-isomorphism $H^i(E^\tau)\simeq H^i(E)\otimes S$ lifting the identity on $H^i(E)$, there is a canonical lifting of the Kodaira-Spencer homomorphism of τ to a homomorphism $R_m(\mathfrak{k}(\mathfrak{g},E))\to S$.

proof. Apply Lemma 4.1 to $g^\cdot=\Gamma(\mathfrak{g}^\cdot), G^\cdot=\Gamma(E^\cdot)$. Because $g^\cdot$ acts trivially on ${}^*G^\cdot$, we have

$$L_m(k(g^\cdot,G^\cdot),\ {}^*G^\cdot\otimes G^\cdot))=\Gamma^*(E^\cdot)\otimes L_m(\Gamma(\mathfrak{k}(\mathfrak{g},E)),\Gamma(E^\cdot)).$$

As $H^j(\Gamma^*(E^\cdot))=0$ for $j\neq -i$, we have

$$H^i(L_m(k(g^\cdot,G^\cdot),\ {}^*G^\cdot\otimes G^\cdot)))=\mathfrak{hom}(H^i(E),H^i(L_m(\Gamma(\mathfrak{k}(\mathfrak{g},E)),\Gamma(E^\cdot)))).$$

Now clearly

$$H^i(L_m(\Gamma(\mathfrak{k}(\mathfrak{g},E)),\Gamma(E^\cdot))))\simeq H^i(L_m(\mathfrak{g},E)))\otimes_{R_m(\mathfrak{g})} R_m(\mathfrak{k}(\mathfrak{g},E)),$$

and by Theorem 1.1 this is just $H^i(M_m(\mathfrak{g},E))\otimes_{R_m(\mathfrak{g})} R_m(\mathfrak{k}(\mathfrak{g},E))$, so the element I_m above yields the required trivialization (4.4).

In terms of cocycles, this trivialization may be seen as follows. Consider the universal $\mathfrak{k}(\mathfrak{g},E)$-deformation over $R=R_m(\mathfrak{k}(\mathfrak{g},E))$. This may be represented by

$$\psi=(\phi,\sum t_j\otimes t_j^*)\in(\Gamma(\mathfrak{g}^1)\oplus\Gamma(E^i)\otimes\Gamma(E^i)^*)\otimes\mathfrak{m},\ \mathfrak{m}=m_R.$$

Letting $(s_k\in\Gamma(E^i))$ be a lift of some basis of H^i and s_k^* be a lift of the dual basis, we may write the integrability condition $\partial\psi=-\frac{1}{2}[\psi,\psi]$ as

$$\begin{array}{rc}\partial\phi= & -\frac{1}{2}[\phi,\phi],\\ \sum(\partial t_j)\otimes t_j^*= & -\sum[\phi,t_j]\otimes t_j^*-\sum[\phi,s_k]\otimes s_k^*,\\ \sum t_j\otimes(\partial t_j^*)= & 0.\end{array} \tag{4.5}$$

Thus, we may assume that $\partial t_j^*=0$ hence we may adjust notation so that $t_j^*=s_j^*$. Then we may write 4.5 in the form

$$\sum\partial(s_j+t_j)\otimes s_j^*)+\sum[\phi,s_j+t_j]\otimes s_j^*=0 \tag{4.6}$$

Recalling that the deformation E^ϕ of E induced by ϕ is just $(E^\cdot,\partial+\mathrm{ad}\phi)$, 4.6 shows precisely that $\sum(s_j+t_j)\otimes s_j^*$ is a lift of $I=\sum s_j\otimes s_j^*$ to $E^\phi\otimes R$, yielding a canonical $R-$valued lift of each s_j.

The latter description makes it easy to establish the universality of $R(\mathfrak{k}(\mathfrak{g},E))$, thus completing the proof. Given E^τ/S, a lifting of the identity on $H^i(E)$ to an S-isomorphism $H^i(E)\otimes S\simeq H^i(E^\tau)$ is given by an element

$$\sum t_j\otimes s_j^*\in\Gamma(E^i)\otimes\Gamma(E^i)^*\otimes\mathfrak{m}_S$$

(i.e s_j+t_j is a lifting of s_j), and writing down the condition that s_j+t_j is a cocycle for $\partial+\mathrm{ad}\tau$ and computing as above shows precisely that $\rho:=(\tau,\sum t_j\otimes s_j^*)$ is an S-valued cocycle for $\mathfrak{k}(\mathfrak{g},E)$, yielding the desired homomorphism $R(\mathfrak{k}(\mathfrak{g},E))\to S$. □

For $m=1$ this result (or rather, its 'relative version') generalizes the Welters-Hitchin construction of connections (see [6], Thm 1.20). For $m\geq 2$ the trivialization we construct amounts to showing that this connection is flat.

5. Relative deformations over a global base

Our purpose in this section is to discuss a more global and relative generalization of the notion of deformation, which occurs not just over a (thickened) point (represented by an Artin local algebra), but over a global base (e.g. a moduli space), suitably thickened. This is closely related -but not identical- to the notion of *family* or *variation* of deformations; the slightly subtle difference is illustrated by the fact that a 'family of trivial deformations' may well be nontrivial as a relative deformation (such subtleties however occur only in the presence of symmetries locally over the base and globally along fibres).

To proceed with the basic definitions, let $f : X_B \to B$ be a continuous mapping of Hausdorff spaces with fibres $X_b = f^{-1}(b)$ and base B which we assume endowed with a sheaf of local $\mathbb{C}-$algebras $\mathcal{O}_B$. A *Lie pair* $(\mathfrak{g}_B, E_B)$ on X_B/S consists of a sheaf $\mathfrak{g}_B$ of $f^{-1}\mathcal{O}_B$-Lie algebras (i.e. with $f^{-1}\mathcal{O}_B-$ linear bracket), a sheaf E_B of $f^{-1}\mathcal{O}_B-$modules with $f^{-1}\mathcal{O}_B-$linear g_B-action. This pair is said to be *admissible* if it admits compatible soft resolutions $(\mathfrak{g}^{\cdot}_B, E^{\cdot}_B)$ such that $\mathfrak{g}^{\cdot}_B$ is a dgla and $E^{\cdot}_B$ is a dg representation of $\mathfrak{g}^{\cdot}_B$, and moreover, $\Gamma(\mathfrak{g}^{\cdot}_B), \Gamma(E^{\cdot}_B)$ may be linearly topologized so that coboundaries (and cocycles) are closed, and the cohomology is of finite type as $\mathcal{O}_B$-module (and in particular vanishes in almost all degrees). Let's call such resolutions *good*. Note that if $(\mathfrak{g}^{\cdot}_B, E^{\cdot}_B)$ is an admissible pair then for any $b \in B$ the 'fibre' $(\mathfrak{g}_b, E_b) := (\mathfrak{g}_B, E_B) \otimes (\mathcal{O}_{B,b}/\mathfrak{m}_{B,b})$ is an admissible pair on X_b.

Now let $\mathcal{S}$ be an augmented $\mathcal{O}_B-$algebra of finite type as $\mathcal{O}_B-$module, with maximal ideal $\mathfrak{m}_{\mathcal{S}}$ (below we shall also consider the case where $\mathcal{S}$ is an inverse limit of such algebras, hence is complete noetherian rather than finite type). By a *relative* $\mathfrak{g}_B-$*deformation of* E_B, *parametrized by* $\mathcal{S}$ we mean a sheaf E^{ϕ}_B of $\mathcal{S}$-modules on X_B, together with a maximal atlas of the following data

(1) An open covering (U_α) of X_B.
(2) -$\mathcal{S}$-isomorphisms $\Phi_\alpha : E^{\phi}|_{U_\alpha} \xrightarrow{\sim} E|_{U_\alpha} \otimes_{\mathcal{O}_B} \mathcal{S}$.
(3) - For each α, β, a lifting of $\Phi_\beta \circ \Phi_\alpha^{-1} \in \ Aut(E|_{U_\alpha \cap U_\beta} \otimes_{\mathcal{O}_B} \mathcal{S})$ to an element $\Psi_{\alpha,\beta} \in \exp(\mathfrak{g}_B \otimes \mathfrak{m}_{\mathcal{S}}(U_\alpha \cap U_\beta))$. If $\mathfrak{g}_B$ acts faithfully on E_B then the $\Psi_{\alpha,\beta}$ are uniquely determined by the Φ_α and form a cocycle; in general we require additionally that the $\Psi_{\alpha,\beta}$ form a cocycle.

Note that if $(\mathfrak{g}_B, E_B)$ is admissible then, as in the absolute case, for any relative deformation ϕ there is a good resolution $(E^{\cdot}, \partial)$ of E and a resolution of E^{ϕ} of the form

$$E^0 \otimes_{\mathcal{O}_B} \mathcal{S} \xrightarrow{\partial+\phi} E^1 \otimes_{\mathcal{O}_B} \mathcal{S} \cdots \tag{5.1}$$

with $\phi \in \Gamma(\mathfrak{g}^1_B) \otimes \mathfrak{m}_{\mathcal{S}}$. We call such a resolution a *good resolution* of E^{ϕ}.

Example 5.1. (i) Let E be a vector bundle on the complex manifold $X = X_B = B$ and let $\mathfrak{g} = \mathfrak{gl}(E)$. Let

$$P^m = P^m_X = \mathcal{O}_{X \times X}/I^{m+1}_{\Delta},$$

which is naturally an $\mathcal{O}_X-$algebra via the first coordinate projection p_1. Likewise the $m-$th jet bundle

$$P^m(E) = p_{1*}(p_2^*(E) \otimes P^m)$$

is a P^m-module and hence a $\mathfrak{g}$-deformation of E parametrized by P^m over X. Locally over the base $B = X$, this deformation is obviously trivial, but it is in general nontrivial as relative deformation. To obtain a good resolution of $P^m(E)$, note that E admits a $\bar{\partial}-$connection (e.g. a Hermitian connection), whose curvature is of type (1,1), i.e. trivial on the (1,0) tangent directions, hence yields a C^∞ isomorphism $P^m(E) \sim P^m \otimes E$, hence the Dolbeault resolution of $P^m(E)$ is a good resolution as in (5.1).

More generally, $P^m(E)$ has a structure of $\mathfrak{g}-$deformation for any $\mathcal{O}_X$-locally free Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(E)$ such that E admits a $\mathfrak{g}-$ *structure* (or 'reduction of the structure algebra to $\mathfrak{g}$'). To recall what that means, let $G(E) = ISO(\mathbb{C}^r, E)$, $r = \mathrm{rk}(E)$ be the associated principal bundle, i.e. the open subset of the geometric vector bundle $\mathfrak{hom}(\mathbb{C}^r, E)$ consisting of fibrewise isomorphisms, with the obvious action of GL_r. Let $\mathfrak{D}(E)$ be the sheaf of GL_r-invariant vector fields on $G(E)$, which may also be identified as the sheaf of relative derivations of $(E, \mathcal{O}_X)$, consisting of pairs $(v, a), v \in T_X, a \in Hom_{\mathbb{C}}(E, E)$ such that

$$a(fe) = fa(e) + v(f)e, \ \forall f \in \mathcal{O}_X, e \in E.$$

Note that $\mathfrak{D}(E)$ is an extension of Lie algebras

$$0 \to \mathfrak{gl}(E) \to \mathfrak{D}(E) \to T_X \to 0 \tag{5.2}$$

This is known as the *Atiyah extesnion* or *Atiyah algebra* of E and goes back to Atiyah (see [1]; an analogue in a general deformation-theoretic setting is given in [13]). Then a $\mathfrak{g}-$ structure on E is a Lie subalgebra $\hat{\mathfrak{g}} \subseteq \mathfrak{D}(E)$ which fits in an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{g} & \to & \hat{\mathfrak{g}} & \to & T_X & \to & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \to & \mathfrak{gl}(E) & \to & \mathfrak{D}(E) & \to & T_X & \to & 0. \end{array}$$

Note that in this case a maximal integral submanifold $\hat{G}$ of $\hat{\mathfrak{g}}$ yields a principal subbundle of $G(E)$ with structure group $G = \exp(\mathfrak{g})$ and conversely such a principal subbundle with Lie algebra $\mathfrak{g}$ yields a $\mathfrak{g}-$structure. Clearly a $\mathfrak{g}-$structure on E yields a structure of $\mathfrak{g}-$ deformation on $P^m(E)$ parametrized by P^m, for any m, and as above this admits a good (Dolbeault) resolution. We denote this deformation by $P^m(E, \mathfrak{g})$.

Similarly, if $f : X_B \to B$ is any smooth morphism of complex manifolds, and E_B is a vector bundle on X_B with a relative $\mathfrak{g}_B$-structure, then there is a relative $\mathfrak{g}_B$-deformation parametrized by P^m_B. We denote this deformation by $P^m(E_B, \mathfrak{g}_B)/B$ or by $P^m(E_B)/B$ if $\mathfrak{g}_B$ is understood. Intuitively, it represents the family of $m-$th order deformations $E_B|_{f^{-1}(N_m(b))} = E_B \otimes (\mathcal{O}_B/\mathfrak{m}^{m+1}_{b,B}), \ b \in B$, where $N_m(b) = \mathrm{Spec}(\mathcal{O}_B/\mathfrak{m}^{m+1}_{B,b})$ is the $m-$th order neighborhood of b in B.

(ii) Similarly, given a smooth morphism of complex manifolds $f : X_B \to B$, there is a natural relative $T_{X_B/B}$-deformation parametrized by P^m_B, namely $\mathcal{O}_X \otimes_{\mathcal{O}_B} P^m_B$ (here $T_{X_B/B}$ denotes the Lie algebra of 'vertical' vector fields, tangent to the fibres of f . We denote this deformation by $P^m(X_B/B)$. Intuitively it represents the family of $m-$th order deformations $f^{-1}(N_m(b)), b \in B$. Since $T_{X_B/B}$ acts on $\mathcal{O}_X$ by $\mathcal{O}_B-$linear derivations, it follows that $P^m(X_B/B)$ is a relative deformation in the category of $\mathcal{O}_B$-algebras.

Now the construction of universal deformations and related objects extends in a straightforward manner to the case of admissible $\mathfrak{g}_B-$ deformations. Thus, there is a relative very symmetric product $X < n > /B \xrightarrow{f_n} B$ which is just the fibre product $X < n > \times_{B<n>} B < 1 > \to B < 1 >= B$, and on this we have a relative Jacobi complex $J_m(\mathfrak{g}_B/B)$ which has a natural relative OS or comultiplicative structure, so that

$$\mathcal{R}_m(\mathfrak{g}_B/B) := \mathcal{O}_B \oplus \mathcal{H}om(\mathbb{R}^0 f_{m*}(J_m(\mathfrak{g}_B/B)), \mathcal{O}_B) =: \mathcal{O}_B \oplus \mathfrak{m}_m(\mathfrak{g}_B/B)$$

is a sheaf of $\mathcal{O}_B$-algebras of finite type as $\mathcal{O}_B-$module. Moreover there is a tautological morphic (comultiplicative) element

$$v_m \in \mathbb{H}^0(X < m > /B, J_m(\mathfrak{g}_B) \otimes \mathfrak{m}_m(\mathfrak{g}_B/B)$$

and there is correspondingly a tautological relative $\mathfrak{g}_B-$ deformation parametrized by $\mathcal{R}_m(\mathfrak{g}_B/B)$, which we denote by u_m/B. Under suitable hypotheses, which we proceed to state, u_m/B and v_m will be universal.

Now the following result generalizes Theorem 3.1 above and Theorem 0.1 of [9], and can be proved similarly.

Theorem 5.2. *Let $\mathfrak{g}_B$ be an admissible differential graded Lie algebra over X/B. Then: (i) to any isomorphism class of relative $\mathfrak{g}_B$-deformation parametrized by an algebra $\mathcal{S}$ of exponent m there are canonically associated a morphic Kodaira-Spencer element*

$$\beta_m(\phi) \in \mathbb{H}^0(J_m(\mathfrak{g}_B/B) \otimes \mathfrak{m}_{\mathcal{S}})$$

and a compatible homomorphism of $\mathcal{O}_B-$algebras

$$\alpha_m(\phi) : \mathcal{R}_m(g_B/B) \to \mathcal{S};$$

conversely, any morphic element $\beta \in \mathbb{H}^0(J_m(\mathfrak{g}_B/B) \otimes \mathfrak{m}_{\mathcal{S}})$ induces a relative $\mathfrak{g}_B$-deformation $\phi_m(\beta)$ parametrized by $\mathcal{S}$;
(ii) if $\mathfrak{g}_B$ has central sections then there is an isomorphism of relative deformations

$$\phi \simeq \phi_m(\beta_m(\phi));$$

any two such isomorphisms differ by an element of $Aut(\phi) = H^0(\exp(\mathfrak{g}_B^{\phi} \otimes \mathfrak{m}_{\mathcal{S}}))$.

Remarks 5.3. (i) As we have seen, there are nontrivial relative deformations even if the fibres of $X_B \to B$ are points, in which case $\mathcal{R}_m(\mathfrak{g}_B/B) = \mathcal{O}_B$ so $\alpha_m(\phi)$ certainly does not determine ϕ.
(ii) Note that in the above situation $\mathcal{R}(g_B/B)$ and $\mathcal{S}$ are not necessarily $\mathcal{O}_B-$flat.

Example 5.4. If $\mathcal{S}$ is of exponent 1, i.e. $\mathfrak{m}_{\mathcal{S}}^2 = 0$, then it is easy to see directly that relative $\mathfrak{g}_B$-deformations parametrized by $\mathcal{S}$ are in 1-1 correspondence with $H^1(X, \mathfrak{g}_B \otimes \mathfrak{m}_S)$. The Kodaira-Spencer homomorphism corresopnding to $\phi \in H^1(X, \mathfrak{g}_B \otimes \mathfrak{m}_S)$ is just the corresponding map $(\mathbb{R}^1 f_*(\mathfrak{g}_B))^v \to \mathfrak{m}_S$.

We might define a 'family of deformations parametrized by $\mathcal{S}$' to be a collection of isomorphism classes of deformations over members of some open cover of B, together with suitable gluing data over the overlaps; this type of object is naturally classified by $H^0(\mathbb{R}^0 f_{m*}(J_m(\mathfrak{g}_B/B)^{vv} \otimes \mathfrak{m}_{\mathcal{S}}))$. There is a natural map to this group from $H^0(J_m(\mathfrak{g}_B/B) \otimes m_{\mathcal{S}})$, and assuming $\mathbb{R}^0 f_{m*}(J_m(\mathfrak{g}_B/B)$ is locally free, this map may be analyzed with the usual Leray spectral sequence, which leads to the following result. First a definition. We will say that a Lie algebra sheaf $\mathfrak{g}_B$ as above has *relatively central sections* if the image of the natural map $f^{-1}f_*(\mathfrak{g}_B) \to \mathfrak{g}_B$ is contained in the center of $\mathfrak{g}_B$. Note that this condition is stronger than saying that $\mathfrak{g}_B$ has central sections, which concerns the image of $H^0(X_B, \mathfrak{g}_B) \to \mathfrak{g}_B$.

Corollary 5.5. *In the situation of Theorem 5.2, assume additionally that $\mathfrak{g}_B$ has relatively central sections, that $\mathbb{R}^0 f_{m*}(J_m(\mathfrak{g}_B/B))$ is $\mathcal{O}_B$-locally free, and that*

$$H^i(f_*(\mathfrak{g}_B) \otimes F) = 0, \forall i > 0,$$

for all coherent $\mathcal{O}_B-$modules F. Then for any relative $\mathfrak{g}_B$-deformation $\phi/\mathcal{S}$, we have

$$\phi \simeq \alpha_m(\phi)^*(u_m) = u_m/B \otimes_{\mathcal{R}_m(\mathfrak{g})} \mathcal{S}.$$

In particular, relative $\mathfrak{g}_B$-deformations are determined by their associated Kodaira-Spencer homomorphisms.

proof. Our hypotheses imply $H^i(\mathbb{R}^j f_{m*}J_m(\mathfrak{g}_B/B))\otimes\mathfrak{m}_{\mathcal{S}}) = 0, \forall j < 0$, so it suffices to apply the usual Leray spectral sequence to compute
$H^0(J_m(\mathfrak{g}_B/B)\otimes\mathfrak{m}_{\mathcal{S}}) = H^0(B, \mathbb{R}^0 f_{m*}(J_m(\mathfrak{g}_B/B))\otimes_{\mathcal{S}})$. □

Note that the hypotheses of the Corollary are satisfied provided first that $\mathbb{R}^0 f_{m*}(J_m(\mathfrak{g}_B/B)$ is locally free (i.e $\mathfrak{g}_B/B$ is 'relatively unobstructed'), and second, either $f_*(\mathfrak{g}_B) = 0$ or B is an affine scheme (provided all sheaves in question are B-coherent). In general however, a relative deformation cannot adequately by thought of as a family of isomorphism classes of deformations, because gluing together isomorphism classes of deformation is weaker than gluing together actual deformations.

Finally we will show that the constructions and results of §8 on connection algebras carry over *mutatis mutandis* to the relative case. Thus, suppose given an relative admissible pair $(\mathfrak{g}_B, E_B)$ on $X_B \xrightarrow{f} B$, with soft $\mathcal{O}_B-$linear resolution $(\mathfrak{g}^{\cdot}_B, E^{\cdot}_B)$, and assume given a finite complex $F^{\cdot}$ of free $\mathcal{O}_B$-modules of finite type such that $H^j(F^{\cdot}\otimes\mathbb{C}(b)) \simeq H^j(X_b, E_b)$, $\forall j, \forall b \in B$. As is well known, such complexes $F^{\cdot}$ always exist locally if f is a proper morphism of algebraic schemes and, as we shall see, the final statement will be essentially independent of the particular complex $F^{\cdot}$. Moreover, if E_B is relatively $i-$ equicyclic (i.e. $H^j(E_b) = 0 \forall j \neq i$) we may assume $F^j = 0 \forall j \neq i, i-1$. Then there is a relative connection algebra

$$\mathfrak{k}(\mathfrak{g}_B, E_B): \ \mathfrak{g}^{\cdot}_B \to f^{-1}({}^*F.)\otimes E^{\cdot}_B$$

where ${}^*F^{\cdot} = Hom^{\cdot}(F^{\cdot}, \mathcal{O}_B)$, which is still admissible and acts on $E^{\cdot}_B$, and the following relative analogue of Theorem 4.2 holds.

Theorem 5.6. *In the above situation, assume additionally that $\mathfrak{g}_B$ has relatively central sections and that E_B is relatively $i-$equicyclic. Then we have a class of isomorphisms*

$$\mathbb{R}^i f_*(M_m(\mathfrak{g}_B, E_B)) \otimes_{\mathcal{R}_m(\mathfrak{g}_B/B)} \mathcal{R}_m(\mathfrak{k}(\mathfrak{g}_B, E_B)) \simeq \mathbb{R}^i f_*(E_B) \otimes_{\mathcal{O}_B} \mathcal{R}_m(\mathfrak{k}(\mathfrak{g}_B, E_B))$$

any two of which differ by a map induced by an element of $\mathrm{Aut}(u_m/B)$ *where u_m/B is the universal relative deformation.* □

Corollary 5.7. *In the situation of Theorem 5.6, assume moreover that f is a smooth proper morphism of complex manifolds and that for some $m \geq 2$ we have that: (i) if ϕ_m is the relative deformation $P^m(E_B, \mathfrak{g}_B)/B$ parametrized by P^m_B (cf. Example 5.1(i)), then the associated Kodaira-Spencer homomorphism*
$\alpha_m(\phi_m): \mathcal{R}_m(\mathfrak{g}_B/B) \to P^m_B$ factors through $\mathcal{R}_m(\mathfrak{k}(\mathfrak{g}_B, E_B))$;
(ii) $f^{-1}f_*(\mathfrak{g}_B)$ acts on E_B as scalars.
Then the vector bundle $\mathbb{R}^i f_*(E_B)$ admits a natural projective connection.

proof. Set $G = \mathbb{R}^i f_*(E_B)$. Then our assumptions give an isomorphism of $P^m(G)$ and $G \otimes P^m_B$ globally defined up to scalars. For any $m \geq 2$, this is equivalent to a projective connection. □

6. Vector bundles on manifolds: the action of base motions

In this section we go back to the situation considered in §2, with a locally fine moduli space $\mathcal{M}$ with associated Lie algebra $\tilde{\mathfrak{g}}$ on $X \times \mathcal{M}$. We assume additionally that X is a compact complex manifold and $\tilde{\mathfrak{g}}$ is an $\mathcal{O}_X-$Lie algebra $\mathfrak{g}$ acting $\mathcal{O}_X$-linearly. We assume that

$$R^0 p_{\mathcal{M}*}(\tilde{\mathfrak{g}}) = 0. \tag{6.1}$$

For convenience, we shall also assume that

$$R^2 p_{\mathcal{M}*}(\tilde{\mathfrak{g}}) = 0, \tag{6.2}$$

which in particular implies that $\tilde{\mathfrak{g}}$ is (relatively) unobstructed, so that $\mathcal{M}$ is smooth (it seems reasonable that similar results can be obtained assuming only the unobstructedness). Of course, condition (6.2) holds automatically when X is a Riemann surface.

Since $\mathcal{M}$ is in a sense a functor of X, it seems intuitively plausible that a motion-say an infinitesimal motion, i.e. global holomorphic vector field on X- should induce a similar motion of $\mathcal{M}$. In this naive form this intuition seems of little use per se, since in cases of interest X will not admit any *global* holomorphic vector fields while *local* vector fields have no obvious relation to $\mathcal{M}$. But there is another, more 'global' way to represent the Lie algebra T_X of holomorphic vector fields on X, namely via the Dolbeault algebra $A^{\cdot}(T_X)$. Then the 'induced motion' idea suggests that there should be (something like) a map

$$\Sigma : A^{\cdot}(T_X) \to A^{\cdot}(T_{\mathcal{M}}). \tag{6.3}$$

Since Σ, at least in some sense, sends a motion of X to the induced motion of $\mathcal{M}$,it should be a dgla homomorphism. Now, at least on cohomology, a map as in (6.3) exists: it is none other than 'cap product with the Atiyah class of the universal bundle' which indeed is given essentially just by differentiating a cocycle defining this universal bundle with respect to the given vector field, then pushing down to $\mathcal{M}$. The upshot, then, is that a suitable version of the map Σ ought to be a Lie homomorphism, i.e. compatible with brackets (as well as, of course, the differential). This is what we aim to show in this section. As one might expect, this fact is important in relating deformations of X and $\mathcal{M}$.

The map Σ is defined as follows. Let $\psi \in \Gamma(\tilde{\mathfrak{g}}^1) \otimes \Omega_{X\times\mathcal{M}}$ be a representative of the Atiyah class $[\psi] = [P^1(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \in H^1(\tilde{\mathfrak{g}} \otimes \Omega_{X\times\mathcal{M}})$, the extension class of the 1st jet of $\tilde{\mathfrak{g}}$ over $X \times \mathcal{M}$. We replace T_X by its Dolbeault resolution $A^{\cdot}(T_X)$ (where $\cdot = (0, \cdot)$), truncated beyond degree 2 (which doesn't affect the deformation theory), and define a map $\Sigma_0 : A^{\cdot}(T_X) \to A^{\cdot+1}_{X\times\mathcal{M}}(\tilde{\mathfrak{g}})$ by

$$\Sigma_0(v) = \psi \neg v, \quad v \in A^{\cdot}(T_X) \tag{6.4}$$

where $\neg$ denotes interior multiplication or contraction. Since ψ is $\bar{\partial}-$closed, clearly Σ_0 commutes with $\bar{\partial}$. On the other hand our assumptions (6.1), (6.2) plus the fact that $\mathcal{M}$ is locally a fine moduli imply that the analogous map
$\Sigma_1 : A^{\cdot}(T_{\mathcal{M}}) \to A^{\cdot+1}_{X\times\mathcal{M}}(\mathfrak{g})$,

$$\Sigma_1(v) = \psi \neg v, \quad v \in A^{\cdot}(T_{\mathcal{M}}) \tag{6.5}$$

is a quasi-isomorphism, so we get a map in the derived category

$$\Sigma = \Sigma_1^{-1} \circ \Sigma_0 \tag{6.6}$$

Our main result concerning Σ is the following

Theorem 6.1. *Σ is a dgla homomorphism, i.e. is compatible with brackets.*

proof. It clearly suffices to prove that if $v_1, v_2 \in A^0(T_X)$, $v_i = \sum a_{i,j}\partial/\partial z_j$, are two type-(1,0) vector fields (not necessarily holomorphic), then

$$[\Sigma(v_1), \Sigma(v_2)] = \Sigma([v_1, v_2]) \tag{6.7}$$

To show that the two sides of (6.7) agree it suffices to check they agree pointwise at each point of $\mathcal{M}$. To this end we will use the recipe of §6 to compute the LHS.

So let us fix a point z of $\mathcal{M}$, corresponding to a particular pair $(\mathfrak{g}, E)$, and fix a $g-$connection of type $\bar{\partial}$ on E and $\mathfrak{g}$. Then first of all, it is clear by standard properties of the Atiyah class that the 'value' of $\Sigma(v)$ at any point $w \in \mathcal{M}$ is given by

$$\Sigma(v)|_w = [\nabla_v, \bar{\partial}_w] \tag{6.8}$$

where $\bar{\partial}_w$ is the $\bar{\partial}$ operator corresponding to w. Next, consider the restriction of $\Sigma(v)$ on the first infinitesimal neighborhood $N_1(z)$ of z in $\mathcal{M}$. By universality, we may identify the restriction of $\mathfrak{g}$ on $X \times N_1(z)$ with $\mathfrak{g}^\phi$, the first-order infinitesimal $\mathfrak{g}$-deformation of $\mathfrak{g}$, and likewise for E. Let $(\phi_i \in \Gamma(\mathfrak{g}^1))$ be a lift of a basis of $H^1(\mathfrak{g})$, and (ϕ_i^*) a lift of a dual basis. Now the prolongation of $\bar{\partial}_z$ in the direction corresponding to ϕ_i is obviously given by $\bar{\partial}_z + \phi_i$, hence may write
$\bar{\partial}|_{N_1(z)} = \bar{\partial}_z + \sum \phi_i^* \otimes \phi_i$. Therefore by (6.8) we have

$$\Sigma(v)|_{N_1(z)} = [\bar{\partial}_z, \nabla_v] + \sum \phi_i^*[\phi_i, \nabla_v]$$

Note that $\sum \phi_i^*[\phi_i, \nabla_v]$ is just the cobracket ${}^t\mathrm{br}(\nabla_v)$. Now by elaboration 2.2 we compute:

$[\Sigma(v_1), \Sigma(v_2)]|_z = <^t \mathrm{br}(\nabla_{v_1}), [\bar{\partial}_z, \nabla_{v_2}] > - <^t \mathrm{br}(\nabla_{v_2}), [\bar{\partial}_z, \nabla_{v_1}] >=$
$= [[\bar{\partial}_z, \nabla_{v_2}], \nabla_{v_1}] - [[\bar{\partial}_z, \nabla_{v_1}], \nabla_{v_2}]$. Applying the Jacobi identity to the first term yields

$[\Sigma(v_1), \Sigma(v_2)]|_z = -[[\nabla_{v_2}, \bar{\partial}_z], \nabla_{v_1}] - [[\nabla_{v_1}, \nabla_{v_2}], \bar{\partial}] - [[\bar{\partial}_z, \nabla_{v_1}], \nabla_{v_2}] = [\bar{\partial}_z, [\nabla_{v_1}, \nabla_{v_2}]]$.

But as our connection of of $\bar{\partial}$ type, its curvature is of type $(1,1)$, while v_1, v_2 are of type $(1,0)$, hence $[\nabla_{v_1}, \nabla_{v_2}] = \nabla_{[v_1, v_2]}$. Consequently we have

$$[\Sigma(v_1), \Sigma(v_2)]|_z = [\bar{\partial}_z, \nabla_{[v_1, v_2]}] = \Sigma([v_1, v_2])|_z.$$

Therefore (6.7) holds and the proof is complete. □

7. Lie atoms

Our purpose here is to give a minimal introduction to a notion which generalizes that of the quotient of a Lie algebra by a subalgebra (more precisely, a pair of Lie algebras viewed in the derived category), and which turns out to possess some of the formal properties of Lie algebras. This notion is suggested by some classical, and apparently disparate, deformation problems such as, on the one hand, the Hilbert scheme; on the other hand, heat-equation deformations in the sense of Welters [W] (see Example 7.3 below). Our treatment of the HKZ connection relies on factoring the moduli map Σ of §6 through a Lie atom, and our discussion of Lie atoms is limited to the minimum needed for that application. For more details, not essential here, on Lie atoms, see [11, 12].

Definition 7.1. *By a* Lie atom *(for 'algebra to module') we shall mean a complex $\mathfrak{g}^\sharp : \mathfrak{g} \xrightarrow{i} \mathfrak{h}$ in degrees 0,1, consisting of a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module $\mathfrak{h}$ and a $\mathfrak{g}$-module homomorphism i .*

The assumption about i being a homomorphism means explicitly that, writing $< , >$ for the $\mathfrak{g}$-action on $\mathfrak{h}$, we have $i([a,b]) = < a, i(b) > = - < b, i(a) >$. It is sometimes convenient to add the assumption that i is injective; this holds in the interesting examples we know. Note that any Lie algebra $\mathfrak{g}$ determines a Lie atom, minus the injectivity hypothesis, by taking $\mathfrak{h} = 0$, and the concept of Lie atom is essentially a generalization of that of Lie algebra. There is an obvious naïve notion of homomorphism of Lie atoms (map of complexes compatible with bracket and action), and for any Lie atom $\mathfrak{g}^\sharp$ the evident map $\mathfrak{g}^\sharp \to \mathfrak{g}$ is such a homomorphism (with kernel the Lie atom $(0, \mathfrak{h})$). We define a morphism of Lie atoms in the derived category sense, i.e. a composition of naïve morphisms and inverses of naive quasi-isomorphisms.

Examples 7.2 **A.** If $j : E_1 \to E_2$ is any linear map of vector spaces, let $\mathfrak{g} = \mathfrak{g}(j)$ be the *interwining algebra* of j, i.e. the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(E_1) \oplus \mathfrak{gl}(E_2)$ given by

$$\mathfrak{g} = \{(a_1, a_2) | j \circ a_1 = a_2 \circ j\}.$$

Thus $\mathfrak{g}$ is the 'largest' algebra acting on E_1 and E_2 so that j is a $\mathfrak{g}-$homomorphism. When j is injective, define

$$\mathfrak{gl}(E_1 < E_2) := (\mathfrak{g}, \mathfrak{gl}(E_2), i),$$

with $i(a_1, a_2) = a_2$. When j is surjective, define

$$\mathfrak{gl}(E_1 > E_2) := (\mathfrak{g}, \mathfrak{gl}(E_1), i),$$

with $i(a_1, a_2) = a_1$ These are Lie atoms. For any Lie atom $\mathfrak{g}^\sharp$, a homomorphism $\mathfrak{g}^\sharp \to \mathfrak{gl}(E_1 < E_2)$ (resp. $\mathfrak{g}^\sharp \to \mathfrak{gl}(E_1 > E_2)$) is called a left (resp. right) representation (or module) of $\mathfrak{g}^\sharp$.

B. If $i : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an injective homomorphism of Lie algebras then $\mathfrak{g}^\sharp := (\mathfrak{g}_1 \xrightarrow{i} \mathfrak{g}_2)$ is a Lie atom. More generally, if $\mathfrak{h}$ is any $\mathfrak{g}_1$ submodule of $\mathfrak{g}_2$ containing $i(\mathfrak{g}_1)$, then $\mathfrak{g}^\sharp := (\mathfrak{g}_1 \xrightarrow{i} \mathfrak{h})$ is a Lie atom.

C. Let E be an invertible sheaf on a ringed space X (such as a real or complex manifold), and let $\mathfrak{D}^i(E)$ be the sheaf of $i-$th order differential endomorphisms of E Then $\mathfrak{g} = \mathfrak{D}^1(E)$ is a Lie algebra sheaf and $\mathfrak{h} = \mathfrak{D}^2(E)$ is a $\mathfrak{g}-$module, giving rise to a Lie atom $\mathfrak{g}^\sharp$ which will be called the *Heat atom* of E and denote by $\mathfrak{D}^{1/2}(E)$. Note that if X is a manifold then $\mathfrak{g}^\sharp$ is quasi-isomorphic as a complex to $\mathrm{Sym}^2(T_X)[-1]$.

D. Let $Y \subset X$ be an embedding of manifolds (real or complex). Let $T_{X/Y}$ be the sheaf of vector fields on X tangent to Y along Y. Then $T_{X/Y}$ is a sheaf of Lie algebras contained in its module T_X, giving rise to a Lie atom

$$N_{Y/X}[-1] = (T_{X/Y} \subset T_X),$$

which we call the *normal atom* to Y in X. Notice that $T_{X/Y} \to T_X$ is locally an isomorphism off Y, so replacing $T_{X/Y}$ and T_X by their sheaf-theoretic restrictions on Y yields a Lie atom that is quasi-isomorphic to, and identifiable with $N_{Y/X}[-1]$.

An important, though elementary, remark about Lie atoms is that a Lie atom indeed constitutes a Lie object in the category of complexes, i.e. that the natural map $\bigwedge^2 \mathfrak{g}^\sharp \to \mathfrak{g}^\sharp$ extends to a complex, called a Jacobi complex $J(\mathfrak{g}^\sharp) : ... \bigwedge^{i+1} \mathfrak{g}^\sharp \to$

$\overset{i}{\bigwedge}\mathfrak{g}^\sharp ... \to \mathfrak{g}^\sharp$, which, to be precise, is a double complex

$$\begin{array}{ccccccc}
\cdots & S^3\mathfrak{h} & & & & \\
 & \uparrow & & & & \\
\cdots & \mathfrak{g}\otimes S^2\mathfrak{h} & \to & S^2\mathfrak{h} & & \\
 & \uparrow & & \uparrow & & \\
\cdots & \overset{2}{\bigwedge}\mathfrak{g}\otimes\mathfrak{h} & \to & \mathfrak{g}\otimes\mathfrak{h} & \to & \mathfrak{h} \\
 & \uparrow & & \uparrow & & \uparrow \\
\cdots & \overset{3}{\bigwedge}\mathfrak{g} & \to & \overset{2}{\bigwedge}\mathfrak{g} & \to & \mathfrak{g}
\end{array}$$

with up arrows defined by

$$g_1\wedge\cdots\wedge g_r\otimes h_1\cdots h_s \mapsto \frac{1}{r}\sum_{j=1}^{r}(-1)^j g_1\wedge\cdots\widehat{g_j}\cdots\wedge g_r\otimes i(g_j)h_1\cdots h_s$$

and right arrows defined by

$$g_1\wedge\cdots\wedge g_r\otimes h_1\cdots h_s\mapsto$$

$$\frac{1}{r!}\sum_{\sigma\in\mathcal{S}_r}\sum_{j=1}^{s}\mathrm{sgn}(\sigma)[g_{\sigma(1)},g_{\sigma(2)}]\wedge g_{\sigma(3)}\cdots g_{\sigma(r)}h_1\cdots h_s$$

$$-g_{\sigma(2)}\wedge\cdots\wedge g_{\sigma(r)}h_1\cdots<g_{\sigma(1)},h_j>\cdots h_s+g_{\sigma(1)}\wedge g_{\sigma(3)}\cdots\wedge g_{\sigma(r)}h_1\cdots<g_{\sigma(2)},h_j>\cdots h_s$$

The $\mathfrak{g}-$linearity of i (plus the $\frac{1}{r}$ factor) make each square commute (NB: if i were a g-derivation rather than $\mathfrak{g}$-linear, each square would commute without the $\frac{1}{r}$ factor, which would yield a differential graded Lie algebra).

Example 7.3. There is an evident notion of differential graded Lie atom that we will leave to the reader to explicate. An example is the following (compare §4), for a vector bundle E:

$$\begin{array}{ccc}
\mathfrak{D}^1(E) & \to & \mathfrak{D}^2(E) \\
\downarrow & & \downarrow \\
E\otimes^* E & = & E\otimes^* E
\end{array}$$

This is obviously quasi-isomorphic to $\mathfrak{D}^{1/2}(E)$, and admits a map to the connection algebra $\mathfrak{k}=\mathfrak{k}(\mathfrak{D}^1(E),E)$. By Theorem 4.2, it follows, when E is i-equicyclic, that for any $\mathfrak{D}^{1/2}(E)$- deformation, the cohomology $H^i(E)$ admits a canonical flat connection. In the case of first-order deformations (where flatness has no meaning), this goes back to Welters [W], as amplified by Hitchin [Hi].

8. Vector bundles on Riemann surfaces: refined action by base motions and Hitchin's connection

Our purpose here is to refine the results of the §6, in the case where X is 1-dimensional, by constructing a lift of Σ to another dgla associated to $\mathcal{M}$. We continue with the notations of that section; in particular, $\mathcal{M}$ is a locally fine moduli space associated to a dgla sheaf $\tilde{\mathfrak{g}}$ on $X\times\mathcal{M}$, and we also fix a $\tilde{\mathfrak{g}}-$ deformation $\tilde{E}$ on $X\times\mathcal{M}$, such that $R^ip_{\mathcal{M}*}(\tilde{E})=0,\quad i\neq 1$, and consequently

$$G := \det R^1 p_{\mathcal{M}*}(\tilde{E})$$

is an invertible sheaf on $\mathcal{M}$.

We note that G itself may be realized as the (sole nonvanishing) direct image of a suitable $\tilde{\mathfrak{g}}-$deformation, as follows. Note that

$$\tilde{\mathfrak{g}}_r := \pi_{r*} p_1^*(\tilde{\mathfrak{g}}),$$

where $\pi_r : X^r \times \mathcal{M} \to X < r > \times \mathcal{M}$, $X^r \times \mathcal{M} \to X \times \mathcal{M}$ are natural projections, naturally has the structure of dgla sheaf acting on $\lambda^r \tilde{E}$, and clearly $G = R^r p_{\mathcal{M}*}(\lambda^r E)$, with all other derived images being zero. There is a pullback map $Rp_{\mathcal{M}*}(\tilde{\mathfrak{g}}) \to Rp_{\mathcal{M}*}(\tilde{\mathfrak{g}}_r)$ which is compatible with brackets and induces isomorphisms on R^0 and R^1. Choosing a fixed base-set $\{x_1, ..., x_{r-1}\} \in X < r-1 >$ yields an embedding $X \to X < r >$ which induces a splitting of the pullback map, showing that this map is injective on R^2. It follows that we have a natural isomorphism

$$R^0 p_{\mathcal{M}*}(\tilde{\mathfrak{g}}) \to R^0 p_{\mathcal{M}*}(\tilde{\mathfrak{g}}_r).$$

Hence we may view G as the direct image of a $\tilde{\mathfrak{g}}$-deformation.

As in Examples 7.2.C, 7.3, we may consider the heat atom $\mathfrak{D}^{2/1}(G)$ associated to the $\mathcal{O}_{\mathcal{M}}-$module G, which is the pair $(\mathfrak{D}^1(G) \to \mathfrak{D}^2(G))$. Note that since G has rank 1, $\mathfrak{D}^{2/1}(G)$ is equivalent as complex on $\mathcal{M}$ to $\mathrm{Sym}^2 T_{\mathcal{M}}[-1]$, which is thus endowed with a Lie bracket. Also, $\mathfrak{D}^{2/1}(G)$ is obviously equivalent to the pair (Lie atom) $\mathfrak{D}^1(G)/\mathcal{O} \to \mathfrak{D}^2(G)/\mathcal{O}$. As we have seen, $\mathfrak{D}^i(G)$ may be naturally identified with the direct image of $J_i(\tilde{\mathfrak{g}}, F^*) \otimes F$ where $F = \lambda^r(\tilde{E})$ hence $\mathfrak{D}^{2/1}(G)$, i.e. $\mathrm{Sym}^2 T_{\mathcal{M}}[-1]$ is the direct image of $\lambda^2(\tilde{\mathfrak{g}})[1]$.

We now assume X is of dimension 1 and that $\mathcal{M}$ is the global fine moduli space $\mathcal{S}U^r_X(d)$ or $\mathcal{S}U^r_X(L)$ of vector bundles of rank r and fixed determinant L on X, where L is a line bundle of degree d. We assume temporarily that $(d, r) = 1$ (the modifications needed to handle the general case will be indicated later. As is well known [7], the assumption $(d, r) = 1$ implies that $\mathcal{M}$ is a fine moduli, in particular a locally fine moduli space associated to the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{sl}(\tilde{E})$, in the sense of §2. We now digress to discuss the Hitchin map (see [Hi, Rrel]). Consider the exact sequence on $X \times X$:

$$0 \to \tilde{\mathfrak{g}} \boxtimes \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \boxtimes \tilde{\mathfrak{g}}(\Delta) \to \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \otimes T_X \to 0$$

Dualizing the trace map yields a map $tr^* : T_X \to \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \otimes T_X$. Pulling back the extension via tr^*, then pushing forward by the natural map $X \times X \to X < 2 >$ and taking anti-invariants, we get a map in the derived category $T_X \to \lambda^2(\tilde{\mathfrak{g}})[1]$, whence a map

$$\Omega : T_X \to \mathfrak{D}^{2/1}(G).$$

A relatively straightforward calculation, already implicit in [Hi] (see also [Rrel]) shows that the composition of Ω with the natural map $\mathfrak{D}^{1/2}(G) \to \mathfrak{D}^1(G)/\mathcal{O} = T_{\mathcal{M}}$ is (a canonical nonzero integer multiple of) the map Σ in §6.

Theorem 8.1. *Ω is a Lie homomorphism.*

proof. Recall that we are identifying T_X with the dgla $A^{\cdot}(T_X)$, which exists in degrees 0,1. In degree 0, $\mathfrak{D}^{2/1}(G)$ can be identified with $\mathfrak{D}^1(G)/\mathcal{O} \simeq T_{\mathcal{M}}$, so the homomorphism property is just Theorem 10.1. Therefore it just remains to prove the homomorphism property in degree 1. For any $v \in T_X$, write $\Omega(v) = (A(v), B(v))$, with $A(v) \in \mathfrak{D}^1(G)/\mathcal{O}, B(v) \in \mathfrak{D}^2(G)/\mathcal{O}$. Then what has to be shown is that for any $v_0 \in A^0(T_X), v_1 \in A^1(T_X)$, we have

$$B([v_0, v_1]) =< A(v_0), B(v_1) - < A(v_1), B(v_0) > . \tag{8.1}$$

Now firstly, $B(v_0) = 0$ since B lowers degree by 1. Next, since $[v_0, v_1]$ is automatically $\bar{\partial}-$closed, we have $\bar{\partial}B([v_0, v_1]) = A([v_0, v_1]) = [A(v_0), A(v_1)]$ the last equality by Theorem 6.1. Again because v_1 is $\bar{\partial}-$closed, we have $A(v_1) = \bar{\partial}B(v_1)$. The upshot is that both sides of (8.1) have the same $\bar{\partial}$, hence their difference yields a global holomorphic section of $\mathfrak{D}^2(G)/\mathcal{O}$ over $\mathcal{M}$. However, it is well known that $\mathfrak{D}^2(G)/\mathcal{O}$ has no nonzero sections: indeed this follows from Hitchin's result that the coboundary map $H^0(\mathrm{Sym}^2 T_{\mathcal{M}}) \to H^1(T_{\mathcal{M}})$ is injective, plus the fact that $H^0(T_{\mathcal{M}}) = 0$ (cf. [7]). This completes the proof. □

Now consider the diagram, as in Example 7.3:

$$\begin{array}{ccc} \mathfrak{D}^1(G)/\mathcal{O} & \to & \mathfrak{D}^2(G)/\mathcal{O} \\ \downarrow & & \downarrow \\ G \otimes^* G/\mathcal{O}I & = & G \otimes^* G/\mathcal{O}I \end{array}$$

where the vertical arrows are induced by the action of $\mathfrak{D}^1(G)$ and $\mathfrak{D}^2(G)$ on ${}^*G$ and I is the identity in $G \otimes^* G$. This diagram itself may be considered a dgla quasi-isomorphic to $\mathfrak{D}^{2/1}(G)$. And of course the left column is quasi-isomorphic to the connection algebra $\mathfrak{k}(\mathfrak{D}^1(G), G)$ (cf. §4). Consequently, we have a Lie homomorphism $\mathfrak{D}^{2/1}(G) \to \mathfrak{k}(\mathfrak{D}^1(G), G)$. Composing this with Ω above, we get a Lie homomorphism

$$\omega : T_X \to \mathfrak{k}(\mathfrak{D}^1(G), G).$$

It follows easily from this that over the deformation space of pairs (X, L) there is a canonical local trivialization or connection on the projective bundle associated to $H^0(G)$, which is the main result of Hitchin [6] (see also [3], [5], [8], [15], [16], [17], [19] and references therein; the connection is sometimes called the Hitchin or Knizhnik-Zamolodchikov connection):

Corollary 8.2. *Let Y be any manifold parametrizing pairs (X, L) where X is a compact Riemann surface of genus $g \geq 3$ and L is a line bundle of degree d on X, and let $\mathcal{H}$ be the vector bundle on Y with fibre $H^0(\mathcal{SU}^r(X, L), G)$. Assume d, r are relatively prime. Then there is a canonical projective connection on $\mathcal{H}$.*

proof. We have a family of smooth curves X_Y/Y and a family of associated moduli spaces which we denote by $\mathcal{M}_Y/Y$, and there is a commutative diagram of $\mathcal{O}_Y$-algebras and homomorphisms:

$$\begin{array}{ccc} R_m(T_{X/Y}/Y) & \to & P_Y^m \\ \uparrow & \nearrow & \\ R_m(T_{\mathcal{M}_Y}/Y) & & \end{array}$$

where the vertical homomorphism is induced by Σ. This diagram represents the intuitive fact that we have a family of $m-$th order deformations of fibres X_y and $\mathcal{M}_y$ for $y \in Y$ (cf. Example 5.1(ii)). As we have seen in Theorem 8.1, the map induced by Σ factors through $\tilde{R} = R_m(\mathfrak{k}(\mathfrak{D}^1(G), G))$. The module $P^m(\mathcal{H})$ comes by extension of scalars from an analogous module over $\tilde{R}$ which by Corollary 3.2 is isomorphic (up to scalars) to $\mathcal{H} \otimes_{\mathcal{O}_B} \tilde{R}$. Hence as $\mathcal{O}_Y$-modules, $P^m(\mathcal{H})$ and $P_Y^m \otimes \mathcal{H}$ are isomorphic up to scalars, so there is a projective connection (cf. Corollary 5.7). □

Now we will indicate the extension of this result to the case where d and r have a common factor, so that we have only a coarse moduli space without a universal family. Fixing r, d, let $U^s \subset SU^r(X,d)$ be the subset corresponding to stable bundles. As is well known, under our assumptions $SU^r(X,d)$ is normal and projective and the complement of U^s has codimension > 1, hence for any line bundle F on $SU^r(X,d)$ the restriction map $H^0(F, SU^r(X,d)) \to H^0(F, U^s)$ is an isomorphism. Now by construction (see [7], [14], [18]), there is a finite collection $\mathcal{U}$ of locally fine moduli spaces U_α, with corresponding rank-r universal bundles $\tilde{E}_\alpha$ on $X \times U_\alpha$, such that the images of the natural maps. $f_\alpha : U_\alpha \to U^s$ form a covering. We may further assume that each U_α is affine and Galois over its image in U^s, and that the collection (U_α, f_α) is 'Galois-stable' in the sense that for each deck transformation $\rho, (U_\alpha, f_\alpha \circ \rho)$ is also in the collection. Now set

$$U_{\alpha\beta} := U_\alpha \times_{U^s} U_\beta$$

and likewise for triple products etc. Let
$p_\alpha := 1_X \times f_\alpha : X \times U_\alpha \to X \times U^s$, let $p_{\alpha\beta,\alpha} : X \times U_{\alpha\beta} \to X \times U_\alpha$ be the obvious projection, and let $p_{\alpha\beta} : X \times U_{\alpha\beta} \stackrel{p_{\alpha\beta,\alpha}}{\to} X \times U_\alpha \stackrel{p_\alpha}{\to} X \times U^s$ be the composite, and again likewise for higher products. Note that for any coherent sheaf F on $X \times U^s$, we may form a Čech-type complex (of sheaves)

$$\check{C}(\mathcal{U}, F) : \bigoplus_\alpha p_\alpha^* F \to \bigoplus_{\alpha,\beta} p_{\alpha\beta}^* F \to \cdots$$

and our saturation condition ensures that the cohomology of F-with H^0 included- may be computed from the hypercohomology of this complex, in other words $\check{C}(\mathcal{U}, F)$ is quasi-isomorphic to F, i.e. to its Čech complex with respect to an ordinary cover of $X \times U^s$ (thus ' étale cohomology coincides with ordinary cohomology for coherent sheaves'). Of course in our case the problem is that we don't have an actual universal bundle E, defined as a sheaf over all of $X \times U^s$ (this is a result of Nori, cf. [14]). However, we shall see that we can still define a complex to play the role of $\check{C}(\mathcal{U}, E)$ for a universal bundle E, and the foregoing discussion shows that this is 'good enough' at least for cohomology.

Note next that up to shrinking our cover, we may assume we have isomorphisms

$$\sigma_{\beta\alpha} : p_{\alpha\beta,\alpha}^* \tilde{E}_\alpha \to p_{\alpha\beta,\beta}^* \tilde{E}_\beta.$$

Indeed the sheaf $p_{U_{\alpha\beta}*}(\mathcal{H}om(p_{\alpha\beta,\alpha}^* \tilde{E}_\alpha, p_{\alpha\beta,\beta}^* \tilde{E}_\beta))$ is invertible by stability, hence after shrinking may be assumed trivial, and a nonvanishing section of it yields the required isomorphism. There is obviously no loss of generality in assuming that $\sigma_{\beta\alpha} = \sigma_{\alpha\beta}^{-1}$. Now note that over a triple product $U_{\alpha\beta\gamma} := U_\alpha \times U_\beta \times U_\gamma$, the map

$$\sigma_{\gamma\alpha}^{-1} \circ \sigma_{\gamma\beta} \circ \sigma_{\beta\alpha} \in \mathrm{Aut}(\tilde{E}_\alpha|_{U_{\alpha\beta\gamma}})$$

must, for the same reason, be a scalar. Consequently, $\sigma_{\gamma\alpha}^{-1} \circ \sigma_{\gamma\beta} \circ \sigma_{\beta\alpha}$ induces the *identity* on

$$\tilde{\mathfrak{g}}_\alpha|_{U_{\alpha\beta\gamma}} = \mathfrak{sl}(\tilde{E}_\alpha)|_{U_{\alpha\beta\gamma}} \subset \tilde{E}_\alpha \otimes \tilde{E}_\alpha^*|_{U_{\alpha\beta\gamma}}.$$

Consequently, we may form a complex which may be considered the 'Čech complex' for $\tilde{\mathfrak{g}}$ with respect to the étale covering $\mathcal{U} := (U_\alpha)$: namely the complex with sheaves

$$\bigoplus \check{C}(\tilde{\mathfrak{g}}, \mathcal{U})_{\alpha\beta\gamma\ldots} = \bigoplus \tilde{\mathfrak{g}}_\alpha|_{U_{\alpha\beta\gamma\ldots}} := \bigoplus p_{\alpha\beta\gamma\ldots,\alpha}^* \tilde{\mathfrak{g}}_\alpha,$$

each of which we identify with its own Dolbeault or Čech complex (using some affine covering of X), and whose differentials are constructed as usual from the pullback maps

$$r_{\alpha\beta\gamma\ldots,\alpha\beta\gamma\ldots\epsilon\ldots} : \tilde{\mathfrak{g}}_\alpha|_{U_{\alpha\beta\gamma\ldots}} \to \tilde{\mathfrak{g}}_\alpha|_{U_{\alpha\beta\gamma\ldots\epsilon\ldots}}$$

and from maps

$$r_{\alpha\beta\gamma\ldots,\epsilon\alpha\beta\gamma\ldots} : \tilde{\mathfrak{g}}_\alpha|_{U_{\alpha\beta\gamma\ldots}} \to \tilde{\mathfrak{g}}_\epsilon|_{U_{\epsilon\alpha\beta\gamma\ldots}}$$

given by restriction to $\tilde{\mathfrak{g}}_\alpha|_{U_{\alpha\beta\gamma\ldots\epsilon\ldots}}$ followed by the isomorphism

$$\tilde{\mathfrak{g}}_\alpha|_{U_{\alpha\beta\gamma\ldots\epsilon\ldots}} \to \tilde{\mathfrak{g}}_\epsilon|_{U_{\epsilon\alpha\beta\gamma\ldots}}$$

induced by $\sigma_{\epsilon\alpha}$. By the above, these indeed form a complex, and this complex automatically inherits the structure of a dgla from $\tilde{\mathfrak{g}}$. For the purposes of our constructions, this complex may be taken as a substitute for $\tilde{\mathfrak{g}}$ itself. Moreover, since the adjoint action of $\tilde{\mathfrak{g}}$ on itself is faithful, we may take $\tilde{\mathfrak{g}}$ as a substitute for the universal $\tilde{\mathfrak{g}}$ deformation $\tilde{E}$.

Now of course the theta-bundle θ itself and its powers such as $F = \det H^1(\tilde{\mathfrak{g}})$ of course exist as actual line bundles on U^s, and all the auxiliary complexes we need are derived from $\tilde{\mathfrak{g}}$ and F. Note that for any line bundle L we have a natural isomorphism of Lie algebras
$\mathfrak{D}^1(L) \overset{\sim}{\to} \mathfrak{D}^1(L^k)$, given by the formula

$$D(s_1 \cdots s_k) = \sum s_1 \cdots D(s_i) \cdots s_k,$$

where D and $s_1, ..., s_k$ are local sections of $D^1(L)$, L respectively. Consequently, we may identify $\mathfrak{D}^1(\theta^k)$ and $D^1(F)$ as Lie algebras. Hence all of our constructions go through in this context and establish the flatness of the connection.

Corollary 8.2 bis. *The conclusion of Corollary 8.2 holds for all d, r.* □

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