CYCLE MAP ON HILBERT SCHEMES OF NODAL CURVES

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ABSTRACT. We study the structure of the relative Hilbert scheme for a family of nodal (or smooth) curves via its natural cycle map to the relative symmetric product. We show that the cycle map is the blowing up of the discriminant locus, which consists of cycles with multiple points. We discuss some applications and connections, notably with birational geometry and intersection theory on Hilbert schemes of smooth surfaces.

INTRODUCTION

An object of central importance in classical algebraic geometry is a family of projective curves, given by a projective morphism

$$\pi : X \to B$$

with smooth general fibre. One wants to take B itself projective, which means one must allow some singular fibres. We will assume our singular fibres al all nodal. Of course, by semistable reduction, etc., any family can be modified so as to have this property, without changing the general fibre $X_b = \pi^{-1}(b)$. Many questions of classical geometry involve point-configurations on fibres X_b with $b \in B$ variable. From a modern standpoint, this means they involve the relative Hilbert scheme

$$X_B^{[m]} = \mathrm{Hilb}_m(X/B).$$

This motivates the interest in studying $X_B^{[m]}$ and setting it up as a tool for studying the geometry, e.g. enumerative geometry, of the family X/B. This paper is a step in this direction. Our focus will be on the *cycle map* (sometimes called the 'Hilb-to-Chow' map) $\mathfrak{c}_m$, which in this case takes values in the relative symmetric product $X_B^{(m)}$.

Our main result is that $\mathfrak{c}_m$ is the blowing-up of the discriminant locus in $X_B^{(m)}$. This result will be proven in §1. As we shall see, the proof amounts to a fairly complete study, locally over $X_B^{(m)}$, of $\mathfrak{c}_m$. We shall see in particular that $\mathfrak{c}_m$ is a small resolution of singularities; in fact in 'most' cases the non-point fibres of $\mathfrak{c}_m$ are chains of rational curves (with at most $m - 1$ components). In §2 we will consider applications of the result of §1 to the further study of $X_B^{[m]}$ and $\mathfrak{c}_m$. We will give a formula for the canonical bundle

of $X_B^{[m]}$ showing that $\mathfrak{c}_m$ 'looks like' a flipping contraction; in fact, $\mathfrak{c}_2$ is none other than the *Francia flip* and admits a natural 2:1 covering by the flop associated to a 3-fold ODP . We will also give a simple formula for the *Euler number* of $X_B^{[m]}$. In §3 we will discuss the Chern classes of *tautological bundles*. These are bundles whose fibre at a point representing a scheme z is $H^0(E \otimes \mathcal{O}_z)$, where E is a fixed vector bundle on X.

This paper has substantial intersection with the Author's papers [7, 8, 9] where some of the results are proven in greater detail.

As to the relevance of this paper to the theme of 'projective varieties with unexpected properties' I can only say that the close links- some exposed below- of the Hilbert scheme, a priori a purely algebraic object, to classical projective geometry were quite unexpected by me, though this is probably due only to my own ignorance.

1. The cycle map as blow-up

Our main object of study is family of projective curves

$$\pi : X \to B$$

whose fibres $X_b = \pi^{-1}(b)$ are smooth for $b \in B$ general. We shall make the following

Essential hypothesis: X_b is nodal for all $b \in B$.

We shall also make the (nonessential, but convenient) hypothesis that X, B are smooth of dimension 2,1 respectively.

Geometry of the family largely amounts to the study of families of sub-varieties (more precisely *subschemes*)

$$\{Z_b \subset X_b, b \in B\}$$

of some fixed degree (length) m over B.

The canonical parameter space for subschemes is the *relative Hilbert scheme*

$$X_B^{[m]} = \mathrm{Hilb}_m(X/B).$$

So (ordinary)points $z \in X_B^{[m]}$ correspond 1-1 with pairs (b, Z) where $b \in B$ and $Z \subset X_b$ is a length-m subscheme. More generally, for any artin local

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$\mathbb{C}$-algebra R and $S = \mathrm{Spec}(R)$, we have a bijection between diagrams

$$\begin{array}{ccc} S & \xrightarrow{f} & X_B^{[m]} \\ & \searrow f_0 & \downarrow \\ & & B \end{array}$$

and

$$\begin{array}{ccccc} Z & \subset & X_S & \rightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S & \xrightarrow{f_0} & B \end{array}$$

with the right square cartesian and Z/S flat of relative length m.

As usual in Algebraic Geometry, we study a complex object like $X_B^{[m]}$ by relating it (mapping it) to other (simpler ?) objects. One approach (not pursued here, but see [7]) is to relate $X_B^{[m]}$ (albeit only by correspondence, not morphism) to $X_B^{[m-1]}$. This leads to studying *flag Hilbert schemes*. These have a rich geometry; they are generally singular.

We focus here on another approach, based on the **cycle map**

$$\mathfrak{c}_m : X_B^{[m]} \rightarrow \mathrm{Sym}_B^m(X) =: X_B^{(m)}$$

$$Z \mapsto \sum_{p \in X} \mathrm{length}_p(Z) p.$$

Clearly, $\mathfrak{c}_m$ is an iso off the locus of cycles whose support meets the critical or singular locus

$$\mathrm{sing}(\pi) = \text{locus in } X \text{ of singular points of fibres of } \pi.$$

Main Theorem. $\mathfrak{c}_m$ *is the blowing-up of the* **discriminant locus**

$$D^m = \{\sum m_i p_i : \exists m_i > 1\} \subset X_B^{(m)}$$

Recall that if I is an ideal on scheme X, we have a surjection of graded algebras from the symmetric algebra on I to the Rees or blow-up algbera

$$\mathrm{Sym}^\bullet(I) \rightarrow \bigoplus_0^\infty I^j$$

Applying the Proj functor, we get a closed embedding (maybe strict) of schemes over X

$$\mathrm{B}\ell_I(X) \subseteq \mathbb{P}(I)$$

of the blow-up into the 'singular projective bundle' $\mathbb{P}(I)$, whose fibres over X are projective spaces of varying dimensions. Note that $\mathbb{P}(I)$ may be reducible, while $\mathrm{B}\ell_I(X)$ is always an integral scheme if X is. Concretely, these schemes may be described, locally over X, as follows: if $f_1, ..., f_r$ generate

I, take formal homogeneous coordinates $T_1, ..., T_r$, then as subschemes of $X \times \mathbb{P}^{r-1}$,

$$\mathrm{B}\ell_I(X) = \mathrm{Zeros}(G(T_1, ..., T_r) : G(f_1, ..., f_r) = 0, G \text{ homogeneous})$$
$$\mathbb{P}(I) = \mathrm{Zeros}(G(T_1, ..., T_r) : G(f_1, ..., f_r) = 0, G \text{ homogeneous linear})$$

Thus, the inclusion $\mathrm{B}\ell_I(X) \subseteq \mathbb{P}(I)$ is strict iff I admits nonlinear syzygies; the case of the discriminant locus, to be studied below, will provide examples of such ideals.

Remark Will see in the proof that
- $X_B^{[m]}$ is smooth (over $\mathbb{C}$) of dimension $m + \dim B$
- $\mathfrak{c}_m$ is a small map (in fact, if each X_b has at most ν nodes– usually, $\nu = 1$ – then fibres of $\mathfrak{c}_m$ have dimension at most $\min(\nu, m/2)$).

Clearly, D^m is a prime Weil divisor on $X_B^{(m)}$, in fact

$$D^m \sim_{bir} X \times_B X_B^{(m-2)}$$

because a general $z \in D$ has the form

$$z = 2p_1 + p_2 + ... + p_{m-1}$$

. On the other hand, near cycles meeting $\mathrm{sing}(\pi)$, esp. 'maximally singular' cycles

$$z = mp, p \in \mathrm{sing}(\pi),$$

it's not clear a priori what (or how many) defining equations D^m has (the proof below will yield a posteriori e set of equations locally at maximally singular cycles).
Note that locally at maximally singular cycles, the relative Cartesian product X_B^m is a complete intersection with equation $x_1 y_1 = ... = x_m y_m$, with the projection to B given by $t = x_1 y_1$, while $X_B^{(m)}$ is a quotient of a complete intersection

$$(x_1 y_1 = ... = x_m y_m)/ \text{ symmetric group } \mathfrak{S}_m.$$

We will see that $X_B^{(m)}$ is not $\mathbb{Q}$-factorial: in fact, D^m is not $\mathbb{Q}$-Cartier; Worse, $X_B^{(m)}$ is not even $\mathbb{Q}$-Gorenstein: we shall see that it admits a small discrepant resolution $X_B^{[m]}$.
Nonetheless, being quotient by a finite group and smooth in codimension 1, $X_B^{(m)}$ is normal and Cohen-Macaulay.

The *plan of proof* is as follows.
◇ Construct explicit (analytic) model of $X_B^{[m]}$ and $\mathfrak{c}_m$, locally over $X_B^{(m)}$; in particular, conclude that $X_B^{[m]}$ is smooth and $\mathfrak{c}_m$ is small, so $\mathfrak{c}_m^{-1}(D^m)$ is Cartier divisor.

$\diamond$ The Universal property of blowing up now yields a factorization

$$X_B^{[m]} \quad \overset{\mathfrak{c}'_m}{\to} \quad \mathrm{B}\ell_{D^m}\, X_B^{(m)}$$
$$\searrow_{\mathfrak{c}_m} \qquad \downarrow \mathrm{b}\ell$$
$$X_B^{(m)}$$

Then we check locally (over the blowup) that $\mathfrak{c}'_m$ is an iso.

To start the proof, fix an analytic neighborhood U of fibre a node p, so the family is given in local analytic coordinates by $xy = t$.

For the local study, the first question is: what are fibres of $\mathfrak{c}_m$?
Now locally in the étale topology, all fibres are (essentially) products of fibres $\mathfrak{c}_{m_i}^{-1}(m_i p_i)$. So suffices to study

$$\mathfrak{c}_m^{-1}(mp), p \in \mathrm{sing}(\pi).$$

Then,

$$\mathfrak{c}_m^{-1}(mp) = \mathrm{Hilb}_m^0(R)$$

where R is the formal power series ring

$$R = \mathbb{C}[[x, y]]/(xy).$$

Here Hilb_m^0 denotes the punctual Hilbert scheme.

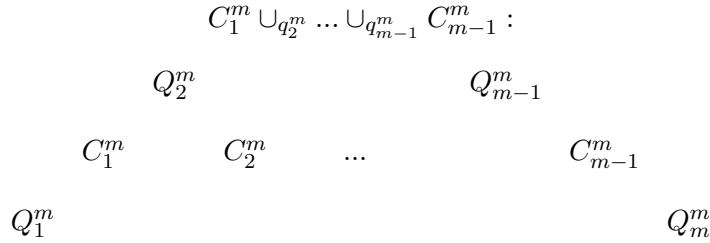**Proposition 1.1.** *$Hilb_m^0(R)$ is a chain of $m - 1$ smooth rational curves meeting normally*

$$C_1^m \cup_{q_2^m} ... \cup_{q_{m-1}^m} C_{m-1}^m :$$

$$\begin{array}{ccccc} & Q_2^m & & Q_{m-1}^m & \\ C_1^m & & C_2^m \quad ... & & C_{m-1}^m \\ Q_1^m & & & & Q_m^m \end{array}$$

Fig. 1

$$q_i^m = (x^{m+1-i}, y^i),$$

$$C_i^m \setminus \{q_i^m, q_{i+1}^m\} = \{I_i^m(a) = (ax^{m-i} + y^i) : a \neq 0\}$$

NB $\lim\limits_{a \to 0} I_i^m(a) = q_i^m,\ \lim\limits_{a \to \infty} I_i^m(a) = q_{i+1}^m.$

Proof. See [8] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Given this, the next question is: what does the full Hilbert scheme look like along Hilb^0, e.g. locally near q_i^m?

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Proposition 1.2. *The universal flat deformation of the ideal* $q_i^m = (x^{m+1-i}, y^i), i = 1, ..., m,$ *rel B, is* (f, g) *where*

$$f = x^{m+1-i} + f_{m-i}^1(x) + vy^{i-1} + f_{i-2}^2(y),$$

$$g = y^i + g_{i-1}^1(y) + ux^{m-i} + g_{m-i-1}^2(x)$$

where each f_b^a, g_b^a *has degree b,* f_{m-i}^1, g_{i-1}^1 *have no constant term, and the following relations, equivalent to flatness, hold*

$$yf = vg$$

$$xg = uf$$

Proof. See [8] □

Concretely, the above relation mean
- the coefficients of $f_{m-i}^1(x), g_{i-1}^1(y)$ are free parameters (no relations);
- the relation $uv = t$ holds;
- f_{i-2}^2, g_{m-i-1}^2 are determined by the other data.

A similar and simpler story holds at the principal ideals $I_i^m(a)$.
We conclude
- $X_B^{[m]}$ is smooth;
- its fibre at $t = 0$, i.e. $\mathrm{Hilb}_m(X_0)$ has, along $\mathrm{Hilb}_m^0(R)$, $(m+1)$ smooth components crossing normally, $D_0, ..., D_m$.

fig. 2

In fact, if $X_0 = X_0' \cup X_0''$ then

$$D_i \sim_{bir} (X_0')^{m-i} \times (X_0'')^i.$$

The next question is: how to glue together the various local deformations ?

Construction Let $C_1, ..., C_{m-1}$ be copies of $\mathbb{P}^1$, with homogenous coordinates u_i, v_i on the i-th copy. Let $\tilde{C} \subset C_1 \times ... \times C_{m-1} \times B$ be the subscheme defined by

$$v_1u_2 = tu_1v_2, ..., v_{m-2}u_{m-1} = tu_{m-2}v_{m-1}$$

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Fibre of $\tilde{C}$ over $0 \in B$ is

$$\tilde{C}_0 = \bigcup_{i=1}^{m-1} \tilde{C}_i,$$

where

$$\tilde{C}_i = [1,0] \times \ldots \times [1,0] \times C_i \times [0,1] \times \ldots \times [0,1]$$

In a neighborhood of $\tilde{C}_0$, $\tilde{C}$ is smooth and $\tilde{C}_0$ is its unique singular fibre over B. We may embed $\tilde{C}$ in $\mathbb{P}^{m-1} \times B$ via

$$Z_i = u_1 \cdots u_{i-1} v_i \cdots v_{m-1}, i = 1, \ldots, m.$$

These satisfy

$$Z_i Z_j = t Z_{i+1} Z_{j-1}, i < j - 1$$

so embed $\tilde{C}$ as a family of rational normal curves $\tilde{C}_t \subset \mathbb{P}^{m-1}, t \neq 0$ specializing to a connected $(m-1)$-chain of lines.

Next consider $\mathbb{A}^{2m}$ with coordinates $a_0, \ldots, a_{m-1}, d_0, \ldots, d_{m-1}$
Let $\tilde{H} \subset \tilde{C} \times \mathbb{A}^{2m}$ be defined by

$$a_0 u_1 = t v_1,$$
$$a_1 u_1 = d_{m-1} v_1, \qquad \ldots \qquad , a_{m-1} u_{m-1} = d_1 v_{m-1}$$
$$d_0 v_{m-1} = t u_{m-1}$$

Fibres of $\tilde{H}$ over $\mathbb{A}^{2m}$ are: a point (generically), or a chain of $i \leq m-1$ rational curves; all values $i = 1, \ldots, m-1$ occur. Consider the subscheme of $Y = \tilde{H} \times_B U$ defined by

$$F_0 := x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$$
$$F_1 := u_1 x^{m-1} + u_1 a_{m-1} x^{m-2} + \ldots + u_1 a_2 x + u_1 a_1$$
$$+ v_1 y$$
$$\ldots$$
$$F_i := u_i x^{m-i} + u_i a_{m-1} x^{m-i-1} + \ldots + u_i a_{i+1} x + u_i a_i$$
$$+ v_i d_{m-i+1} y + \ldots + v_i d_{m-1} y^{i-1} + v_i y^i$$
$$\ldots$$
$$F_m := d_0 + d_1 y + \ldots + d_{m-1} y^{m-1} + y^m$$

The following is proven in [9]

Theorem 1.1. *(i) $\tilde{H}$ is smooth and irreducible.*
(ii) The ideal sheaf $\mathcal{I}$ generated by $F_0, \ldots, F_m$ defines a subscheme of $\tilde{H} \times_B U$ that is flat of length m over $\tilde{H}$
(iii) The classifying map

$$\Phi = \Phi_{\mathcal{I}} : \tilde{H} \to Hilb_m(U/B)$$

is an isomorphism.

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The proof shows furthermore that $\tilde{H}$ is covered by opens
$$U_i = \{Z_i \neq 0\}, i = 1, ..., m$$

fig.3

On U_i, we have
$$F_j = u_j x^{i-j-1} F_{i-1}, j < i - 1$$
$$F_j = v_j y^{j-i} F_i, j > i$$
hence F_{i-1}, F_i generate $\mathcal{I}$ on U_i (they yield the f, g in the universal deformation of Proposition 2 above).

Also, $a_i = (-1)^i \sigma^x_{m-i}$ are the elementary symmetric functions in the roots of F_0, and ditto for d_i, σ^y_{m-i}, F_m. So the projection $\tilde{H} \to \mathbb{A}^{2m}_B$ factors through the cycle map

$$\tilde{H}$$
$$\mathfrak{c} \downarrow \qquad \searrow$$
$$X^{(m)}_B \quad \overset{\sigma}{\to} \quad \mathbb{A}^{2m}_B$$
$$\sigma = (\sigma^x_1, ..., \sigma^x_m, \sigma^y_1, ..., \sigma^y_m)$$

(one can show σ is embedding). To prove the Main Theorem, we must show:
$\mathfrak{c}$ is the blow-up of D^m
It is convenient to pass to an 'ordered' model, defined by the following Cartesian diagram:

$$X^{[m]}_B \quad \to \quad X^{[m]}_B$$
$$\downarrow \qquad\qquad \downarrow$$
$$X^m_B \quad \to \quad X^{(m)}_B$$

In this diagram, the right vertical arrow is the cycle map, the bottom horizontal arrow is the natural map between the Cartesian and symmetric products, and the other arrows are defined by the fibre product construction. Recall the description of the blowup of an ideal I as subscheme of $\mathbb{P}(I)$. Let us rewrite the defining local equations for $X^{[m]}_B$ in terms of the homogeneous coordinates Z_i on $\mathbb{P}^{m-1}$: they are

linear:

$$\sigma^y_{m-j} Z_i = t^{m-j-i} \sigma^x_j Z_{i+1}, \ i = 1, ..., m-1, j = 0, ..., m-1;$$

$$\sigma^x_{m-j} Z_i = t^{m-j-i} \sigma^y_j Z_{i-1}, \ i = 2, ..., m, j = 0, ..., m-1.$$

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quadratic:

$$Z_i Z_j = t Z_{i+1} Z_{j-1}, i < j - 1$$

Our task at this point is to 'reverse engineer' an ideal whose generators $G_1, ..., G_m$ satisfy (precisely) these relations. Actually, the choice of G_1 determines $G_2, ..., G_m$ via the linear relations, though a priori, $G_2, ..., G_m$ are only *rational* functions. Now recall that Z_1 generates $\mathcal{O}(1)$ over the open U_1 which meets the special fibre $t = 0$ in the locus of m-tuples entirely on x-axis. On that locus, an equation for the discriminant is given by the Van der Monde determinant:

$$v_x^m = \det(V_x^m),$$

$$V_x^m = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_m \\ \vdots & & \vdots \\ x_1^{m-1} & \cdots & x_m^{m-1} \end{bmatrix}.$$

Thus motivated, set

$$G_1 = v_x^m.$$

This forces

$$G_i = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} v_x^m = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} G_1, \quad i = 2, ..., m.$$

If the construction is to make sense, these better be regular. In fact,

$$G_i = \pm \det(V_i^m),$$

$$V_i^m = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_m \\ \vdots & & \vdots \\ x_1^{m-i} & \cdots & x_m^{m-i} \\ y_1 & \cdots & y_m \\ \vdots & & \vdots \\ y_1^{i-1} & \cdots & y_m^{i-1} \end{bmatrix}$$

(we call this the 'Mixed' Van der Monde matrix). The G_i satisfy same relations as the Z_i, so we can map isomorphically

$$\mathcal{O}(1) \rightarrow J = \text{Ideal}(G_1, ..., G_m)$$

$$Z_i \mapsto G_i.$$

Then J is an invertible ideal defining a Cartier divisor Γ. The Main Theorem's assertion that $\mathfrak{c}$ is the blowup of D^m means

$$J = \mathfrak{c}^*(I_{D^m})$$

i.e.

$$\Gamma = \mathfrak{c}^*(D^m)$$

Containment $\supseteq$ of schemes is clear. Equality is clear off the special fibre $t = 0$. Now this special fibre is sum of components

$$\Theta_I = \mathrm{Zeros}(x_i, i \notin I, y_i, i \in I), I \subseteq \{1, ..., m\}.$$

Set

$$\Theta_i = \bigcup_{|I|=i} \Theta_I.$$

Note that the open set U_i meets only Θ_i, Θ_{i-1}. One can check that the vanishing order of G_i on any $\Theta_I, |I| = k$, is

$$\mathrm{ord}_{\Theta_I}(G_i) = (k - i)^2 + (k - i)$$

$$= 0 \quad \text{if} \quad k = i, i - 1$$

So $\mathrm{Zeros}(G_i) = \mathfrak{c}^*(D^m)$ on U_i, i.e.

$$\Gamma|_{U_i} = \mathfrak{c}^*(D^m)|_{U_i}, \quad \forall i$$

$$\therefore \quad \Gamma = \mathfrak{c}^*(D^m)$$

This concludes the proof of the Main Theorem.

One point of interest is the interpretation of mixed Van der Monde matrices, whose determinants played a large role in the proof : The universal subscheme

$$\Xi = \mathrm{Zeros}(\mathcal{I}) \subset X_B^{[m]} \times X$$

contains sections

$$\Psi_i = \mathrm{graph}(p_i : X_B^{[m]} \to X)$$

The universal quotient

$$\mathfrak{Q}_m = p_{X_B^{[m]}*}(\mathcal{O}/\mathcal{I})$$

maps to $\mathcal{O}_{X_B^{(m)}}$ via restriction on Ψ_i. Assembling together, get map

$$V : \mathfrak{Q}_m \to m\mathcal{O}_{X_B^{[m]}}.$$

Then $V_i^m =$ is just the matrix of V with respect to the basis $1, x, ..., x^{m-i}, y, ..., y^{i-1}$ of $\mathfrak{Q}_m$ on U_i.

A somewhat mysterious point that comes up in the above proof is: as the Z_i are interpreted as the equations of the discriminant, what, if any, is the interpretation of u_i, v_i ?

2. Applications

Canonical bundle. A first application is a formula for the canonical bundle of $X_B^{[m]}$. For any class α on X, denote

$$\alpha^{[m]} = q_* p^*(\alpha)$$

where $\Omega \subset X_B^{[m]} \times_B X$ is universal subscheme and $p : \Omega \to X, q : \Omega \to X_B^{[m]}$ are natural maps. Here q_* denotes the cohomological direct image, sometimes called the norm or denoted $q_!$, *not* the sheaf-theoretic direct image.

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Another way to construct $\alpha^{[m]}$ is as follows. First note the natural isomorphism over $\mathbb{Q}$

$$H^*(X^{(m)}) \simeq \mathrm{Sym}^m(H^*(X))$$

This yields a class $\alpha^m \in H^*(X^{(m)})$, and $\alpha^{[m]}$ is the image of the latter via the composite

(1) $$H^*(X^{(m)}) \to H^*(X_B^{(m)}) \overset{\mathfrak{c}_m^*}{\to} H^*(X_B^{[m]})$$

Also set

$$\mathcal{O}_{X_B^{[m]}}(1) = \mathcal{O}(-\Gamma)$$

(the canonical $\mathcal{O}(1)$ as blowup, via Proj).

Corollary 2.1. $K_{X_B^{[m]}/B} = (K_{X/B})^{[m]} \otimes \mathcal{O}_{X_B^{[m]}}(1)$

Proof. It suffices to note that both sides agree off the exceptional locus of $\mathfrak{c}_m$. $\qquad\square$

In particular, $K_{X_B^{[m]}/B}.C_i^m = +1$, so $\mathfrak{c}_m$ 'looks like' a flipping contraction. The following example partially confirms this.

Example: $m = 2$. We have a diagram

$$\begin{array}{ccc} X_B^{[2]} & \to & X_B^{[2]} \\ \mathfrak{c}_2' \downarrow & & \downarrow \mathfrak{c}_2 \\ X_B^2 & \to & X_B^{(2)} \end{array}$$

with horizontal maps of degree 2. Local equations for X_B^2 are:

$$x_1 y_1 = x_2 y_2 = t$$

(so this is a 3-fold ODP);
for $X_B^{[2]}$ in $X_B^2 \times \mathbb{P}^1$:

$$x_1 u = y_2 v$$
$$x_2 u = y_1 v$$

so $\mathfrak{c}_2'$ is a small resolution of the ODP, known as a flopping contraction; it can be flopped to yield X^{**} smooth that is the source of the 'opposite' flopping contraction.
Local equations for $X_B^{(2)}$ are:

$$\sigma_2^y \sigma_1^x = t\sigma_1^y$$
$$\sigma_2^x \sigma_1^y = t\sigma_1^x$$
$$\sigma_2^x \sigma_2^y = t^2$$

Viewed in $\mathbb{A}^5$, this is a cone over a cubic scroll in $\mathbb{P}^4$. $X_B^{[2]}$ is small resolution of the cone, with exceptional locus $C_1^2 = \mathbb{P}^1$. Equations for $X_B^{[2]}$ in $X_B^{(2)} \times \mathbb{P}^1$ are

$$\sigma_2^x u_1 = t v_1$$
$$\sigma_1^x u_1 = \sigma_1^y v_1$$

$$\sigma_2^y v_1 = t u_1.$$

A well-known procedure, due to Francia, yields a flip, called Francia's flip, of $\mathfrak{c}_2$: blow up C_1^2 in $X_B^{[2]}$ (which is the same as blowing up the vertex of the cone); the exceptional divisor is a scroll of type F_1; then blow up the negative curve of F_1 to get a new exceptional surface of type F_0; then blow down F_0 in the other direction to $C^* = \mathbb{P}^1$ so the F_1 becomes a $\mathbb{P}^2$; then finally blow down $\mathbb{P}^2$ to a (singular) point on a new 3-fold X^*, which is 2:1 covered by X^{**}.

This situation is intriguing in view of recent work of Bridgeland [2] and Abramovich and Chen [1] which shows that the flop X^{**} and the flip X^* can be interpreted as moduli spaces of certain '1-point perverse sheaves' on $X_B^{[2]}$ and $X_B^{[2]}$, respectively. This raises the question of finding a natural interpretation of X^*, X^{**} and their higher-order analogues, if they exist, in terms of our family of curves X/B.

Euler number. As an application of our study of $\mathfrak{c}_m$, we can compute (topological) Euler number $e(X_B^{[m]}) = c_{m+1}(T_{X_B^{[m]}})$, at least for case of ≤ 1 node in any fibre:

Corollary 2.2. *If X/B has σ singular fibres and each has precisely 1 node, then the topological Euler number of $X_B^{[m]}$ is given by*

$$(2) \qquad e(X_B^{[m]}) = (-1)^m \binom{2g-2}{m} (2 - 2g(B)) + \sigma \binom{m - 2g + 2}{m - 1}$$

Proof. Let

$$(X_i, p_i, X_{i,0} = X_i \setminus p_i), i = 1, ..., \sigma$$

be the singular fibres with their respective unique singular point and smooth part, and

$$X_0 = X \setminus (X_1 \cup \ldots \cup X_\sigma), B_0 = \pi(X_0).$$

Then $X_B^{(m)}$ admits a (locally closed) stratification with big stratum

$$(X_0)_{B_0}^{(m)}$$

and other strata

$$\Sigma_{i,j} = i p_j + (X_{j,0})^{(m-i)}, i = 0, ..., m, j = 1, ..., \sigma.$$

The fibre of $\mathfrak{c}_m$ over each of these strata is, respectively, a point over the big stratum, and over the $\Sigma_{i,j}$, a point for $i = 0, 1$, a chain of $(i - 1)$ $\mathbb{P}^1$ s for $i = 2, ..., m$. Since the Euler number is multiplicative in fibrations and additive over strata, we get

$$e(X_B^{[m]}) = e((X_0)_{B_0}^{(m)}) + \sum e((X_{j,0})^{(m)})$$
$$+ \sum_{i>0} i e((X_{j,0})^{(m-i)})$$

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Now MacDonald's formula [5] says that for any X, the Euler number of its mth symmetric product is given by

$$e(X^{(m)}) = (-1)^m \binom{-e(X)}{m}.$$

Plugging this into the above and using multiplicativity for the fibration $(X_0)_{B_0}^{(m)}$ over B_0 yields

$$(3) \quad e(X_B^{[m]}) = (-1)^m \binom{2g-2}{m}(2 - 2g(B)) + \sigma \sum_{k=0}^{m-1}(-1)^k(m-k)\binom{2g-2}{k}$$

Now, as pointed out by L.C. Wang, (2) follows from (3) by the elementary formula

$$\sum_{k=0}^{b}(-1)^k \binom{a}{k} = (-1)^b \binom{a-1}{b}$$

which in turn is an easy consequence of Pascal's relation $\binom{a}{k} = \binom{a-1}{k} + \binom{a-1}{k-1}$. $\qquad\square$

Remark 2.3. Suppose our family X/B is a blowup

$$\beta : X \to Y$$

of a *smooth* $\mathbb{P}^1$ bundle; equivalently, each singular fibre of X/B has consists of two $\mathbb{P}^1$ components. Then there is another way to construct $X_B^{[m]}$ and obtain formula (3) above, as follows. Note that the natural map

$$\eta : Y_B^{[m]} = Y_B^{(m)} \to B$$

is a $\mathbb{P}^m$-bundle. Blow up a $\mathbb{P}^{m-1}$ in each fibre of η over a singular value of π, giving rise to exceptional divisors $E_{1,i}, i = 1, ..., \sigma$; then blow up a $\mathbb{P}^{m-2}$ in general position in each exceptional divisor $E_{1,i}$, giving rise to new exceptional divisors $E_{2,i}$, etc. Finally, blow up general point on each exceptional divisor $E_{m-1,i}$. This yields $X_B^{[m]}$. In these blowups, the change in Euler number is easy to analyze, yielding (2). $\qquad\square$

Further developments (under construction). We mention some natural questions and possible extensions.

- What is the total Chern class $c(T_{X_B^{[m]}})$?

- Develop intersection calculus for diagonal loci of all codimensions in $X_B^{[m]}$, i.e. degeneracy loci

$$\Gamma_r^m = \mathrm{rk}(V_i^m) \le m + 1 - r$$

(locus where r points come together)
More generally, loci $\Gamma_{(m.)}^m$, $m_1 + ... + m_k = m$,

$$\Gamma_{(m.)}^m = \{z : \mathfrak{c}_m(z) = \sum m_i p_i\}.$$

In particular, the small diagonal

$$\Gamma^m_{(m)} = \text{locus of length-}m\text{ schemes supported at 1 point}$$

which coincides with the blowup of X, locally at each fibre node, in a punctual subscheme of type

$$(x^{\binom{m}{2}}, ..., x^{\binom{m-i}{2}}y^{\binom{i}{2}}, ..., y^{\binom{m}{2}})$$

A potential application of this calculus is is to enumerative geometry (multiple points, multisecant spaces, special divisors on stable curves...)

A *Sample corollary* which however can also be derived by other means) is the following **relative triple point formula:** for a map $f : X \to \mathbb{P}^2$, the number of relative triple points is

$$N_{3,X}(f) = (\frac{(d-2)(d-4)}{2} + g - 1)L^2 + (3 - \frac{d}{2})\omega L + 2\omega^2 - 4\sigma$$

where $L = f^*\mathcal{O}(1), L^2 = \deg(f), d = \deg(\text{fibre}), g = \text{genus(fibre)}$.
See[9] for some progress on this.

   • If X/B is of *compact type* (assume for simplicity there exists a section), we have an Abel-Jacobi morphism to the Jacobian:

$$X^{[m]}_B \to J(X/B)$$

Fibres give a notion of 'generalized linear system' on reducible fibres. How is this related to other approaches to such notions in the literature ?

3. Chern classes of tautological bundles

In [7] we gave a simple formula for the Chern classes of the tautological bundles $\lambda_m(L)$, where L is a vector bundle on X. Here X need not be a surface; we just need a family of nodal curves X/B. More precisely, we gave in [7] a formula for the pullback of $\lambda_m(L)$ on the (full) flag relative Hilbert scheme, denoted $W^m(X/B)$. The formula is simple and involves only divisor classes plus classes coming from X, but has the disadvantage that these classes, unlike $\lambda_m(L)$ itself, do not descend to the Hilbert scheme $X^{[m]}_B$. Though it is, broadly speaking, obvious that a formula on $X^{[m]}_B$ can be derived from the one on $W^m(X/B)$, it is still of some interest, in view of possible applications, to work this out. It turns out that for X a surface, a formula for the Chern classes of tautological bundles was already derived, in the context of the (absolute) Hilbert scheme $X^{[m]}$, by Lehn [4], using the Fock space formalism introduced earlier by Nakajima [6, 3]. Since our tautological bundles $\lambda_m(L)$ are pullbacks of the analogous bundles on $X^{[m]}$ via the natural inclusion

$$X^{[m]}_B \subset X^{[m]},$$

Lehn's formula yields an analogous one on $X^{[m]}_B$. Our purpose here, then, is to verify that when X is a surface, the push-down from $W^m(X/B)$ to $X^{[m]}_B$

of the formula of [7] coincides with the restriction of Lehn's formula, at least when L is a line bundle. Thus, we have compatibility in the natural diagram

$$W^m(X/B)$$
$$w_m \downarrow$$
$$X_B^{[m]} \quad \rightarrow \quad X^{[m]}$$

We begin with some formalism. First, we have the operation of *exterior multiplication* $\star$ of cohomology classes on various $X_B^{[m]}$, defined as follows. Let

$$Z_{m,n} \subset X_B^{[m]} \times_B X_B^{[n]} \times_B X_B^{[m+n]}$$

be the closure of the locus

$$\{(z_m, z_n, z_m \coprod z_n) : z_m \cap z_n = \emptyset\},$$

and let

$$p : Z_{m,n} \rightarrow X_B^{[m]} \times_B X_B^{[n]}, q : Z_{m,n} \rightarrow X_B^{[m+n]}$$

be the projections, both generically finite. For $\alpha \in H^r(X_B^{[m]}), \beta \in H^s(X_B^{[n]})$, identifying homology and cohomology, set

$$\alpha \star \beta = q_* p^*(\alpha \times \beta) \in H^{r+s}(X_B^{[m+n]}).$$

This operation is obviously associative and commutative on even (in particular, algebraic) classes. In particular, taking $\beta = 1$, we get a natural way of mapping $H^r(X_B^{[m]})$ to $H^r(X_B^{[m+n]})$ for each $n \geq 0$.

Next, consider the small diagonal

$$\Gamma_{(m)}^m \overset{i_m}{\hookrightarrow} X_B^{[m]}.$$

The restriction of the cycle map yields a birational morphism

$$\beta_m : \Gamma_{(m)}^m \rightarrow X.$$

For any $\alpha \in H^r(X)$, we set

$$q_m[\alpha] = i_{m*}(\beta_m^*(\alpha)) \in H^{r+2m-2}(X_B^{[m]}).$$

Via $\star$ multiplication, $q_m[\alpha]$ may be viewed as with an operator on

$$\bigoplus_{s,n=0}^{\infty} H^s(X_B^{[n]})$$

which has operator bidegree $(r + 2m - 2, m)$. This is known as Nakajima's creation operator (cf. [6, 3]).

Lehn's formula is as follows

Theorem. *(Lehn [4]) For a line bundle L, the total Chern class of $\lambda_m(L)$ is the part in bidegrees $(*, m)$ of*

$$\exp(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q_n[c(L)]).$$

Now our formula is the following

Theorem 3.1. *For a line bundle L, we have*
$c(\lambda_m(L)) =$
$$\sum_{\substack{I = (1 \leq i_1 < ... < i_k) \\ |I| \leq m}} (-1)^{|I|-k} \frac{(i_1 - 1)!...(i_k - 1)!}{|I|!(m - |I|)!} q_{i_1}[c(L)]...q_{i_k}[c(L)].$$

[The multiplication involved in Theorem 3.1, as well as in Lehn's Theorem, is ordinary cup product.] It is elementary to derive Theorem 3.1 from Lehn's theorem (whose proof is rather long). Our purpose here, however, is to derive Theorem 2 from a result in [7], as follows. Let

$$w_m : W = W^m(X/B) \to X_B^{[m]}$$

be the natural morphism from the flag Hilbert scheme to the ordinary one, let

$$p_i : W \to X$$

be the ith projection, mapping a filtered scheme $z_1 < ... < z_m$ to the support of z_i/z_{i-1}, and let

$$\Delta_{ij} \subset W, i < j$$

denote the (reduced) locus where the p_i and p_j coincide; also set, for any class $c \in H^*(X)$,

$$c_i = p_i^*(c).$$

It is shown in [7] that each sum $\sum_{i=1}^{j-1} \Delta_{ij}$ is a Cartier divisor (even though W is in general singular and each summand individually is not Cartier). It is also shown there that the following result holds (for a line bundle L):

(4)
$$c(w^*\lambda_m(L)) = \prod_{j=1}^{m}(1 + L_j - \sum_{i=1}^{j-1}\Delta_{ij})$$

Deriving Theorem 3.1 from (4) is a matter of expanding the product as a sum of monomials, applying w_* and dividing by $m! = \deg(w)$. In doing so, it is useful to observe the following. Let's call a *connected* monomial on an index set I one which, after a permutation, can be written in the form

$$q_I[c] = c_{i_1}\Delta_{i_1 i_2}\Delta_{i_2 i_3}...\Delta_{i_{k-1} i_k}, I = (i_1 < ... < i_k)$$

where c is either 1 or $[L]$. The intersection implicit in the above product is transverse, hence well-defined even though the divisors are not Cartier. It is easy to see by induction that there are $(k - 1)!$ unordered monomials in the expansion of (4) yielding the same $q_I[c]$. Moreover it is clear that

$$w_*(q_I[c]) = q_{|I|}[c].$$

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Now we note that each monomial appearing in the expansion of (4) may be decomposed uniquely as a product of connected monomials on pairwise disjoint index sets (its 'connected components'), yielding a term

$$(-1)^{\sum\limits_{j=1}^{k}(|I_j|-1)} q_{I_1}[c_1]...q_{I_k}[c_k],$$

with each c_j equal to either 1 or $[L]$; for fixed $I = I_1 \coprod ... \coprod I_k$, this term appears $(|I_1| - 1)!...(|I_k| - 1)!$ times. Applying w_*, we get, for each choice of $I \subseteq \{1,...,m\}$ and c_j, a term in w_* applied to (4):

$$(i_1 - 1)!...(i_k - 1)!(-1)^{i-k} q_{i_1}[c_1]...q_{i_k}[c_k],$$

$i_j = |I_j|, i = \sum\limits_j i_j$. Then multiplying by $\binom{m}{i}$ for the choice of subset I with $|I| = i$, and dividing by $m!$ yields the result. $\square$

REFERENCES

[1] D. Abramovich, J.C. Chen, Flops, flips and perverse point sheaves on threefold stacks (math.AG/0304354).

[2] T. Bridgeland, Flops and derived categories, Invent. math 147 (2002), 613-632 (math.AG/0009053).

[3] G. Ellingsrud, L. Göttsche, Hilbert schemes of points on surfaces and Heisenberg algebras, ICTP lectures, 1999.

[4] M. Lehn, Chern classes of tautological sheaves on Hilbert schemes of points on surfaces Invent. math 136,pp. 157-207, 1999.

[5] I.G. Macdonald, The Poincaré polynomial of a symmetric product, Proc. Camb. Phil. Soc. 58 (1962), 563-568.

[6] H. Nakajima, Heisenberg algebra and Hilbert schemes of points on projective surfaces, Ann. Math. 145 (1997), 379-388.

[7] Z. Ran, Geometry on nodal curves (math.AG/0210209; appear in Compositio Math. (2005)).

[8] ____ , A note on Hilbert schemes of nodal curves (arXiv.org/math.AG/0410037; to appear in J. Alg.)

[9] ____ , Geometry on nodal curves II: cycle map and intersection calculus (preprint available at http://math.ucr.edu/~ ziv/papers/cyclemap.pdf or at arXiv.org/math.AG/0410120)

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