STRUCTURE OF THE CYCLE MAP FOR HILBERT SCHEMES OF FAMILIES OF NODAL CURVES

ZIV RAN

ABSTRACT. We study the relative Hilbert scheme of a family of nodal (or smooth) curves, over a base of arbitrary dimension, via its (birational) *cycle map*, going to the relative symmetric product. We show the cycle map is the blowing up of the discriminant locus, which consists of cycles with multiple points. We determine the structure of certain projective bundles called *node scrolls* which play an important role in the geometry of Hilbert schemes.

CONTENTS

Date: August 23, 2015.
1991 *Mathematics Subject Classification.* 14N99, 14H99.
Key words and phrases. Hilbert scheme, cycle map.
http://arxiv.org/0903.3693.

INTRODUCTION

In the classical (pre-1980) theory of (smooth) algebraic curves, a dominant role is played by divisors– equivalently, finite subschemes– and their parameter spaces, i.e. symmetric products. Notably, one of the first proofs of the existence of special divisors [6] was based on intersection theory on symmetric products, developed earlier by Macdonald [10]. In more recent developments however, where the focus has been on moduli spaces of stable curves, subscheme methods have been largely absent, replaced by tools related to stable maps and their moduli spaces (see [18] for a sampling stressing Vakil's work). Our purpose in this paper (and others in this series) is to develop and apply global subscheme methods suitable for the study of stable curves and their families, aiming eventually, inter alia, to extend Macdonald's theory to the case of families of curves with at most nodal singularities.

In this paper we always work over the complex numbers. Fix a family of curves given by a flat projective morphism

$$\pi : X \to B$$

over an irreducible base, with fibres

$$X_b = \pi^{-1}(b), b \in B$$

which are nonsingular for the generic b and at worst nodal for every b. For example, X could be the universal family of automorphism-free curves over the appropriate open subset of $\overline{\mathcal{M}}_g$, the moduli space of Deligne-Mumford stable curves. Consider the *relative Hilbert scheme*

$$X_B^{[m]} = \mathrm{Hilb}_m(X/B),$$

which parametrizes length-m subschemes of X contained in fibres of π. This comes endowed with a *cycle map* (also called 'Hilb-to-Chow'– in this case, 'Hilb-to-Sym'– map) to the relative symmetric product

$$c_m : X_B^{[m]} \to X_B^{(m)}.$$

See §1 for a review. Because $X_B^{(m)}$ may be considered 'elementary' (though it's highly singular- see [14]), c_m is a natural tool for studying $X_B^{[m]}$. The structure of c_m is the object of this paper. Our first main result is the following theorem which was announced with a sketch of proof in [13], where some applications are given as well.

Blowup Theorem. *c_m is equivalent to the blowing up of the discriminant locus*

$$D^m \subset X_B^{(m)},$$

which is the Weil divisor parametrizing nonreduced cycles.

In particular, we obtain an effective Cartier divisor

$$2\Gamma^{(m)} = c_m^{-1}(D^m)$$

so that $-2\Gamma^{(m)}$ can be identified with the natural $\mathcal{O}(1)$ polarization of the blowup. In fact, we shall see that $\Gamma^{(m)}$ also exists as a Cartier divisor, not necessarily effective, and the dual of the associated line bundle, i.e. $\mathcal{O}(-\Gamma^{(m)})$, will be (abusively) called the *discriminant polarization* (though 'half discriminant' is more accurate); we will also refer to $\Gamma^{(m)}$ itself sometimes the discriminant polarization. We emphasize that the Blowup Theorem is valid without dimension restrictions on B. As suggested by the Theorem, the discriminant polarization encodes the additional information in Hilb vis-a-vis the unwieldy Sym and so, unsurprisingly, plays a central role in subsequent developments of geometry and intersection theory on the Hilbert schemes $X_B^{[m]}$.

The proof of the Blowup Theorem occupies §§2-4 and may be outlined as follows.

(i) A preliminary reduction is made to the local case (§2);

(iI) we construct an explicit local model H for the relative Hilbert scheme (§3);

(ii) we construct an ideal G in the relative Cartesian product, whose *syzygies* correspond, essentially, to the defining equations of the pullback OH of H over the Cartesian product; this yields a map γ from the blowup of G to OH (§4);

(iv) using the local analysis, it is shown that γ is an isomorphism and that G is the ideal of the ordered discriminant (big diagonal);

(v) consequently γ descends to an isomorphism from the blowup of the ideal of the discriminant to H.

The usefulness of the local model H extends far beyond the Blowup Theorem; in particular, it yields information about the *singularity stratification*, of $X_B^{[m]}$, which may be defined as follows. Let $\theta_1, ..., \theta_r$ be a collection of distinct, hence disjoint, relative nodes of the family, each living in the total space over its own boundary component, and let $n_1, ..., n_r$ be integers. Set

$$\mathcal{S}^{m,n.}(\theta_.; X/B) = \{z : c_m(z) \geq \sum n_i\theta_i\} \subset X_B^{[m]}$$

This is mainly interesting when all $n_i \geq 2$. In this case, we construct a surjection

$$\bigcup_{1\leq j_i\leq n_i-1, \forall 1\leq i\leq r} F_{j.}^{m,n.}(\theta.; X/B) \twoheadrightarrow S^{m,n.}(\theta_.; X/B)$$

where each $F_{j.}^{m,n.}(\theta.; X/B)$, called a *node polyscroll* (or node scroll, when $r = 1$), is a $(\mathbb{P}^1)^r$-bundle over the smaller Hilbert scheme $(X^{\theta.})^{[m-\sum n_i]}$, where $X^{\theta.}$ denotes the blowup (=partial normalization) of X in $\theta_1, ..., \theta_r$, defined over the intersection of the boundary components corresponding to the θ_i. The fibre parameter of i-th factor of the node polyscroll encodes a sort of higher-order (more precisely, $(n_i - 1)$-st order) 'slope', locally at the i-th node, and these together constitute the additional information contained in the Hilbert scheme beyond what's in the symmetric product.

In the following section §6 we give an analogue of the blowup theorem in the case of flag-Hilbert schemes, which are often important in inductive arguments and procedures.

Our next main results (see Theorems 9.3, 9.5) determine the structure of node polyscrolls as $(\mathbb{P}^1)^r$-bundles. In fact, the disjointness of the nodes (in the total space) implies that the $\mathbb{P}^1$ factors 'vary independently', which allows us to reduce to the case of node scrolls, i.e. $r = 1$.

Actually, what's essential for the enumerative theory of the Hilbert scheme, as studied e.g. in [16], and in which node scrolls play an essential role, is the structure of the node scroll F as a *polarized* $\mathbb{P}^1$ bundle, that is, the rank-2 vector bundles E so that there is an isomorphism $\mathbb{P}(E) \simeq F$, under which the canonical $\mathcal{O}(1)$ polarization on $\mathbb{P}(E)$ associated to the projectivization corresponds to the restriction of the discriminant polarization $-\Gamma^{(m)}$ on F. To state the result (approximately), denote by θ_x, θ_y the node preimages on X^θ, and by ψ_x, ψ_y the relative cotangent spaces to X^θ/T along them, and by $[m-n]_*D$, for any divisor D on X^θ, the 'norm' of D, considered as a divisor on $(X^\theta)^{[m-n]}$.

Node Scroll Theorem. *There is a polarized isomorphism*

$$F_j^{m,n}(\theta) = \mathbb{P}(\mathcal{O}(-D_j^n(\theta)) \oplus \mathcal{O}(-D_{j+1}^n(\theta)))$$

where

$$D_j^n(\theta) = -\binom{n-j+1}{2}\psi_x - \binom{j}{2}\psi_y + (n-j+1)[m-n]_*\theta_x + j[m-n]_*\theta_y + \Gamma^{[m-n]}.$$

This result, and its polyscroll analogue, reduce intersection theory on polyscrolls to that of the Mumford tautological classes, about which a great deal is now known thanks to the work of Witten, Kontsevich, Faber and many others (see e.g. [18] and references therein). The Node Scroll Theorem is one of the main ingredients of a complete 'Hilbert-tautological' intersection calculus, developed in [16], which allows us to extend the intersection theory and enumerative geometry of a single smooth curve, as developed notably by Macdonald [10] and presented in [2], to the case of families of curves with at most nodal singularities, extending work of Cotteril [3] in low degrees. As described in [16], this intersection calculus has now been implemented on the computer, in the form of a Java program called macnodal [9], due to Gwoho Liu and available from the author's web page. See also [12] for an application to the class of the closure of the hyperelliptic class in $\overline{\mathcal{M}}_g$.

Acknowledgments I thank Mirel Caibar for asking some stimulating questions early on, and the referee for many helpful, detailed comments and suggestions.

Convention In this paper we always work over $\mathbb{C}$.

Part 1. Blowup theorem and discriminant polarization

1. Review of cycle map

See [1], [8] or [17] for more informatrion.

1.1. **Norms and multisections.** Let $Z = \mathrm{Spec}_T(A) \to T$ be a finite, flat, degree-m morphism of algebraic $\mathbb{C}$-schemes, corresponding to a sheaf of T- algebras A that is locally T-free of rank m. The action of the algebra $\mathrm{Sym}^m_T A$ on the invertible T-module $\bigwedge^m_T(A)$ yields a T-homomorphism of algebras

$$\mathrm{Sym}^m_T(A) \to \mathcal{O}_T = \mathrm{End}_T(\bigwedge^m_T(A)).$$

This is a symmetric-tensor version of the norm map, usually given as a homogeneous polynomial; it can be written locally it terms of determinants. Applying Spec, we get a T-map, called the *canonical multisection* of Z/T,

$$\sigma_{Z/T} : T \to Z_T^{(m)} = \mathrm{Spec}_T(\mathrm{Sym}_T^m(A)).$$

This map is obviously compatible with base-change and satisfies a 'locality' property, namely if $Z = \coprod Z_i$ with each Z_i flat of degree m_i, ,then $\sigma_{Z/T}$ factors through

$$\prod \sigma_{Z_i/T} : T \to \prod Z_T^{(m_i)}.$$

Consequently, if $t \in T$ and the fibre $Z(t) = \coprod Z_i(t)$ and each Z_i is supported at a unique point p_i, then $\sigma_{Z_i/T}(t)$ is the unique point of $(Z_i)_T^{(m_i)}$, usually denoted $m_i p_i$, and $\sigma_{Z/T}(t) = \sum m_i p_i$.

1.2. **Cycle map.** Let $X \to B$ be a quasi-projective morphism, $T \to B$ a morphism and Z a T-valued point of the relative Hilbert scheme $X_B^{[m]}$, i.e. a closed subscheme of $X \times_B T$ that is finite flat of degree m over T. Examples of possible T include the Hilbert scheme $X_B^{[m]}$ itself and any scheme mapping to it. We have the canonical multisection, which is a T-morphism

$$\sigma_{Z/T} : T \to Z_T^{(m)} \subset (X \times_B T)_T^{(m)} = X_B^{(m)} \times_B T.$$

Composing with the projection, we get the cycle map, a B-morphism

$$c_Z : T \to X_B^{(m)}.$$

Again, this is compatible with base-change $B' \to B$ and has a locality property. Moreover, it depends only on Z quasi-intrinsically in the sense that if $Y \subset X$ is any locally closed subscheme such that $Y \times_B T$ contains Z scheme-theoretically, then c_Z factors through $Y_B^{(m)}$. Also, there is an analogous and compatible construction in the analytic category.

2. Blowup Theorem: Set-up and preliminary reductions

2.1. **Set-up.** Let

$$\pi : X \to B$$

be a flat family of nodal, generically smooth curves with X, B reduced and irreducible. Let $X_B^m, X_B^{(m)}$, respectively, denote the mth Cartesian and symmetric fibre products of X relative to B. Thus, there is a natural map

$$\omega_m : X_B^m \to X_B^{(m)}$$

which realizes its target as the quotient of its source under the permutation action of the symmetric group $\mathfrak{S}_n$. Let

$$\mathrm{Hilb}_m(X/B) = X_B^{[m]}$$

denote the relative Hilbert scheme parametrizing length-m subschemes of fibres of π, and

$$c = c_m : X_B^{[m]} \to X_B^{(m)}$$

the natural cycle map constructed above, associated to the universal subscheme $Z \subset X_B^{[m]} \times_B X$. Let $D^m \subset X_B^{(m)}$ denote the discriminant locus or 'big diagonal', consisting of cycles supported on $< m$ points (endowed with the reduced scheme structure). Clearly, D^m is a prime Weil divisor on $X_B^{(m)}$, birational to $X \times_B X_B^{(m-2)}$ (though it is less clear what the defining equations of D^m on $X_B^{(m)}$ are near singular points). The main result of Sections 1-4 is the

Theorem 2.1 (Blowup Theorem). *The cycle map*

$$c_m : X_B^{[m]} \to X_B^{(m)}$$

is equivalent to the blowing up of $D^m \subset X_B^{(m)}$.

The proof presented here is an elaboration of the one sketched in [13].

2.2. **Reductions.** We begin with some preliminary remarks and reductions. To begin with, recall that the cycle map is compatible with base-change, as was observed in §1, and note now that the same is true of the blowup of D^m: indeed given a base-change $X_{B'} = X \times_B B'$, we have $I_{D^m(X_{B'}/B')} = I_{D^m} \otimes \mathcal{O}_{B'}$, hence also $I^n_{D^m(X_{B'}/B')} = I^n_{D^m} \otimes \mathcal{O}_{B'}$, so

$$\bigoplus_n I^n_{D^m(X_{B'}/B')} = (\bigoplus_n I_{D^m}) \otimes \mathcal{O}_{B'},$$

and applying Proj we get

$$\mathcal{B}\ell_{D^m}(X_B^{(m)}) \times_B B' = \mathcal{B}\ell_{D^m(X_{B'}/B')}(X_{B'})^{(m)}_{B'}.$$

Because the Theorem is local over B and locally any family is a base-change from a versal one, we may as well assume X/B is a versal deformation of a nodal curve X_0, and in particular X and B are smooth.

Next, the Theorem is the statement that the natural birational correspondence between $X_B^{[m]}$ and $\mathcal{B}\ell_{D^m}(X_B^{(m)})$ projects isomorphically both ways (in particular $X_B^{[m]}$ is irreducible). By GAGA, it suffices to prove for the corresponding analytic spaces. Then, since the statement is local over $X_B^{(m)}$, we may work over a neighborhood of a given cycle $Z = \sum_{i=1}^{k} m_i p_i$, of the form $\prod_B (U_i)_B^{(m_i)}$ where U_i is a suitable analytic neighborhood of p_i. The corresponding open subset of $X_B^{[m]}$ is just $\prod_B (U_i^{[m_i]})_B$, where for an analytic open $U \subset X$, $U_B^{[m]} \subset X_B^{[m]}$ is the set of schemes contained in U. We note that this depends only on U/B up to analytic isomorphism: e.g. because it can be identified with a Douady space of finite subspaces of U; or more directly, by GAGA, there is a natural correspondence between analytic families of finite subschemes X/B contained in U and finite analytic subspaces of U/B. Now choosing U_i appropriately, we may assume there is an open subset $V \subset \mathbb{C}^2$ such that U_i/B is a base-change of the family V/T given by $xy = t$ (the 'standard model').

Now suppose we could show that $V_T^{[m]}, \forall m$, is the blowup of $V_T^{(m)}$ in D^m. Then the same is true for $(U_i)_B^{[m_i]}, \forall i$. To conclude that $\prod_B (U_i)_B^{[m_i]} \simeq \mathcal{B}\ell_{D^m} \prod_B (U_i)_B^{(m_i)}$, it would suffice to show that

$$\mathcal{B}\ell_{D^m} \prod_B (U_i)_B^{(m_i)} \simeq \prod_B (\mathcal{B}\ell_{D^m} (U_i)_B^{(m_i)})$$

or equivalently,

$$\mathcal{B}\ell_{D^m} \prod_B (U_i)_B^{(m_i)} \simeq \prod_{\prod_B (U_i)_B^{(m_i)}} (\mathcal{B}\ell_{D^m} (U_i)_B^{(m_i)}) \tag{2.1}$$

(2.1) holds because:

(i) The local analysis of the next two sections will show, in particular, that $\mathcal{B}\ell_{D^m} V_T^{(m)}$ is a small blowup, centered over the locus of schemes with multiplicity ≥ 2 at the node, therefore so is $\mathcal{B}\ell_{D^m}(U_i)_B^{(m_i)}$.

(ii)

$$I_{D^m(\coprod U_i/B)}|_{\prod_B (U_i)_B^{(m_i)}} = \prod I_{D^{m_i}(U_i/B)}.$$

(iii) The blowup centers are transverse for different i.

(iv) The following general remark.

Remark 2.2. Let $I_1, ..., I_k$ be an arbitrary collection of ideals on a variety X, not necessarily mutually transverse or even distinct.

(i) The blowup $\mathcal{B}\ell_{I_1 \dots I_k} X$ of the product ideal is the unique X-dominating component of the fibre product $\mathcal{B}\ell_{I_1} X \times_X ... \times_X \mathcal{B}\ell_{I_k} X$. For simplicity we check this for $k = 2$. We may work locally over X. If f_i, g_i are generators for I_1, I_2 respectively, then the blowup of $I_1 I_2$ is covered by open affines $U_{i,j}$ whose coordinate rings are generated over X by symbols $[f_{i'} g_{j'}/f_i g_j]$ satisfying the obvious relations $f_i g_j [f_{i'} g_{j'}/f_i g_j] = f_{i'} g_{j'}$, $\forall i', j'$. Similarly with open affines V_i^1, V_j^2 for the blowup of I_1, I_2, with generators $[f_{i'}/f_i], [g_{j'}/g_j]$ are regular. There are obvious maps $U_{i,j} \rightleftarrows V_i^1 \times_X V_j^2$, defined by $[f_{i'} g_{j'}/f_i g_j] \rightleftarrows [f_{i'}/f_i] \otimes [g_{j'}/g_j]$, leading to maps over X

$$\mathcal{B}\ell_{I_1 I_2} X \rightleftarrows \mathcal{B}\ell_{I_1} X \times_X \mathcal{B}\ell_{I_2} X.$$

These clearly give an isomorphism as claimed.

Note that the foregoing argument makes no assumption regarding transversality of I_1, I_2. In general, if I_1, I_2 are not transverse, e.g. $I_1 = I_2 = I$, then $\mathcal{B}\ell_{I_1} X \times_X \mathcal{B}\ell_{I_2} X$ is reducible: e.g. $[f_1/f_2][f_2/f_1] - 1$ is a zero-divisor (usually nonzero) on $\mathcal{B}\ell_I X \times_X \mathcal{B}\ell_I X$. The dominating component of $\mathcal{B}\ell_I X \times_X \mathcal{B}\ell_I X$ is $\mathcal{B}\ell_{I^2} X \simeq \mathcal{B}\ell_I X$.

(ii) In the above situation, if the $\mathcal{B}\ell_{I_i} X$ are small blowups, i.e. for each i the exceptional locus on X (the center), i.e. the non-invertible locus of I_i, is of codimension ≥ 3 and its inverse image is of codimension ≥ 2, and if for different i the centers are mutually transverse, then the fibre product is in fact irreducible, i.e. has no non-dominating components. This is because any non-dominating component would have to be of smaller dimension, whereas by semi-continuity, in the fibre product, which is the inverse image of the small diagonal in X^k by the natural map

$$\prod \mathcal{B}\ell_{I_i} X \to X^k,$$

every component is of dimension $\geq \dim(X)$.

We have now reduced the Theorem to the case where X/B is the standard family $xy = t$, which we assume till further notice; we also let U denote any neighborhood of the origin in X.

3. A LOCAL MODEL

We now give an explicit construction in coordinates of the relative Hilbert scheme of the standard family. This construction will have many applications beyond the proof of the Blowup Theorem. We begin with some preliminaries.

3.1. **Symmetric product.** Assuming U/B has the local form $xy = t$, the relative Cartesian product U^m_B, as a subscheme of $U^m \times B$, is given locally by

$$x_1 y_1 = ... = x_m y_m = t.$$

Let $\sigma^x_i, \sigma^y_i, i = 0, ..., m$ denote the elementary symmetric functions in $x_1, ..., x_m$ and in $y_1, ..., y_m$, respectively, where we set $\sigma_0 = 1$. We note that these functions satisfy the relations

$$\sigma^y_m \sigma^x_j = t^j \sigma^y_{m-j}, \quad \sigma^x_m \sigma^y_j = t^j \sigma^x_{m-j}, \tag{3.1}$$

$$t^{m-i} \sigma^y_{m-j} = t^{m-i-j} \sigma^x_j \sigma^y_m, \quad t^{m-i} \sigma^x_{m-j} = t^{m-i-j} \sigma^y_j \sigma^x_m \tag{3.2}$$

(of course the relations in second set follow from those of the first). Putting the sigma functions together with the projection to B, we get a map

$$\sigma : U^{(m)}_B = \mathrm{Sym}^m(U/B) \to \mathbb{A}^{2m}_B = \mathbb{A}^{2m} \times B$$

$$\sigma = ((-1)^m \sigma^x_m, ..., -\sigma^x_1, (-1)^m \sigma^y_m, ..., -\sigma^y_1, \pi^{(m)})$$

where $\pi^{(m)} : X^{(m)}_B \to B$ is the structure map.

Lemma 3.1. *σ is an embedding locally near mp where $p = (0, 0)$ is the origin in U.*

Proof. It suffices to prove this formally, i.e. to show that $\sigma^x_i, \sigma^y_j, i, j = 1, ..., m$ generate the completion $\hat{\mathfrak{m}}$ of the maximal ideal of mp in $X^{(m)}_B$. To this end it suffices to show that any $\mathfrak{S}_m$-invariant polynomial in the x_i, y_j is a polynomial in the σ^x_i, σ^y_j and t. Let us denote by R the averaging or symmetrization operator with respect to the permutation action of $\mathfrak{S}_m$, i.e.

$$R(f) = \frac{1}{m!} \sum_{g \in \mathfrak{S}_m} g^*(f).$$

Then it suffices to show that the elements $R(x^I y^J)$, where x^I (resp. y^J) range over all monomials in $x_1, ..., x_m$ (resp. $y_1, ..., y_m$) are polynomials in the σ^x_i, σ^y_j and t. Because $x_i y_i = t$, we may assume I, J are disjointly supported in the sense that $I_k > 0 \Rightarrow J_k = 0$. On the other hand, expanding the product $R(x^I)R(y^J)$ we get a sum of monomials $x^{I'} y^{J'}$ times a rational number; those with $I' \cap J' = \emptyset$ add up to $\frac{1}{m!} R(x^I y^J)$, while those with I', J' not disjointly supported are divisible by t. Thus,

$$R(x^I y^J) - m! R(x^I) R(y^J) = tF$$

where F is an $\mathfrak{S}_m$-invariant polynomial in the x_i, y_j of bidegree $(|I| - 1, |J| - 1)$, hence a linear combination of elements of the form $R(x^{I'} y^{J'}), |I'| = |I| - 1, |J'| = |J| - 1$. By induction, F is a polynomial in the σ^x_i, σ^y_j and clearly so is $R(x^I)R(y^J)$. Hence so is $R(x^I y^J)$ and we are done. $\square$

Remark 3.2. It will follow from the Blowup Theorem 2.1 and its proof that the equations (3.1-3.2) actually define the image of σ scheme-theoretically (see Cor. 4.5 below); we won't need this, however.

3.2. **A projective family.** Now we present a construction of our local model $\tilde{H}$. This is motivated by our study in [15] of the relative Hilbert scheme of a node. As we saw there, the fibres of the cycle map are chains consisting of n rational curves where n takes the values from $n = 0$ for the generic fibre (meaning the fibre is a singleton) to $n = m - 1$ for the most special fibre. Therefore, it is reasonable to try to model the cycle map on a standard pencil of rational normal $(m - 1)$-tics specializing to a chain of lines. Further motivation for the construction that follows comes from [14], where an explicit construction is given for the full-flag Hilbert scheme.

Let $C_1, ..., C_{m-1}$ be copies of $\mathbb{P}^1$, with homogenous coordinates u_i, v_i on the i-th copy. Let

$$\tilde{C} \subset C_1 \times ... \times C_{m-1} \times B/B$$

be the subscheme over B defined by

$$v_1 u_2 = t u_1 v_2, ..., v_{m-2} u_{m-1} = t u_{m-2} v_{m-1}. \tag{3.3}$$

This construction is motivated (cf. [14]) by viewing u_i/v_i as a stand-in for y_I/x_{I^c} where $I \subset [1, m]$ is of cardinality i and $x_I = \prod\limits_{a \in I} x_a$ etc; the *ratio* is independent of I for fixed $|I|$. That said, $\tilde{C}$ is in any event a reduced complete intersection of divisors of type

$$(1, 1, 0, ..., 0), (0, 1, 1, 0, ..., 0), ..., (0, ..., 0, 1, 1)$$

(relatively over B) and it is easy to check that the fibre of $\tilde{C}$ over $0 \in B$ is

$$\tilde{C}_0 = \bigcup_{i=1}^{m-1} \tilde{C}_i, \tag{3.4}$$

where

$$\tilde{C}_i = [1, 0] \times ... \times [1, 0] \times C_i \times [0, 1] \times ... \times [0, 1]$$

and that in a neighborhood of the special fibre $\tilde{C}_0$, $\tilde{C}$ is smooth and $\tilde{C}_0$ is its unique singular fibre over B. We may embed $\tilde{C}$ in $\mathbb{P}^{m-1} \times B$, relatively over B using the plurihomogeneous monomials

$$Z_i = u_1 \cdots u_{i-1} v_i \cdots v_{m-1}, i = 1, ..., m. \tag{3.5}$$

These satisfy the relations

$$Z_i Z_j = t^{j-i-1} Z_{i+1} Z_{j-1}, i < j - 1 \tag{3.6}$$

so they embed $\tilde{C}$ as a family of rational normal curves $\tilde{C}_t \subset \mathbb{P}^{m-1}, t \neq 0$ specializing to $\tilde{C}_0$, which is embedded as a nondegenerate, connected chain of $m - 1$ lines.

3.3. **To Hilb.** Next consider an affine space $\mathbb{A}^{2m}$ with coordinates $a_0, ..., a_{m-1}$, $d_0, ..., d_{m-1}$. The a_i, d_j are to play the roles of $\sigma^x_{m-i}, \sigma^y_{m-j}$ respectively (where as we recall u_i/v_i plays that of $y_{m-i+1}...y_m/x_1...x_{m-i}$). With this and the relations (3.1), (3.2) in mind, let $\tilde{H} \subset \tilde{C} \times \mathbb{A}^{2m}$ be the subscheme defined by

$$a_0u_1 = tv_1, d_0v_{m-1} = tu_{m-1} \tag{3.7}$$
$$a_1u_1 = d_{m-1}v_1, ..., a_{m-1}u_{m-1} = d_1v_{m-1}.$$

Note that $\tilde{H}$ comes equipped with a map to B (via the projection to $\tilde{C}$), whence a projection

$$p_{\mathbb{A}^{2m}_B} : \tilde{H} \to A^{2m}_B.$$

Set $L_i = p^*_{C_i}\mathcal{O}(1)$. Then consider the subscheme of $Y = \tilde{H} \times_B U$ defined by the equations

$$F_0 := x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0 \in \Gamma(Y, \mathcal{O}_Y) \tag{3.8}$$

$$F_1 := u_1x^{m-1} + u_1a_{m-1}x^{m-2} + ... + u_1a_2x + u_1a_1 + v_1y \in \Gamma(Y, L_1) \tag{3.9}$$

$$...$$

$$F_i := u_ix^{m-i} + u_ia_{m-1}x^{m-i-1} + ... + u_ia_{i+1}x + u_ia_i + v_id_{m-i+1}y + ... + v_id_{m-1}y^{i-1} + v_iy^i$$
$$\in \Gamma(Y, L_i) \tag{3.10}$$

$$...$$

$$F_m := d_0 + d_1y_1 + ... + d_{m-1}y^{m-1} + y^m \in \Gamma(Y, \mathcal{O}_Y). \tag{3.11}$$

The following statement essentially summarizes results from [15].

Theorem 3.3. (i) *$\tilde{H}$ is smooth and irreducible.*

(ii) *The ideal sheaf $\mathcal{I}$ generated by $F_0, ..., F_m$ defines a subscheme of $\tilde{H} \times_B X$ that is flat of length m over $\tilde{H}$ and flat over X.*

(iii) *The classifying map*

$$\Phi = \Phi_{\mathcal{I}} : \tilde{H} \to \mathit{Hilb}_m(U/B)$$

is an isomorphism and via Φ, the projection $p_{\mathbb{A}^{2m}_B} : \tilde{H} \to \mathbb{A}^{2m}_B$ corresponds to the cycle map.

(iv) *Φ induces an isomorphism*

$$\tilde{C}_0 = (\tilde{C})_0 = p^{-1}_{\mathbb{A}^{2m}_B}(0) \to \mathit{Hilb}^0_m(X_0) = \bigcup_{i=1}^{m-1} C^m_i$$

(cf. [15]) of the fibre of $\tilde{H}$ over $0 \in \mathbb{A}^{2m}_B$ with the punctual Hilbert scheme of the node on the special fibre X_0, in such a way that the point $[u, v] \in \tilde{C}_i \simeq C^m_i \simeq \mathbb{P}^1$ corresponds to

- *the subscheme with ideal $I^m_i(u/v) = (x^{m-i}+(u/v)y^i) \in C^m_i \subset \mathit{Hilb}^0_m(X_0)$ if $uv \neq 0$,*
- *the subscheme $(x^{m+1-i}, y^i) \in C^m_i$ if $[u, v] = [0, 1]$,*
- *the subscheme $(x^{m-i}, y^{i+1}) \in C^m_i$ if $[u, v] = [1, 0]$.*

In particular, $\tilde{C}_i$ corresponds to C^m_i.

(v) *over the standard open $U_i = (Z_i \neq 0) \subset \mathbb{P}^{m-1}$, a co-basis for the universal ideal $\mathcal{I}$ (i.e. a basis for $\mathcal{O}/\mathcal{I}$) is given by*

$$1, ..., x^{m-i}, y, ..., y^{i-1}.$$

(vi) Φ *induces an isomorphism of the special fibre* $\tilde{H}_0$ *of* H *over* B *with Hilb*$_m(X_0)$*, and* $\tilde{H}_0 \subset \tilde{H}$ *is a divisor with global normal crossings* $\bigcup_{i=0}^{m} D_i^m$ *where each* D_i^m *is smooth, birational to* $(x-\mathrm{axis})^{m-i} \times (y-\mathrm{axis})^i$*, and for* $i = 1, ..., m-1$ *has special fibre* C_i^m *under the cycle map* $p_{\mathbb{A}_B^{2m}}$*.*

Proof. Assertions (i), (ii) are clear from the defining equations To prove (iii) and (iv) consider the point $q_i, i = 1, ..., m$, on the special fibre of $\tilde{H}$ over $\mathbb{A}_B^{2m}$ with coordinates

$$v_j = 0,\ \forall j < i; u_j = 0,\ \forall j \geq i.$$

Then q_i has an affine neighborhood U_i in $\tilde{H}$ defined by

$$U_i = \{u_j = 1,\ \forall j < i;\ v_j = 1,\ \forall j \geq i\}, \tag{3.12}$$

and these $U_i, i = 1, ..., m$ cover a neighborhood of the special fibre of $\tilde{H}$. Now the generators F_i admit the following relations:

$$u_{i-1}F_j = u_j x^{i-1-j} F_{i-1},\ 0 \leq j < i-1;\ v_i F_j = v_j y^{j-i} F_i,\ m \geq j > i$$

where we set $u_i = v_i = 1$ for $i = 0, m$. Hence $\mathcal{I}$ is generated on U_i by F_{i-1}, F_i and assertions (iii), (iv) follow directly from Theorems 1,2 and 3 of [15] .

As for (v), it follows immediately from the definition of the F_i, plus the fact just noted that, over U_i, the ideal $\mathcal{I}$ is generated by F_{i-1}, F_i, and that on U_i, we can set $u_{i-1} = v_i = 1$. Finally (vi) is contained in [15], Thm. 2.

□

At this point it's worth noting the following consequences of Theorem 3.3, (i). First, recall that a deformation X/B of a nodal curve X_0 is said to be *locally versal* (or *locally versal at the nodes*) if the natural map of B to the product of local deformation spaces is smooth.

Corollary 3.4. *Let* X/B *be a family of nodal or smooth curves.*

(i) $X_B^{[m]}/B$ *is a normal crossings morphism, i.e. fibres have normal crossings.*

(ii) If X/B *is locally versal at the nodes, then* $X_B^{[m]}$ *and the universal subscheme over* $X_B^{[m]}$ *are smooth.*

(iii) If X *is irreducible then so is* $X_B^{[m]}$

Remark. *In (ii), the smoothness claimed is of course in the absolute sense, i.e. over* $\mathbb{C}$*, not over* B*.*

Proof. We first prove (ii) as (i) is similar and simpler. Working near a fibre X_0, there is a standard coordinate neighborhood U_i of each node $p_i, i = 1, ..., k$, which is a pullback of $V/T : xy = t$, and such that the product map $B \to T^k$ is smooth. Then $\prod_B (U_i)_B^{[m_i]}$ is smooth over $\prod_{\mathbb{C}} V_T^{[m_i]}$, and the latter is smooth. Therefore $\prod_B (U_i)_B^{[m_i]}$ is smooth, hence so is $X_B^{[m]}$.

(iii) It follows from the local models that the every fibre component of X/B is m dimensional and dominates a fibre component of $X_B^{(m)}$. Since $X_B^{(m)}$ is irreducible, so is $X_B^{[m]}$. □

In light of Theorem 3.3, we identify a neighborhood H_m of the special fibre in $\tilde{H}$ with a neighborhood of the punctual Hilbert scheme (i.e. $c_m^{-1}(mp)$) in $X_B^{[m]}$, and note that the projection $H_m \to \mathbb{A}^{2m} \times B$ coincides generically, hence everywhere, with $\sigma \circ c_m$. Hence H_m may be viewed as the subscheme of $U_B^{(m)} \times_B \tilde{C}$ defined by the equations

$$\begin{aligned} \sigma^x_m u_1 &= t v_1, \\ \sigma^x_{m-1} u_1 = \sigma^y_1 v_1, \ldots, \sigma^x_1 u_{m-1} &= \sigma^y_{m-1} v_{m-1}, \\ t u_{m-1} &= \sigma^y_m v_{m-1}. \end{aligned} \tag{3.13}$$

Alternatively, in terms of the Z coordinates, H_m may be defined as the subscheme of $U_B^{(m)} \times \mathbb{P}^{m-1} \times B$ defined by the relations (3.6), which define $\tilde{C}$, together with

$$\sigma^y_i Z_i = \sigma^x_{m-i} Z_{i+1},\ i = 1, \ldots, m-1. \tag{3.14}$$

4. Reverse engineering and proof of Blowup Theorem

Reverse-engineering an ideal means finding generators with given syzygies. Our task now is effectively to reverse-engineer an ideal (discriminant ideal) in the σ's whose syzygies, for suitable generators, are given by (3.14) and (3.6). This will be achieved by passing to the ordered version of Hilb, i.e. $X_B^{[m]} \times_{X_B^{(m)}} X_B^m$. The sought-for generators will be given by certain 'mixed Van der Monde' determinants. The proof of the Blowup Theorem is then concluded, essentially by showing explicitly that, locally over Hilb, all the generators are multiples of one of them.

4.1. **Order.** Let $OH_m = H_m \times_{U_B^{(m)}} U_B^m$, so we have a cartesian diagram

$$\begin{array}{ccc} OH_m & \xrightarrow{\varpi_m} & H_m \\ oc_m \downarrow & \square & \downarrow c_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array}$$

and its global analogue

$$\begin{array}{ccc} X_B^{\lceil m \rceil} & \xrightarrow{\varpi_m} & X_B^{[m]} \\ oc_m \downarrow & \square & \downarrow c_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array}$$

Here the horizontal maps are all $\mathfrak{S}_m$-quotients, hence flat. Note that $X_B^{(m)}$ is normal and Cohen-Macaulay: this follows from the fact that it is a quotient by $\mathfrak{S}_m$ of X_B^m, which is a locally complete intersection with singular locus of codimension ≥ 2 (in fact, > 2, since X is smooth). Alternatively, normality of $X_B^{(m)}$ follows from the fact that H_m is smooth and the fibres of $c_m : H_m \to X_B^{(m)}$ are connected and reduced (being products of connected chains of rational curves), using the following general fact: if $f : A \to B$ is a proper surjective morphism with connected reduced fibres between integral algebraic schemes over an algebraically closed field and A is normal, then so is B [proof: For any closed point $b \in B$, the inclusion $\mathcal{O}_{B,b}/\mathfrak{m}_{B,b} \to H^0(\mathcal{O}_{f^{-1}(b)}) = H^0(\mathcal{O}_A/\mathfrak{m}_{B,b}\mathcal{O}_A)$ is an isomorphism because $f^{-1}(b)$ is reduced and connected. By an easy composition series argument, the analogous statement holds with $\mathfrak{m}_{B,b}$ replaced by $\mathfrak{m}_{B,b}^n$ for any $n \geq 1$. Consequently, by the Formal Function Theorem ([5],§3.11), we have $f_*(\mathcal{O}_A) = \mathcal{O}_B$. Then since the local rings of A are integrally closed, the same is true of $\mathcal{O}_A(U), U \subset A$ open, hence also for the local rings of $\mathcal{O}_B$].

Now a few remarks are in order.

- The map ω_m is simply ramified generically over D^m and we have
$$\omega_m^*(D^m) = 2OD^m$$
where
$$OD^m = \sum_{i<j} D^m_{i,j}$$
where $D^m_{i,j} = p^{-1}_{i,j}(OD^2)$ is the locus of points whose ith and jth components coincide.
- $\mathcal{B}\ell_{OD^m} X^m_B = \mathcal{B}\ell_{2OD^m} X^m_B$ as blowups, because blowing up an ideal and its powers are the same (see [5], Ex. II.7.11.a).
- We have
$$(\mathcal{B}\ell_{D^m} X^{(m)}_B) \times_{X^{(m)}_B} X^m_B = \mathcal{B}\ell_{\omega^*_m(D^m)} X^m_B = \mathcal{B}\ell_{2OD^m} X^m_B.$$
So if the natural map $X^{\lceil m\rceil}_B \to \mathcal{B}\ell_{2OD^m} X^m_B$ is an isomorphism, then (obviously) so is the $\mathfrak{S}_m$-equivariant map
$$f : X^{\lceil m\rceil}_B \to (\mathcal{B}\ell_{D^m} X^{(m)}_B) \times_{X^{(m)}_B} X^m_B,$$
which is just the pullback of the natural map
$$c'_m : X^{[m]}_B \to \mathcal{B}\ell_{D^m} X^{(m)}_B$$
by the finite flat surjective map ϖ_m, therefore so is c'_m itself (which is the $\mathfrak{S}_m$-quotient of f).
- Therefore finally we are reduced to showing that oc_m is equivalent to the blowing up of OD^m.

The advantage of working with OD^m rather than its unordered analogue is that at least some of its equations are easy to write down: let
$$v^m_x = \prod_{1\leq i<j\leq m} (x_i - x_j),$$
and likewise for v^m_y. As is well known, v^m_x is the determinant of the Van der Monde matrix
$$V^m_x = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-1} & \dots & x_m^{m-1} \end{bmatrix}.$$
Also set
$$\tilde{U}_i = \varpi_m^{-1}(U_i),$$
where U_i is as in (3.12), being a neighborhood of q_i on H_m. Then in U_1, the universal ideal $\mathcal{I}$ is defined by
$$F_0, \quad F_1 = y + (\text{function of } x)$$

and consequently the length-m scheme corresponding to $\mathcal{I}$ maps isomorphically to its projection to the x-axis. Therefore over $\tilde{U}_1 = \varpi_m^{-1}(U_1)$, where F_0 splits as $\prod(x - x_i)$, the equation of OD^m is simply given by

$$G_1 = v_x^m.$$

Similarly, the equation of OD^m in $\tilde{U}_m$ is given by

$$G_m = v_y^m.$$

Now let

$$\Xi : OH_m \to \mathbb{P}^{m-1}$$

be the morphism corresponding to (the pullback of) $[Z_1, ..., Z_m]$ (cf. (3.5); note that this is by definition essentially a product projection, hence a morphism); set

$$L = \Xi^*(\mathcal{O}(1)). \tag{4.1}$$

Note that $\tilde{U}_i$ coincides with the open set where $Z_i \neq 0$, so Z_i generates L over $\tilde{U}_i$. Let

$$O\Gamma^{(m)} = oc_m^{-1}(OD^m).$$

We shall see below that this is a Cartier divisor, in fact we shall construct an isomorphism

$$\gamma : \mathcal{O}(-O\Gamma^{(m)}) \to L. \tag{4.2}$$

This isomorphism will easily imply Theorem 1. To construct γ, it suffices to specify it on each $\tilde{U}_i$.

4.2. **Mixed Van der Mondes.** A clue as to how the latter might be done comes from the relations (3.14). Thus, set

$$G_i = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} v_x^m = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} G_1, \quad i = 2, ..., m. \tag{4.3}$$

Thus,

$$G_2 = \frac{\sigma_m^y}{t^{m-1}} G_1, G_3 = \frac{\sigma_m^y}{t^{m-2}} G_2, ..., G_{i+1} = \frac{\sigma_m^y}{t^{m-i}} G_i, i = 1, ..., m-1. \tag{4.4}$$

In light of (3.1),(3.2), we deduce

$$\sigma_{m-i}^x G_{i+1} = \sigma_i^y G_i. \tag{4.5}$$

Comparing this with (3.14) certainly suggests solving our reverse-engineering problem by assigning Z_i to G_i, which is what we will do eventually.

Remark 4.1. Clearly v_x^m, hence all the G_i, are invariant under the alternating group $\mathfrak{A}_m$, hence are well-defined on the 'Orientation product', i.e. the quotient of X_B^m by the action of $\mathfrak{A}_m$, which coincides with the double cover of $X_B^{(m)}$ branched on D^m.

Now an elementary calculation shows that if we denote by V_i^m the 'mixed Van der Monde' matrix

$$V_i^m = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-i} & \dots & x_m^{m-i} \\ y_1 & \dots & y_m \\ \vdots & & \vdots \\ y_1^{i-1} & \dots & y_m^{i-1} \end{bmatrix}$$

then we have

(4.6) $$G_i = \pm \det(V_i^m), i = 1, ..., m.$$

⌜ Indeed for $i = 1$ this is standard; for general i, it suffices to prove the analogue of (4.4) for the mixed Van der Monde determinants. For this, it suffices to multiply each jth column of V_i^m by y_j, and factor a $t = x_j y_j$ out of each of rows $2, ..., m-i+1$, which yields

(4.7) $$\sigma_m^y \det(V_i^m) = (-1)^{m-i} t^{m-i} V_{i+1}^m.$$

⌞ From (4.6) it follows, e.g., that G_m as given in (4.3) coincides with $\pm v_y^m$.

4.3. **Conclusion of proof.** The following result is key for the Blowup Theorem.

Lemma 4.2. *G_i generates $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_i$. In particular, $O\Gamma^{(m)}$ is Cartier.*

Proof of Lemma. This is clearly true where $t \neq 0$ and it remains to check it along the special fibre $OH_{m,0}$ of OH_m over B. Note that $OH_{m,0}$ is a sum of components of the form

(4.8) $$\Theta_I = \text{Zeros}(x_i, i \notin I, y_i, i \in I), I \subseteq \{1, ..., m\},$$

none of which is contained in the singular locus of OH_m. Set

$$\Theta_i = \bigcup_{|I|=i} \Theta_I.$$

Note that

$$\tilde{C}_i \times 0 \subset \Theta_i, i = 1, ..., m-1$$

and therefore

$$\tilde{U}_i \cap \Theta_j = \emptyset, j \neq i-1, i.$$

Note that y_i vanishes to order 1 (resp. 0) on Θ_I whenever $i \in I$ (resp. $i \notin I$). Similarly, $x_i - x_j$ vanishes to order 1 (resp. 0) on Θ_I whenever both $i, j \in I^c$ (resp. not both $i, j \in I^c$). From this, an elementary calculation shows that the vanishing order of G_j on every component Θ of Θ_k is

(4.9) $$\text{ord}_\Theta(G_j) = (k-j)^2 + (k-j).$$

We may unambiguously denote this number by $\text{ord}_{\Theta_k}(G_j)$. Since this order is nonnegative for all k, j, it follows firstly that the rational function G_j has no poles, hence is in fact regular on X_B^m near mp (recall that X_B^m is normal); of course, regularity of G_j is also immediate from (4.6). Secondly, since this order is zero for $k = j, j-1$, and Θ_j, Θ_{j-1} contain all the components of $OH_{m,0}$ meeting $\tilde{U}_j$, it follows that in $\tilde{U}_j$, G_j has no zeros besides $O\Gamma^{(m)} \cap \tilde{U}_j$, so G_j is a generator of $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_j$. QED Lemma. $\square$

The Lemma yields a set of generators for the ideal of OD^m:

Corollary 4.3 (of Lemma). *The ideal of OD^m is generated, locally near $(p,...,p)$, by $G_1,...,G_m$.*

Proof. If Q denotes the cokernel of the map $\bigoplus_m \mathcal{O}_{X_B^m} \to \mathcal{O}_{X_B^m}(-OD^m)$ given by $G_1,...,G_m$, then $oc_m^*(Q) = 0$ by the Lemma, hence $Q = 0$, so the G's generate $\mathcal{O}_{X^m}(-OD^m)$. $\square$

Now we can construct the desired isomorphism γ as in (4.2), as follows. Since Z_j is a generator of L over $\tilde{U}_j$, we can define our isomorphism γ over $\tilde{U}_j$ simply by specifying that

$$\gamma(G_j) = Z_j \text{ on } \tilde{U}_j.$$

Now to check that these maps are compatible, it suffices to check that

$$G_j/G_k = Z_j/Z_k$$

as rational functions (in fact, units over $\tilde{U}_j \cap \tilde{U}_k$). But the ratios Z_j/Z_k are determined by the relations (3.14), while G_j/G_k can be computed from (4.5), and it is trivial to check that these agree.

Now we can easily complete the proof of Theorem 1. The existence of γ, together with the universal property of blowing up, yields a morphism

$$Bc_m : OH_m \to B_{OD^m} X_B^m$$

which is clearly proper and birational, hence surjective. On the other hand, the fact that the G's generate the ideal of OD^m, and correspond to the Z coordinates on $OH_m \subset X_B^m \times \mathbb{P}^{m-1}$, implies that Bc_m is a closed immersion. Therefore Bc_m is an isomorphism. $\square$

Remark 4.4. It follows from the foregoing proof that the cycle map is a *nonlinear* blowup, i.e. that the inclusion $\mathrm{Proj}(\bigoplus I_{D^m}^n) \subset \mathbb{P}(I_{D^m})$ is proper.

4.4. **Complements and consequences.** These concern the standard family V/T given by $xy = t$:

Corollary 4.5. *For the standard family, the image of the relative symmetric product $V_T^{(m)}$ under the elementary symmetric functions embedding σ (cf. Lemma 3.1) is schematically defined by the equations (3.1-3.2).*

Proof. We have a diagram locally

$$\begin{array}{ccc} H^m & \subset & \mathbb{P}^{m-1} \times \mathbb{A}^{2m} \times T \\ \downarrow & & \downarrow \\ V_T^{(m)} & \stackrel{\sigma}{\hookrightarrow} & \mathbb{A}^{2m} \times T. \end{array} \tag{4.10}$$

We have seen that the image of the top inclusion is defined by the equations (3.6), (3.14). The equations of the schematic image of σ are obtained by eliminating the Z coordinates from the latter equations, and this clearly yields the equations as claimed. $\square$

Now as one byproduct of the proof of Theorem 2.1, we obtained generators of the ideal of the ordered half-discriminant OD^m. As a further consequence, we can determine the ideal of the discriminant locus D^m in the symmetric product $X_B^{(m)}$ itself: let δ_m^x denote the discriminant of F_0, which, as is well known [7], is a polynomial in the σ_i^x such that

$$\delta_m^x = G_1^2. \tag{4.11}$$

Set

$$\eta_{i,j} = \frac{(\sigma_m^y)^{i+j-2}}{t^{(i-1)(m-i/2)+(j-1)(m-j/2)}}\delta_x^m. \tag{4.12}$$

It is easy to see that this is a polynomial in the $\sigma_.^x$ and the $\sigma_.^y$, such that $\eta_{i,j} = G_i G_j$.

Corollary 4.6. *For the standard family $xy = t$, the ideal of D^m is generated, locally near mp, by $\eta_{i,j}, i, j = 1, ..., m$.*

Proof. This follows from the fact that ϖ_m is flat (being a $\mathfrak{S}_m$- quotient) and that

$$\varpi_m^*(\eta_{i,j}) = G_i G_j, i, j = 1, ..., m$$

generate the ideal of $2OD^m = \varpi_m^*(D^m)$. □

Because any family of nodal curves X/B is locally isomorphic to a pullback of the standard family V/T , it follows that analogues of the previous two corollaries hold for $X_B^{(m)}$ over a neighbourhood of a point $m\theta$, where θ is a relative node.

5. Discriminant polarization

We now return to the case of a general family X/B of nodal-or-smooth curves. We study some natural sheaves, including the discriminant polarization , on the Hilbert schemes $X_B^{[m]}$.

Note that the ideal of the Cartier divisor $c_m^*(D^m)$ on $X_B^{[m]}$, that is, $\mathcal{O}_{X_B^{[m]}}(-c_m^*(D^m))$, is isomorphic in terms of our local model $\tilde{H}$ to $\mathcal{O}(2)$ (i.e. the pullback of $\mathcal{O}(2)$ from $\mathbb{P}^{m-1}$). This suggests that $\mathcal{O}(-c_m^*(D^m))$ is divisible by 2 as line bundle on $X_B^{[m]}$. This is indeed so, and is subsumed in the definition of *discriminant polarization* which follows, together with that of *tautological sheaf*. Consider the tautological subscheme

$$D^{m,1} \subset X_B^{[m]} \times_B X$$

with maps $p_X : D^{m,1} \to X, p_{X_B^{[m]}} : D^{m,1} \to X_B^{[m]}$.

Definition 5.1. *(i) For any sheaf A on X, the associated tautological sheaf is defined by*

$$\Lambda_m(A) = p_{X_B^{[m]}*}(p_X^*(A))$$

(ii) The discriminant polarization on $X_B^{[m]}$ is defined by

$$\mathcal{O}_{X_B^{[m]}}(1) = \mathcal{O}(-\Gamma^{(m)}) := \det(\Lambda_m(\mathcal{O}_X))$$

Note that if A is locally free, then by flatness so is $\Lambda_m(A)$. These bundles are obviously compatible with base-change. Moreover, note that the trace pairing

$$\Lambda_m(\mathcal{O}_X) \otimes \Lambda_m(\mathcal{O}_X) \to \mathcal{O}_{X_B^{[m]}}$$

yields a generically injective map $\Lambda_m(\mathcal{O}_X) \to \Lambda_m(\mathcal{O}_X)^*$ which drops rank precisely on the discriminant $c_m^*(D^m)$, therefore $2\Gamma^{(m)} \sim_{\text{lin}} c_m^*(D^m)$.

We will also use the notation

$$\mathcal{O}(\Gamma^{(m)}) = \mathcal{O}_{X_B^{[m]}}(-1).$$

Note that $\Gamma^{(m)}$ is defined as an effective Weil divisor, and as a line bundle, but not necessarily as an effective Cartier divisor, though $2\Gamma^{(m)}$ and $\Gamma^{\lceil m \rceil}$ are effective (the latter because the symmetrization map $X_B^m \to X_B^{(m)}$ is generically ramified with multiplicity 2 along D^m). In fact, $\Gamma^{(m)}$ is essentially never effective Cartier, as Remark 5.3 below shows. Nonetheless, $-\Gamma^{(m)}$ is relatively ample on the Hilbert scheme $X_B^{[m]}$ over the symmetric product $X_B^{(m)}$, hence the name discriminant polarization.

Further light on the discriminant is shed by the notion of *norm*:

Definition 5.2. *For a line bundle A on X, its m-th norm on $X_B^{[m]}$ is defined by*

$$[m]_*(A) = \det(\Lambda_m(A)) \otimes \mathcal{O}(\Gamma^{(m)})$$

If $A = \mathcal{O}(Y)$ for an effective divisor Y, the exact sequence

$$0 \to \Lambda_m(A^*) \to \Lambda_m(\mathcal{O}_X) \to \Lambda_m(\mathcal{O}_Y) \to 0$$

shows that in this case $[m]_*(A) = -[m]_*(A^*) = \det(\Lambda_m(\mathcal{O}_Y))$ is an effective divisor supported on the locus of schemes whose support meets that of Y.

Remark 5.3. Let X be a smooth curve of genus $g \geq 2$ and fix $m \geq 2$. Then the discriminant $D \subset X^{(m)}$ is not algebraically equivalent to $\sum a_i A_i$ where each $a_i > 0$, $\sum a_i \geq 2$ and the A_i are effective and nontrivial; thus, D is neither splittable nor divisible as effective divisor up to algebraic equivalence.

Proof. Else, it follows that D, being a prime divisor, meets each A_i properly, hence $\mathcal{O}_D(A_i)$ is effective, therefore $\mathcal{O}_D(D)$ is effective up to algebraic equivalence on D. Letting $f : X \times X^{(m-2)} \to D$ denote the obvious (normalization) map, $f(x,z) = 2x + z$, it follows that $f^*(D)$ is effective. But

$$f^*(D).(X \times \text{pt}) = -\deg(\omega_X) = 2 - 2g < 0,$$

which contradicts effectivity.

For $g \leq 1$, D is effectively divisible by 2, at least for a single curve. For $g = 1$, X is an elliptic curve with group law $*$ and D is algebraically equivalent to $2D_a, a \in X$, where

$$D_a = \{x + x * a + \sum_{i=1}^{m-2} x_i\}$$

The algebraic equivalence becomes linear when a has order 2 in the group.

6. Flags

See [17] for Flag- Hilbert schemes in general. Flag-Hilbert schemes for points on nodal curves were studied in [14, 15]. In [14], a construction is given for the full-flag Hilbert scheme via an explicit blowup procedure, different in flavor from the above discriminant blowup. In [15], a model analogous to H_m was constructed for the relative Hilbert scheme $X_B^{[m,m+1]}$ of $(m, m+1)$-flags, i.e. pairs of ideals $(z_1 \supset z_2)$ of respective lengths $(m, m+1)$. Here we try to reconcile the two viewpoints by showing that the full-flag Hilbert scheme can also be represented as a blowup of a discriminant-like (viz. incidence) variety, in analogy with the case of the ordinary Hilbert scheme.

Consider the flag-Hilbert scheme, which fits in a diagram

$$\begin{tikzcd} & X_B^{[m,m+1]} \arrow[dl, "p_{[m]}"'] \arrow[d, "p_{[m+1]}"] & \subset \quad X_B^{[m]} \times X_B^{[m+1]} \\ X_B^{[m]} & X_B^{[m+1]} & \end{tikzcd} \tag{6.1}$$

Via this, $X_B^{[m,m+1]}$ is endowed with divisors denoted $\Gamma^{(m)}, \Gamma^{(m+1)}$, which are pullbacks of analogous divisors on $X_B^{[m]}, X_B^{[m+1]}$ respecively. There is a natural morphism (where X is identified with the set of colength-1 ideals)

$$X_B^{[m,m+1]} \to X$$
$$(z_1 \supset z_2) \mapsto \mathrm{Ann}(z_1/z_2)$$

whence a map

$$c_{m,1} : X_B^{[m,m+1]} \to X_B^{[m]} \times_B X \tag{6.2}$$

Theorem 6.1. *$c_{m,1}$ is the blowing-up of the incidence variety $D^{(m,1)} = \{(z, x) : x \in z\}$*

Proof. Let

$$b : Y \to X_B^{[m]} \times_B X$$

be the blowing up of $D^{(m,1)}$ and $\Gamma^{(m,1)}$ the exceptional (Cartier) divisor, i.e. the inverse image of $D^{(m,1)}$. Because $c_{m,1}^{-1}(D^{(m,1)}) = \Gamma^{(m+1)} - \Gamma^{(m)}$ is Cartier, it follows from the universal property of blowing up that we get a diagram

$$\begin{tikzcd} X_B^{[m,m+1]} \arrow[rr, "c'"] \arrow[dr, "c_{m,1}"'] & & Y \arrow[dl, "b"] \\ & X_B^{[m]} \times_B X & \end{tikzcd}$$

On the other hand, there is an obvious map

$$Y \to X_B^{(m+1)}$$

and the pullback of $D^{(m+1)}$ is just $\Gamma^{(m)} + \Gamma^{(m,1)}$, hence Cartier. So by the Blowup Theorem we get a map $Y \to X_B^{[m+1]}$. Together with the projection $Y \to X_B^{[m]}$, this gives a map $Y \to X_B^{[m]} \times_B X_B^{[m+1]}$ whose image is clearly contained in $X_B^{[m,m+1]}$, whence a map

$$d : Y \to X_B^{[m,m+1]},$$

which together with c' fits in a diagram

$$\begin{array}{ccccc} X_B^{[m,m+1]} & & \overset{c'}{\underset{d}{\rightleftarrows}} & & Y \\ & c_{m,1} \searrow & & \swarrow b & \\ & & X_B^{[m]} \times_B X & & \end{array}$$

As both vertical maps are birational, c', d are mutually inverse isomorphisms. $\square$

As a consequence, we obtain recursively a presentation of the full-flag Hilbert scheme as a blowup of incidence varieties. This slightly generalizes a result proven in ([14], Thm. 2.1) by more explicit means.

Corollary 6.2. *Denote by $W^{m.}(X/B)$ the flag Hilbert scheme parametrizing flags of subschemes of fibres $(z_{m_1} < z_{m_2} ... < z_{m_k})$ of respective lengths $m_1 < m_2 < ... < m_k$. Then $W^{m.,m_k+1}(X/B)$ is the blowup of $W^{m.}(X/B) \times_B X$ in the incidence variety*

$$D^{(m.,1)} = \{(z., x) : x \in z_{m_k}\}.$$

Remark 6.3. We don't know if the analogues of Theorem 6.1 or Corollary 6.2 hold for arbitrary flags, e.g. of type $[m, m+2]$. Those Hilbert schemes seem to be worse behaved: inter alia, the fibres of the cycle map on $X^{[m,m+2]}$ can have dimension 2 if $m > 1$. For instance, a generic length-2 subscheme of a node is contained in just two length-3 subschemes, but in an entire 1-paramater family of length-4 subschemes.

Part 2. Node scrolls

7. Study of H_m

We continue our study of the cycle map over a neighborhood of a maximally singular cycle $m\theta$ with θ a fibre node, using the model H_m. The results will be applied in the Node Scroll theorem. Having previously determined the structure of c_m along its 'most special' fibre $c_m^{-1}(m\theta)$ (which corresponds in the model H_m to the fibre over the origin $0_{\mathbb{A}^{2m}}$), our purpose in this section is to determine its structure along nearby fibres and their variation. Thus we will assume for the rest of this section, unless otherwise stated, that we are in the local situation where B is a smooth curve, with local coordinate t, and the family U/B is the standard degeneration $xy = t$. Our purpose is to prove the following result, which serves as the foundation for our study of node scrolls. The notation will be explained below; suffice it to recall here that on a node with equation $xy = 0$, an ideal of type C_j^n (resp. Q_j^n) is generated by $x^{n-j} + ty^j, t \neq 0$ (resp. x^{n-j+1} and y^j).

Lemma 7.1. *For each $1 \leq j \leq n-1$, there exists a $\mathbb{P}^1$-bundle $F_j^{m,n}$ over $(U^\theta)^{(m-n)}$, together with a pair of disjoint sections $Q_j^{m,n}, Q_{j+1}^{m,n}$ and a map*

$$p_{j,[m]} : F_j^{m,n} \to H_m,$$

such that

(i) *the image of $p_{j,[m]}$ coincides with the closure of the locus of schemes having length n and type C_j^n at θ;*

(ii) *the combined image of*

$$\coprod_{j=1}^{n-1} F_j^{m,n} \to H_m$$

coincides with the locus of schemes of length at least n *at* θ

(iii) *the image* $p_{j,[m]}(Q_\bullet^{m,n}), \bullet = j, j+1$, *coincides with the closure of the locus of schemes having length* n *and type* $Q_\bullet^n$ *at* θ.

7.1. **Nearby fibres.** Let $U', U"$ denote the x, y axes, respectively in $U_0 = X_0 \cap U$, with their respective origins $\theta', \theta"$ mapping to $\theta \in U$. Set $U^\theta = U' \coprod U"$, the normalization of U_0. If the special fibre X_0 is reducible, then $U', U"$ globalize to (i.e. are open subsets of) the two components of the normalization. If X_0 is irreducible, then both U' and $U"$ globalize to the normalization. For any pair of natural numbers $(a, b), 0 < a + b < m$, set

$$U^{(a,b)} = U'^{(a)} \times U"^{(b)}$$

(which globalizes to a component –the unique one, if X_0 is irreducible– of the normalization of X_0^{a+b}). Then we have a natural map

$$U^{(a,b)} \to (U_0)_B^{(m)} \subset (U)_B^{(m)}$$

given by

$$(\sum m_i x_i, \sum n_j y_j) \mapsto \sum m_i(x_i, 0) + \sum n_j(0, y_j) + (m - a - b)\theta.$$

This map is clearly birational to its image, which we denote by $\bar{U}^{(a,b)}$. Thus $U^{(a,b)}$ coincides with the normalization of $\bar{U}^{(a,b)}$. It is clear that $\bar{U}^{(a,b)}$ is defined by the equations

$$\sigma_m^x = ... = \sigma_{a+1}^x = 0, \sigma_m^y = ... = \sigma_{b+1}^y = 0.$$

A point

$$c \in \bar{U}^{(a,b)} - (\bar{U}^{(a+1,b)} \cup \bar{U}^{(a,b+1)}),$$

i.e. a cycle in which $(0, 0)$ appears with multiplicity exactly $n = m - a - b$, is said to be of *type* (a, b). Type yields a natural stratification of the symmetric product $U_0^{(m)}$. Now let $\bar{H}^{(a,b)}$ be the closure of the locus of schemes whose cycle is of type (a, b), i.e.

$$\bar{H}^{(a,b)} = \mathrm{closure}(c_m^{-1}(\bar{U}^{(a,b)} - (\bar{U}^{(a+1,b)} \cup \bar{U}^{(a,b+1)}))) \subset H_m \tag{7.1}$$

Also let

$$H^{(a,b)} = \bar{H}^{(a,b)} \times_{\bar{U}^{(a,b)}} U^{(a,b)}. \tag{7.2}$$

Clearly the restriction of c_m on $\bar{H}^{(a,b)}$ factors through a map

$$\tilde{c}_m : \bar{H}^{(a,b)} \to \bar{U}^{(a,b)},$$
$$\tilde{c}_m = ((\sigma_1^x, ..., \sigma_a^x), (\sigma_1^y, ..., \sigma_b^y))$$

Approaching the 'origin cycle' $m(0, 0)$ through cycles of type (a, b), on $\bar{U}^{(a,b)}$, means that a (resp. b) points are approaching the origin θ' (resp. $\theta"$) along the x (resp. y)-axis. For a general cycle c of type (a, b), we have, for all $j \leq b$, that $\sigma_j^y \neq 0, \sigma_{m-j}^x = 0$, hence by the equations (3.7) (setting each $a_i = \sigma_{m-i}^x, d_i = \sigma_{m-i}^y$), we conclude $v_j = 0$; thus

$$v_1 = ... = v_b = 0; \tag{7.3}$$

similarly, for all $j \le a$, we have $\sigma^y_{m-j} = 0, \sigma^x_j \ne 0$ (c being general), hence again by the equations (3.7) , we conclude $u_{m-j} = 0$; thus

(7.4) $$u_{m-1} = ... = u_{m-a} = 0.$$

Consequently, the fibre of c_m over this point is schematically

(7.5) $$c_m^{-1}(c) = \tilde{c}_m^{-1}(c) \simeq \bigcup_{i=b+1}^{m-a-1} C_i^m,$$

provided $a+b \le m-2$ (where the C_i^m are the components of the punctual Hilbert scheme, as in the basic construction of the model H_m, see Theorem 3.3). If $a+b = m-1$, the fibre is the unique point given by

$$v_1 = ... = v_b = u_{b+1} = ... = u_{m-1} = 0$$

(as a subscheme of X/B, this point is the one denoted Q^m_{b+1} in [15], and has ideal (x^{m-b}, y^{b+1})). As c approaches the 'origin' $(m\theta)$ in $\overline{U}^{(a,b)}$, or for that matter any point c', the equations (7.3),(7.4) persist, so we conclude

(7.6) $$\tilde{c}_m^{-1}(c') = \begin{cases} \bigcup\limits_{i=b+1}^{m-a-1} C_i^m, a+b \le m-2, \\ Q^m_{b+1}, a+b = m-1. \end{cases}$$

[Informally, this is a priori plausible: because schemes in C_i^m represent i points coalesced through the y-axis and $m-i$ points coalesced through the x-axis. Then moving 'out' to c represents generalizing $b < i$ (resp. $a < m-i$) of the i (resp. $m-i$) points over the y (resp. x) axis.]

Of particular interest naturally is the case where the union above is a single $\mathbb{P}^1$, in other words when $b = i-1, a = m-i-1 = m-b-2$. In this case

$$\bar{H}^{(m-i-1,i-1)} \to \bar{U}^{(m-i-1,i-1)}$$

is just a $\mathbb{P}^1$-bundle, with fibre C_i^m at the origin. Of course the same is true with the bars removed (i.e. after pullback over $U^{(m-i-1,i-1)}$). [Informally again, this says C_i^m as a component of the punctual Hilbert scheme (schemes of length m concentrated at θ) extends most generically by freeing up $i-1$ and $m-i-1$ points respectively over the two axes.]

More generally, for any $1 \le j < n \le m-1, a+b = m-n$, we have a natural map

$$\alpha(n-j-1, j-1) : U^{(a,b)} \to U^{(a+n-j-1,b+j-1)},$$
$$(.,.) \mapsto (. + (n-j-1)\theta', . + (j-1)\theta")$$

Pulling back over $H^{(a+n-j-1,b+j-1)}$, we obtain $\mathbb{P}^1$-bundles

$$F_j^{m,n}(a,b) \to U^{(a,b)}$$

(7.7) $$F_j^{m,n} = \coprod_{a+b=m-n} F_j^{m,n}(a,b) \to (U^\theta)^{(m-n)} = \coprod_{a+b=m-n} U^{(a,b)}.$$

We call $F_j^{m,n}$ a 'model node scroll'. It is a special case of the general *node scroll*, to be studied further below. Note that $F_j^{m,n}$ comes equipped with a map $F_j^{m,n} \to H_m$, whose combined image for $j = 1, .., n-1$ by definition is the closure of the locus of schemes having length n at the node θ. Note that any subscheme z having length n locally at θ sits over a cycle c of type $(a, b), a+b = m-n$ and therefore occurs in (7.5) for some i, hence also in in $F_j^{m,n}$ with $j = i - b$. Furthermore, if z' is a subscheme having length $n' \geq n$ at θ, it occurs on $F_j^{m,n}(a', b'), a' + b' = m - n'$ for some j. Then choosing $a \geq a', b \geq b'$ with $a + b = m - n$, we can factor $\alpha(n' - j - 1, j - 1)$ via $U^{(a,b)}$:

$$U^{(a',b')} \to U^{(a,b)} \stackrel{\alpha(n-j-1,j-1)}{\to} U^{(a'+n'-j-1,b'+j-1)}$$

to conclude that z' occurs on $F_j^{m,n}(a, b)$ and in particular on $F_j^{m,n}$. Thus, the image of $F_j^{m,n}$ in H_m corresponds to the closure of the locus of schemes which are of length n and type C_j^n (i.e. local equation $x^{n-j} + \alpha y^j$), $\alpha \in \mathbb{C}^*$) at the node θ.

Also, referring to (7.6), we see that $F_j^{m,n}(a, b)$ and also $F_j^{m,n}$ contain two special, mutually disjoint cross sections corresponding to Q_j^m, Q_{j+1}^m, which come respectively from

$$\bar{H}^{(m-i,i-1)}, \bar{H}^{(m-i-1,i)} \subset \bar{H}^{(m-i-1,i-1)}.$$

We denote these by $Q_j^{m,n}(a, b), Q_{j+1}^{m,n}(a, b)$ and $Q_j^{m,n}, Q_{j+1}^{m,n}$, respectively. This notation is slightly imprecise in that there is a $Q_j^{m,n}$ on both $F_j^{m,n}$ and $F_{j-1}^{m,n}$. But both of them have the same image in the Hilbert scheme, viz. the closure of the locus of schemes having length n and type $Q_j^{m,n}$ (i.e. local equations (x^{n-j+1}, y^j) at θ. The reason is the same as given above in the case of $F_j^{m,n}$. This completes the proof of Lemma 7.1.

7.2. **Node scrolls: an optional preview.** This subsection is not needed anywhere. It presents an alternative, more 'qualitative' perspective on a property of node scrolls that is subsumed in the Node Scroll Theorem 9.3. This property has to do with the intrinsic, as opposed to polarized, structure of these scrolls.

Fixing m, n, a, b for now, the $F_j = F_j^{m,n}(a, b)$ are components of special (but typical) cases of what are to be called *node scrolls*. It follows from Lemma 7.1 that we can write

$$F_j = \mathbb{P}(L_j^n \oplus L_{j+1}^n)$$

for certain line bundles L_j^n, L_{j+1}^n on $U^{(a,b)}$, corresponding to the disjoint sections $Q_j^{m,n}, Q_{j+1}^{m,n}$, where the difference $L_j^n - L_{j+1}^n$ is uniquely determined (we use additive notation for the tensor product of line bundles and quotient convention for projective bundles). The identification of a natural choice for both these line bundles, using methods to be developed later in this section, will be taken up in the next section and plays an important role in the enumerative geometry of the Hilbert scheme. But the difference $L_j^n - L_{j+1}^n$, and hence the intrinsic structure of the node scroll F_j, may already be computed now, as follows.

Write

$$Q_j = \mathbb{P}(L_j), Q_{j+1} = \mathbb{P}(L_{j+1})$$

for the two special sections of type $Q_j^{m,n}, Q_{j+1}^{m,n}$ respectively. Let

$$D_{\theta'}, D_{\theta"} \subset U^{(a,b)}$$

be the divisors comprised of cycles containing θ' (resp. θ"). In the local model, these are given locally by the respective equations

$$D_{\theta'} = (\sigma^x_a), D_{\theta"} = (\sigma^y_b).$$

Lemma 7.2. *We have, using the quotient convention for projective bundles,*

$$F_j = \mathbb{P}_{U^{(a,b)}}(\mathcal{O}(-D_{\theta'}) \oplus \mathcal{O}(-D_{\theta"})), j = 1, ..., n-1. \tag{7.8}$$

Proof. Our key tool is a $\mathbb{C}^*$- parametrized family of sections 'interpolating' between Q_j and Q_{j+1}. Namely, note that for any $s \in \mathbb{C}^*$, there is a well-defined section I_s of F_j whose fibre over a general point $z \in X^{(a,b)}$ is the scheme

$$I_s(z) = (sx^{n-j} + y^j) \amalg \mathrm{sch}(z),$$

where $\mathrm{sch}(z)$ is the unique subscheme of length $a+b$, disjoint from the nodes, corresponding to z, and we are identifying a (principal) ideal with the corresponding subscheme..

Claim: The fibre of I_s over a point $z \in D_{\theta'}$ (resp. $z \in D_{\theta"}$) is a scheme of type $Q^{m,n}_j$, i.e. (x^{n-j+1}, y^j) (resp. $Q^{m,n}_{j+1}$).

Proof of claim. Indeed set-theoretically the claim is clear from the fact thar this fibre corresponds to a length-n punctual scheme meeting the x-axis (resp. y-axis) with multiplicity at least $n-j+1$ (resp. $j+1$).

To see the same thing schematically, via equations in the local model H_{n+1}, we proceed as follows. We work near a *generic* point $z_0 \in D_{\theta'}$, necessarily of multiplicity 1 at the origin. Then we can, discarding distal factors supported away from the nodes, write the singleton (length-1) scheme corresponding to a nearby cycle z as $\mathrm{sch}(z) = (x-c, y)$ where $c \to 0$ as $z \to z_0$, and then

$$I_s(z) = (sx^{n-j} + y^j)(x-c, y) = (sx^{n-j+1} - csx^{n-j} - cy^j, y^{j+1}).$$

Thus, in terms of the system of generators (3.8) et seq., $I_s(z)$ is defined locally by

$$cu_j - sv_j = 0 \tag{7.9}$$

(with other $[u_k, v_k]$ coordinates either $[1,0]$ for $k < j$ or $[0,1]$ for $k > j$. The limit of this as $c \to 0$ is $[u_j, v_j] = [1,0]$, which is the point Q_j. *QED Claim.*

Clearly I_s doesn't meet Q_j or Q_{j+1} away from $D_{\theta'} \cup D_{\theta"}$. Therefore, denoting the scroll projection by π, we have

$$I_s \cap Q_j = Q_j.\pi^*(D_{\theta'}), \tag{7.10}$$

$$I_s \cap Q_{j+1} = Q_{j+1}.\pi^*(D_{\theta"}); \tag{7.11}$$

an easy calculation in the local model shows that the intersection is transverse. Because $Q_j \cap Q_{j+1} = \emptyset$, it follows that

$$I_a \sim Q_j + \pi^*(D_{\theta'}) \tag{7.12}$$

$$I_a \sim Q_{j+1} + \pi^*(D_{\theta"}). \tag{7.13}$$

These relations also follow from the fact, which comes simply from setting $s = 0$ or dividing by s and setting $s = \infty$ in (7.9), that

$$\lim_{s\to 0} I_s = Q_j + \pi^*(D_{\theta'}), \lim_{s\to\infty} I_s = Q_{j+1} + \pi^*(D_{\theta"}) \tag{7.14}$$

It then follows that

$$(Q_j)^2 = Q_j.(I_s - \pi^*(D_{\theta'})) = Q_j.(Q_{j+1} + \pi^*(D_{\theta''} - D_{\theta'})),$$

hence

$$(Q_j)^2 = Q_j.\pi^*(D_{\theta''} - D_{\theta'}), \tag{7.15}$$

therefore finally

$$L_j^n - L_{j+1}^n = \pi^*(D_{\theta''} - D_{\theta'}). \tag{7.16}$$

This proves the Lemma. □

8. Definition of node scrolls and polyscrolls

We now begin to extend our scope to a global proper family X/B of nodal curves, with possibly higher-dimensional base and fibres with more than one node. Our main interest is in the node scrolls in this generality, where, rather than living over a symmetric product, they become $\mathbb{P}^1$-bundles over a relative Hilbert scheme (of lower degree) associated to a 'boundary family' of X/B, i.e a family obtained, essentially, as the partial normalization of the subfamily of X/B lying over the normalization of a component of the locus of singular curves in B (viz. the boundary of B). For our purposes, it will be convenient to work 'node by node', associating to each a boundary family. We begin by making the appropriate notion of boundary family precise.

8.1. **Boundary data.** Let $\pi : X \to B$ now denote an arbitrary flat family of nodal curves of arithmetic genus g over an irreducible base, with smooth generic fibre. In order to specify the additional information required to define a node scroll, we make the following definition.

Definition 8.1. *A* <u>*boundary datum*</u> *for X/B consists of*

(i) *an irreducible variety T with a map $\delta : T \to B$ unramified to its image;*
(ii) *a 'relative node' over T, i.e. a lifting $\theta : T \to X$ of δ such that each $\theta(t)$ is a node of $X_{\delta(t)}$;*
(iii) *a labelling, continuous in t, of the two branches of $X_{\delta(t)}$ along $\theta(t)$ as x-axis and y-axis, denoted X', X''.*

Given such a datum, the <u>*associated boundary family X_T^θ*</u> *is the normalization (= blowup) of the base-changed family $X \times_B T$ along the section θ, i.e.*

$$X_T^\theta = \mathcal{B}\ell_\theta(X \times_B T),$$

viewed as a family of curves of arithmetic genus $g-1$ with two smooth, everywhere distinct, individually defined marked points θ_x, θ_y on the respective branches X', X''. We denote by ϕ the natural map fitting in the diagram

$$\begin{tikzcd}
X_T^\theta \arrow[d] \arrow[dr, "\phi"] & \\
X \times_B T \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r, "\delta"] & B.
\end{tikzcd}$$

Remark 8.2. Note that the fibres of X_T^θ are disconneted (e.g. a disjoint union of smooth curves of genera $i, g-i$) whenever θ is a separating node; still they always have arithmetic genus $g-1$, where the arithmetic genus of a curve X is defined as $1-\chi(\mathcal{O}_X)$.

Note that a boundary datum indeed lives over the boundary of B; in the other direction, we can associate to any component T_0 of the boundary of B a finite number of boundary data in this sense: first consider a component T_1 of the normalization of $T_0 \times_B \operatorname{sing}(X/B)$, which already admits a node-valued lifting θ_1 to X, then further base-change by the normal cone of $\theta_1(T_1)$ in X (which is 2:1 unramified, possibly disconnected, over T_1), to obtain a boundary datum as above. 'Typically', the curve corresponding to a general point in T_0 will have a single node θ and then the degree of δ will be 1 or 2 depending on whether the branches along θ are distinguishable in X or not (they always are distinguishable if θ is a separating node and the separated subcurves have different genera). Proceeding in this way and taking all components which arise, we obtain finitely many boundary data which 'cover', in an obvious sense, the entire boundary of B. Such a collection, weighted so that each boundary component T_0 has total weight $=1$ is called a *covering system of boundary data.*

8.2. **Node scrolls: definition.**

Proposition-definition 8.3. *Given a boundary datum (T, δ, θ) for X/B and natural numbers $1 \le j < n$, there exists a $\mathbb{P}^1$-bundle $F_j^{m,n}(\theta)$, called a* <u>*node scroll*</u> *over the Hilbert scheme $(X_T^\theta)^{[m-n]}$, endowed with two disjoint sections $Q_{j,j}^{m,n}(\theta), Q_{j+1,j}^{m,n}(\theta)$, together with a surjective map generically of degree equal to $\deg(\delta)$ of*

$$\bigcup_{j=1}^{n-1} F_j^{m,n}(\theta) := \coprod_{j=1}^{n-1} F_j^{m,n}(\theta) / \coprod_{j=1}^{n-2} (Q_{j+1,j}^{m,n}(\theta) \sim Q_{j+1,j+1}^{m,n}(\theta))$$

onto the closure in $X_B^{[m]}$ of the locus of schemes having length precisely n at θ, so that a general fibre of $F_j^{m,n}(\theta)$ corresponds to the family C_j^n of length-n schemes at θ generically of type C_j^n, with the two nonprincipal schemes Q_j^n, Q_{j+1}^n corresponding to $Q_{j,j}^{m,n}(\theta), Q_{j,j+1}^{m,n}(\theta)$ respectively. We denote by δ_j^n the natural map of $F_j^{m,n}(\theta)$ to $X_B^{[m]}$.

Proof-construction. We fix m and θ (and suppress them when convenient). The scroll $F_j^{m,n}(\theta)$ is defined as follows. Fixing the boundary data, consider first the locus

$$\bar{F}_j^n \subset T \times_B X_B^{[m]}$$

consisting of compatible pairs (t, z) such that z is in the closure of the set of schemes which are of type I_j^n (i.e. $x^{n-j} + ay^j, a \in \mathbb{C}^*$) at $\theta(t)$, with respect to the branch order (θ_x, θ_y). The discussion of §7 shows that the general fibre of $\bar{F}_j$ under the cycle map is a $\mathbb{P}^1$, namely a copy of C_j^n; moreover the closure of the locus of schemes having multiplicity n at θ is the union $\bigcup_{j-1}^{n-1} \bar{F}_j^{m,n}$. In fact locally over a neighborhood of a cycle having multiplicity precisely $n+e$ at θ, $\bar{F}_j^{m,n}$ is a union of components $\bar{F}_j^n(a,b) \times U^{(m-e)}, a+b=e$, where U is an open set disjoint from θ, $\bar{F}_j^n(a,b) \subset H_{n+e}$ maps to $(U')^a \times (U")^b$ and is defined in H_{n+e}

by by the vanishing of all $Z_i, i \neq j+b, j+b+1$ or alternatively, in terms of u, v coordinates, by

$$v_1 = ... = v_{j+b} = u_{j+b+1} = ... = u_{n+e} = 0$$

Then $F_j^{m,n}(\theta)$ is the locus

$$\{(w,t,z) \in (X_T^\theta)^{[m-n]} \times_T \bar{F}_j^n : \phi_*(c_{m-n}(w)) + n\theta = c_m(z)\}, \tag{8.1}$$

where $\phi : X^\theta \to X$ is the natural map, clutching together θ_x and θ_y, and ϕ_* is the induced push-forward map on cycles. Then the results of the previous section show that $F_j^{m,n}(\theta)$ is locally defined near a cycle having multiplicity b at θ_y, e.g. by the vanishing of the $Z_i, i \neq j+b, j+b+1$ on

$$\{(w,u,Z) \in (X_T^\theta)^{[m-n]} \times X_B^{(e)} \times \mathbb{P}^{n+e} : \phi_*(c_{m-n}(w))_\theta + n\theta = u\}$$

where $\__\theta$ indicates the portion near θ. The latter locus certainly projects isomorphically to its image in $(X_T^\theta)^{[m-n]} \times \mathbb{P}^{n+e}$, hence $F_j^{m,n}(\theta)$ is a $\mathbb{P}^1$-bundle over $(X_T^\theta)^{[m-n]}$. Since $F_j^{m,n}(\theta)$ admits the two sections $Q_{j,j}^{m,n}(\theta), Q_{j+1,j}^{m,n}(\theta)$, it is the projectivization of a decomposable rank-2 vector bundle. □

Note that the node scroll $F_j^{m,n}(\theta)$ also depends on m, and is by construction a subscheme of the 'flag-like' Hilbert scheme

$$\begin{array}{ccc} F_j^{m,n}(\theta) \subset & \{(z_1, z_2) : \phi(z_1) \subset z_2\} & \to X_B^{[m]} \\ & \downarrow & \\ & (X_T^\theta)^{[m-n]} & \end{array} \tag{8.2}$$

Of course z_1, z_2 live on different families so this is not the usual flag-Hilb. We will denote the two Hilbert-scheme targeted projections on $F_j^{m,n}(\theta)$ by $p_{[m-n]}, p_{[m]}$ respectively. When the dependence on $\theta, m, ...$ is obvious, we will omit the corresponding designator. The following simple technical point will be needed below.

Lemma 8.4. *Let $T' \to T$ be a base change and θ' a section of $X_{T'}^\theta$ disjoint from the distinguished sections $(\theta_x)_{T'}, (\theta_y)_{T'}$ and identified with the corresponding section of $X_{T'}$. Then on the pulled- back node scroll $F_j^{m,n}(\theta)_{T'}$,*

$$p_{[m]}^*[m]_*\theta' = p_{[m-n]}^*[m-n]_*\theta'$$

Proof. It suffices to verify this on the ordered version where, e.g. $[m]_*\theta' = \sum\limits_{i=1}^m p_i^*\theta'$ and the projection $p_{[m-n]}$ corresponds to projection on the first $m-n$ coordinates. But then for $i > m-n$, we have $p_i^*\theta' \cap F = \emptyset$ as the nodes are disjoint. This gives our assertion. □

Obviously, $Q_{j,j-1}^{m,n}(\theta)$ and $Q_{j,j}^{m,n}(\theta)$ coincide in $(X_T^\theta)^{[m-n]} \times X_B^{[m]}$ and when convenient we will write them as $Q_j^{m,n}(\theta)$ or $Q_j^{m,n}(\theta)$, omitting θ when harmless. It is noteworthy that the map from $Q_j^{m,n}(\theta)$ can be written down explicitly:

Lemma 8.5. *The map $(X_T^\theta)^{[k]} \simeq Q_j^{m,n}(\theta) \to X_B^{[m]}$ is given by*

$$z_0 + a_x\theta_x + a_y\theta_y \mapsto \phi(z_0) + Q_{j+a_y}^{n+a_x+a_y} \tag{8.3}$$

where z_0 is supported off $\theta_x \cup \theta_y$.

Proof. To begin with, as θ_x, θ_y are smooth sections of X_T^θ, any length-k subscheme of it can indeed be expressed uniquely as in the formula. The formula is clearly true when $a_x = a_y = 0$. Then the general case follows by taking limits, in view of the explicit local description of the schemes of type Q_r^p as (x^{p-r+1}, y^r). □

8.3. **Polyscrolls.** Consider now a collection $\theta_. = (\theta_1, ..., \theta_r)$ of distinct relative nodes of X/B and $T = T(\theta_1, ..., \theta_r) \to B$ a common boundary locus for them, compatible with the boundary data for each θ_i. Thus, X_T is endowed with r distinct relative nodes that we still denote by $\theta_1, ..., \theta_r$. Let $X_T^{\theta_.}$ be the blowup or partial normalization of X_T in $\theta_1, ...\theta_r$. As the θ_i are disjoint, the blowing up may be done inductively, in any order, or simultaneously. Let $(j.), (n.)$ be sequences of r positive integers with $(j.) < (n.)$ in the sense that $j_i < n_i, \forall i$. We aim to define a node polyscroll $F := F_{j.}^{m,n.}(\theta.; X/B)$. This can be done using induction on r. Assume the $(r-1)$- polyscroll $F' = F_{j_2,...,j_r}^{m-n_1,n_2,...,n_r}(\theta_2, ..., \theta_r; X_{T(\theta_1)}^{\theta_1})$ is defined, together with maps

$$\begin{array}{ccc} F' & \stackrel{p_{[m-n_1]}}{\to} & (X^{\theta_1})_{T(\theta_1)}^{[m-n_1]} \\ p_{[m-|n.|]} \downarrow & & \\ (X_T^{\theta_.})^{[m-|n.|]} & & \end{array}$$

the horizontal one being generically finite and the vertical one a $(\mathbb{P}^1)^{r-1}$-bundle projection. Of course, the node scroll $F_{j_1}^{m,n_1}(\theta_1; X/B)$ is a $\mathbb{P}^1$-bundle over $(X^{\theta_1})_{T(\theta_1)}^{[m-n_1]}$. Define F as the fibre product

$$\begin{array}{ccccc} & & F & & \\ & \swarrow & & \searrow & \\ F_{j_1}^{m,n_1}(\theta_1; X/B) & & \diamondsuit & & F' \\ & \searrow & & \swarrow & \downarrow \\ & & (X^{\theta_1})_{T(\theta_1)}^{[m-n_1]} & & (X_T^{\theta_.})^{[m-|n.|]} \end{array} \tag{8.4}$$

Then F comes equipped with a $(\mathbb{P}^1)^r$-bundle projection $p_{[m-|n.|]}F \to F' \to (X_T^{\theta_.})^{[m-|n.|]}$, as well as a generically finite map $p_{[m-n_1]} : F \to F_{j_1}^{m,n_1}(\theta_1; X/B) \to X_B^{[m]}$. Writing, suggestively, F' as $F(\hat{\theta}_1)$ and assuming inductively maps $F' \to F'(\hat{\theta}_i), \forall i > 1$, we can identify $F_{j_1}^{m,n_1}(\theta_1; X/B) \times F'(\hat{\theta}_i)$ as $F(\hat{\theta}_i)$ and obtain an induced map $F \to F(\hat{\theta}_i)$. Then taking fibre product with $F_{j_i}^{m,n_i}(\theta_i; X/B)$, we obtain a morphism, easily seen to be an isomorphism, from F to a similar node polyscroll with θ_1, θ_i interchanged. Continuing in this way, it is easy to see that F is independent of the ordering and the composite $F \to F' \to (X_T^{\theta_.})^{[m-|n.|]}$ is a $(\mathbb{P}^1)^r$-bundle.

We summarize some of the important properties of node polyscrolls as follows

Proposition 8.6. (i) *The r-polyscroll $F = F_{j.}^{m,n.}(\theta..X/B)$ is a $(\mathbb{P}^1)^r$-bundle over the Hilbert scheme $(X_T^{\theta_.})^{[m-|n.|]}$.*

(ii) *F parametrizes subschemes of X/B having length at least n_i at $\theta_i, i = 1, ..., r$.*

(iii) *F is independent of the order of $(\theta., n., j.)$ and admits a $(\mathbb{P}^1)^{r-s}$-bundle projection to a pullback of the s-polyscroll based on any s of the (θ_i, n_i, j_i).*

9. STRUCTURE OF NODE SCROLLS

We fix a boundary datum (T, δ, θ) as above. Our aim now is to determine the structure of a node scroll as $\mathbb{P}^1$ bundle together with the relative polarization induced by minus the discriminant. The following result is critical:

Proposition 9.1. *Let $Q_j^{m,n} = Q_j^{m,n}(\theta)$ be the canonical section of type $Q_j^{m,n}$ in the node scroll*

$$F_j^{m,n} = F_j^{m,n}(\theta) \subset X_B^{[m]}.$$

Then up to linear equivalence, we have, where $k = m - n$, $Q_j^{m,n}$ is identified with $(X^\theta)_T^{[k]}$ and $\Gamma^{(k)}$ is the discriminant on the latter:

$$\Gamma^{(m)}.Q_j^{m,n} \equiv_{\text{lin}} -\binom{n-j+1}{2}\psi_x - \binom{j}{2}\psi_y + (n-j+1)[k]_*\theta_x + j[k]_*\theta_y + \Gamma^{(k)} := D_j^{m,n}(\theta) \quad (9.1)$$

Proof. We begin with the *special case* $n = 2$. Here the possible values of j are 1 and 2 and by symmetry it suffices to consider $j = 1$, where the formula reads

$$\Gamma^{(m)}.Q_1^{m,2} \sim -\psi_x + 2[k]_*\theta_x + [k]_*\theta_y + \Gamma^{(k)}. \quad (9.2)$$

Recall that $Q = Q_1^{m,2}$ is the graph of the morphism $q : (X^\theta)_T^{[k]} \to X_B^{[m]}$ given as in Lemma 8.5. Every scheme in the image of q contains the length-2 scheme along the x-axis, $(2\theta_x)$, locally defined by (y, x^2). Therefore q clearly factors through a map

$$q' : (X^\theta)_T^{[k]} \to X_B^{[2,m]}$$

to the Hilbert scheme of $(2, m)$-flags. Moreover the projection of q' to $X_B^{[2]}$ is the relatively constant map with value $(2\theta_x)$ (the unique length-2 scheme contained in the x-branch).

Now, $X_B^{[2,m]}$ carries the pullback of $-\Gamma^{(2)}$ from $X_B^{(2)}$, which clearly pulls back via q' to the cotangent space in the x direction, i.e. ψ_x. So we get an injection

$$\mathcal{O}_{Q_1^{m,2}}(-\Gamma^{(m)}) \subset \psi_x \otimes \mathcal{O}_{Q_1^{m,2}}(-\Gamma^{(k)})$$

(where $\Gamma^{(k)} = \Gamma^{(k)}_{X_T^\theta}$ denotes as above the discriminant associated to the boundary family X_T^θ). This injection is clearly an iso over the open set of subschemes of X^θ disjoint form $\theta_x \cup \theta_y$. Therefore

$$\mathcal{O}_{Q_1^{m,2}}(-\Gamma^{(m)}) = \psi_x \otimes \mathcal{O}_{Q_1^{m,2}}(-\Gamma^{(k)} - \alpha[k]_*\theta_x - \beta[k]_*\theta_y)$$

for some nonnegative integers α, β. To identify these, we can work at a general point of their support, which corresponds to a scheme with a length-3 portion near θ. By the usual support decomposition of Hilbert schemes as in §2, we are reduced to the case $m = 3$, working near a scheme of type $Q_2^3 = (y^2, x^3)$ for β or $Q_1^3 = (y^1, x^2)$ for α. Moreover, pulling back by the finite flat morphism from the ordered Hilb $X_B^{\lceil 3 \rceil}$, we are reduced to working there with the 3rd coordinate being the one from X^θ and the first two corresponding to the map to $X_B^{[2]}$ (so that $y_1 = y_2 = x_1^2 = x_2^2 = 0$).

Then finally, in the first case, the generator G_2^3 (i.e. the mixed Van der Monde) can be expanded along the last row, which shows that it maps to $y\psi_x$, therefore $\beta = 1$. In the second case, the generator G_1^3 maps to $x^2\psi_x$, so $\alpha = 2$. This completes the proof in the case $n = 2$.

Passing to the general case, recall from §7 that $Q_j^{m,n}$ is the pullback of $Q_1^{2,m}$ via the map

$$f : (X^\theta)_B^{[k]} \to (X^\theta)_B^{[m-2]}$$
$$z \mapsto z + (n-j-1)\theta_x + (j-1)\theta_y$$

Then given (9.2), the desired formula (9.1) is a consequence of following elementary formulas (recall $k = m - n$)

$$f^*([m-2]\theta_x) = [k]_*\theta_x - (n-j-1)\psi_x$$
$$f^*([m-2]_*\theta_y) = [k]_*\theta_y - (j-1)\psi_y$$
$$f^*(\Gamma^{(m-2)}) = \Gamma^{(k)} + (n-j-1)[k]_*\theta_x - \binom{n-j-1}{2}\psi_x + (j-1)[k]_*\theta_y - \binom{j-1}{2}\psi_y.$$

Note that because θ_x, θ_y map to a node of X/B, they are contained in the smooth part of X^θ/B. Then, note that f is an iterate of maps of the following form, associated to a section $\sigma : B \to Y$ of a nodal family

$$i_\sigma : Y_B^{[k]} \to Y_B^{[k+1]}$$
$$i_\sigma(z) = z + \sigma$$

To prove the above formulas, it suffices by an evident recursion to prove the following Lemma, which will conclude the proof of Lemma 9.1. □

Lemma 9.2. *For σ as above and a section σ' disjoint from σ, assume σ, σ' are contained in the smooth part of Y/B. Then we have, where $\psi_\sigma = \omega_{Y/B}|_\sigma$*

$$i_\sigma^*([k+1]_*\sigma) = [k]_*\sigma - \psi_\sigma, \quad i_\sigma^*([k+1]_*\sigma') = [k]_*\sigma'$$
$$i_\sigma^*(\Gamma^{(k+1)}) = \Gamma^{(k)} + [k]_*\sigma \tag{9.3}$$

Proof of 9.2. It suffices to prove the analogous fact on the relative symmetric product, where = becomes linear equivalence of Weil divisors. Because such linear equivalence descends via a finite flat map like the symmetrization, it suffices to prove the analogous fact over the relative Cartesian product. There, the first two assertions are obvious (keeping in mind the the image of our sections is disjoint from the nodes). The last assertion becomes obvious as well once we recall that the big diagonal on the Cartesian product is the sum of pullbacks from 2-fold products. □

We are now in position to determine the polarized structure of the node scroll $F_j^{m,n}(\theta)$. This means finding a vector bundle E such that $F_j^{m,n} = \mathbb{P}(E)$ and such that the canonical $\mathcal{O}(1)$ polarization on $\mathbb{P}(E)$ corresponds to $-\Gamma^{(m)}$. We recall (see EGA or [5], Ch. II.7 or [4] which unfortunately uses the opposite sign convention) that for any vector bundle E, there is a canonically defined (depending on E) line bundle on $\mathbb{P}(E)$, denoted $\mathcal{O}(1)$ or $\mathcal{O}_E(1)$, which restricts to the usual (Grothendieck, or quotient) $\mathcal{O}(1)$ on (geometric) fibres.

Theorem 9.3 (Node scroll theorem)**.** *For any boundary datum* (T, δ, θ), *and any* $1 \le j < n \le m$, *there is an isomorphism*

$$F_j^{m,n}(\theta) \simeq \mathbb{P}(\mathcal{O}(D_j^n(\theta)) \oplus \mathcal{O}(D_{j+1}^n(\theta))) \tag{9.4}$$

which pulls back the canonical $\mathcal{O}(1)$ *polarization on the RHS to the restriction of* $-p^*_{X_B^{[m]}}\Gamma^{(m)} + p^*_{(X_T^{\theta.})^{[m-n]}}\Gamma^{(m-n)}$ *on the LHS.*

Proof. As $F_j^{m,n}(\theta)$ admits the two disjoint sections $Q_j^{m,n}, Q_{j+1}^{m,n}$, the result is immediate from Proposition 9.1. $\square$

Corollary 9.4. *On* $F_j^{m,n}(\theta)$, *we have*

$$\begin{aligned} -\Gamma^{(m)} &\sim Q_j^{m,n} + p^*_{[m-n]}(D^n_{j+1}) \\ &\sim Q_{j+1}^{m,n} + p^*_{[m-n]}(D^n_j). \end{aligned} \tag{9.5}$$

Proof. Follows from the elementary fact that on any $\mathbb{P}^1$-bundle $\mathbb{P}(A \oplus B)$ with projection π, we have

$$c_1(\mathcal{O}(1)) \sim \mathbb{P}(A) + \pi^*(c_1(B)).$$

Indeed the natural map $\pi^*(B) \to \mathcal{O}(1)$ vanishes precisely on the divisor $\mathbb{P}(A) \subset \mathbb{P}(A \oplus B)$. $\square$

The extension to polyscrolls is direct from the definition in §8.3 once we note that thanks to the disjointness of the nodes, the divisors $D_{j_i}^{m,n_i}(\theta_i)$ correspond naturally to a similarly-denoted divisor on $(X_T^{\theta.})^{[m-|n.|]}$, with $[m-n_i]_*\theta_{x,i}$ corresponding to $[m-|n.|]_*\theta_{x,i}$, e.g. on $F_{j_1}^{n_1}(\theta_1)$,

$$p^*_{[m]}[m]_*\theta_{2,x} = p^*_{[m-n_1]}[m-n_1]_*\theta_{2,x}$$

etc. where $p_{[k]}$ denotes the natural map to the length-k Hilbert scheme (of X or X^{θ_1}) (compare Lemma 8.4).

Theorem 9.5 (Node Polyscroll Theorem)**.** *There is an isomorphism*

$$F_{j.}^{m,n.}(\theta.; X/B) \sim \prod_{(X^{\theta.}_{T(\theta.)})^{[m-|n.|]}} \mathbb{P}(\mathcal{O}(D_{j_i}^{m,n_i}(\theta_i)) \oplus \mathcal{O}(D_{j_i+1}^{m,n_i}(\theta_i))) \tag{9.6}$$

under which $-\Gamma^{(m)} + \Gamma^{(m-|n.|)}$ *corresponds to the canonical* $\mathcal{O}(1, ..., 1)$.

Proof. We use the setting and notations of §8.4. Consider the natural projection $F \to F'$, which is just a base-change of the scroll projection

$$p_{[m-n_1]} : F_1 = F_{j_1}^{m,n_1}(\theta_1, X/B) \to (X^{\theta_1})^{[m-n_1]}.$$

Via this, we have

$$\mathcal{O}_{F_1}(1) = \binom{n_1 - j_1 + 1}{2}\psi_{1,x} + \binom{j_1}{2}\psi_{1,y} - (n_1 - j_1 + 1)[m-n_1]_*(\theta_{1,x}) - j_1[m-n_1]_*\theta_{1,y}.$$

On F this becomes, using Lemma 8.4 (essentially, the disjointness of the sections θ_i),

$$\mathcal{O}_{F_1}(1)|_F =$$
$$\binom{n_1 - j_1 + 1}{2}\psi_{1,x} + \binom{j_1}{2}\psi_{1,y} - (n_1 - j_1 + 1)[m-|n.|]_*(\theta_{1,x}) - j_1[m-|n.|]_*(\theta_{1,y}).$$

and by Theorem 9.3, this coincides on F with $-\Gamma^{(m)} + \Gamma^{(m-n_1)}|_{F'}$. By induction, $-\Gamma^{(m-n_1)}|_{F'} + \Gamma^{(m-|n.|)}$ coincides with the appropriate $\mathcal{O}(1, ..., 1)$ on the $(r-1)$-polyscroll F', and the Theorem follows.

□

Remark 9.6. As mentioned in the Introduction, the paper [16] and the related software Macnodal [9] contain numerous numerical examples and applications of the Node Scroll and Polyscroll Theorems.

Remark 9.7. Define a *smudgy curve* of type g, p, k to be a nodal, p-pointed, genus-g curve together with a length-k subscheme such that the entire object has finite automorphism group, and let $\overline{\mathcal{M}}_{g,p}^{[k]}$ denote the moduli space (DM stack) of smudgy curves of this type (assuming it exists). Some interesting questions about (ordinary) curves (for example, Brill-Noether loci) can be formulated in terms of smudgy curves. The node scrolls define correspondences between smudgy moduli spaces:

$$\overline{\mathcal{M}}_{g_1,p_1+1}^{[k_1]} \times \overline{\mathcal{M}}_{g_2,p_2+1}^{[k_2]} \leftarrow \quad F_j^{m,n} \quad \to \overline{\mathcal{M}}_{g_1+g_2,p_1+p_2}^{[k_1+k_2+n]},$$

$$\overline{\mathcal{M}}_{g-1,p+2}^{[k]} \leftarrow \quad F_j^{m,n} \quad \to \overline{\mathcal{M}}_{g,p}^{[k+n]}$$

where $k = k_1 + k_2, p = p_1 + p_2, g = g_1 + g_2$ (identifying the LHS with a boundary component of $\overline{\mathcal{M}}_{g,p}^{[k]}$). These are analogous to the correspondences used by Nakajima [11] to define creation-annihilation operators on the cohomology of Hilbert schemes of surfaces.

References

1. B. Angéniol, *Familles de cycles algébriques- schéma de Chow*, LNM, vol. 896, Springer.
2. E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves*, Grundl. d. math. Wiss., vol. 267, Springer, 1985.
3. E. Cotteril, *Geometry of curves with exceptional secant planes*, Math. Z. **267** (2011), 549–582.
4. W. Fulton, *Intersection theory*, Ergeb. d. Math. u. i. Grenzgeb. 3. Folge, Bd. 2, Springer, Berlin, 1984.
5. R. Hartshorne, *Algebraic Geometry*, Springer, Berlin, 1977.
6. S. Kleiman and D. Laksov, *On the existence of special divisors*, Amer. J. Math (1972), 431–436.
7. M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. math **136** (1999), 157–207.
8. ______, *Lectures on Hilbert schemes*, CRM notes, Centre de Recherches Mathématiques, Montreal, 2004.
9. Gwoho Liu, *The macnodal package for intersection theory on hilbert schemes of nodal curves*, web interface (small jobs only) at `http://gwoho.com/macnodal/index.html.a?format=html`; source + executable + instructions at `http://math.ucr.edu/~ziv/`.
10. I. G. MacDonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.
11. H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University lecture series, Amer. Math. Soc., 1999.
12. Z. Ran, *Boundary modifications of Hodge bundles and enumerative geometry*, `http:arXiv.org/abs/1011.0406v2`.
13. ______, *Cycle map on Hilbert schemes of nodal curves*, Projective varieties with unexpected properties (Ciliberto et al., ed.), De Gruyter, Berlin, 2005, pp. 363–380.
14. ______, *Geometry on nodal curves*, Compositio math **141** (2005), 1191–1212.
15. ______, *A note on Hilbert schemes of nodal curves*, J. Algebra **292** (2005), 429–446.
16. ______, *Tautological module and intersection theory on Hilbert schemes of nodal curves*, Asian J. Math. **17** (2013), 193–264, `arxiv:0905.2229v5`.
17. E. Sernesi, *Deformations of algebraic schemes*, Grundl. d. math. Wiss., vol. 334, Springer International, Berlin, Heidelberg, 2006.
18. R. Vakil, *The moduli space of curves and Gromov-Witten theory*, Enumerative invariants in algebraic geometry and string theory (K. Behrend and M. Manetti, eds.), Springer-Verlag, 2008, `arxiv.org/math.AG/0602347`.

UC Math Dept.
Surge Facility, Big Springs Road
Riverside CA 92521 US
ziv.ran @ ucr.edu