UNIVERSAL VARIATIONS OF HODGE STRUCTURE AND CALABI-YAU-SCHOTTKY RELATIONS

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Varieties generally come to life through the maps between them and the modules (e.g. functions) that live on them. Moduli spaces are no exception to this rule. An important class of maps on moduli spaces (for compact Kähler manifolds) is that of period maps, which are substantially the same thing as the modules known as variations of Hodge structure.

The purpose of this paper is twofold. First, we give a canonical formula for the variation of Hodge structure associated to the m-th order universal deformation of an arbitrary compact Kähler manifold without vector fields. Second, we specialize to the case of a Calabi-Yau manifold X where we give a formula for the m-th differential of the period map of X and deduce formal defining equations for its image (Schottky relations); these are (necessarily infinite, in dimension ≥ 3) power series in the middle cohomology.

We will use the method of canonical infinitesimal deformations, developed by the author in earlier papers [R1, R2]. This method gives a canonical description of infinitesimal moduli spaces and, what's more, natural maps involving them. While it might be argued that a germ of a smooth space-such as the moduli of an unobstructed manifold-is a rather rigid featureless object, making a canonical description of it uninteresting, on the contrary *maps* involving such germs can be quite interesting; in the case of moduli, the method of canonical infinitesimal deformations provides a vehicle for studying such maps. For instance in the case at hand the m-th derivative of the period map of Calabi-Yau n-fold X is a filtered map

$$T_X^m M \to H_{DR}^n(X)/F^n$$

whose associated gradeds $S^i H^1(T_X) \to H_X^{n-i,i}$ are the so-called Yukawa- Green forms (cf. [G]). We will develop cohomological formulas for this and other derivative maps (Theorems 4.1, 4.3 below), which will allow us to determine their image and derive (Schottky) relations defining this image, essentially in terms of some generalized Yukawa-Green type forms (Theorem 4.1, Corollary 4.4) . For $n = 2$ we recover the celebrated 'period quadric' of $K3$ surface theory; for $n \geq 3$ the relations seem to be new. For $n = 3$ this situation is particularly interesting assuming the 'mirror conjecture' because then the higher derivatives of the period map, here computed, are related -in fact, carry equivalent information to- the 'quantum cohomology'(esp. numbers of rational curves, etc.) of the mirror of X. We hope to return to this in greater detail elsewhere.

The present methods should be applicable in other Schottky-type problems: the case of curves has been developed by G. Liu (UCR dissertation 1997).

This paper is a revised version of a manuscript entitled 'linear structure on Calabi-Yau moduli spaces' (May 1993). We are grateful to Professors P. Deligne and M. Green for some enlightening comments early on. We are especially grateful

to the referee, for his dedication above and beyond the call of duty, and for his detailed and insightful comments which have greatly improved this paper.

1. Modular coalgebra

In [R2] we gave a characterization of the dual vector space of an artin local $\mathbb{C}$-algebra as a so-called OS (Order-Symbolic) structure. This characterization was used in constructing the base ring of the universal formal deformation (e.g. of a compact complex manifold). Now as might be expected, an important role in the geometry of moduli spaces, both local and global, is played by certain canonical 'modular' sheaves and modules over them which are naturally associated to the deformation problem. Thus motivated we now extend the aforementioned dual characterization from algebras to modules. The latter give rise on the other, ' coalgebra' side to a so-called Modular Order-Symbolic or MOS structure. An important guiding principle to keep in mind is that the process of passing from a module E to the associated MOS structure $B(E)$ should involve *no dualising*, i.e. be covariant functorial in E, because we shall want to apply this when E is a cohomology group and cohomology and dualising don't mix. Thus while the algebra-OSS correspondence is essentially a matter of brute-force arrow reversal, the same is not true for module-MOSS.

Now let (R_n, m_n) be an artin local $\mathbb{C}$-algebra of exponent n (which to us means $m_n^{n+1} = 0 \neq m_n^n$) and $V = V^n = m_n^*$ the associated (standard) OS structure. It will be convenient also to consider

$$V_0 = V_0^n = R_n^* = V \oplus V_0^0$$

where $V_0^0 = \mathbb{C}1^*$ and $V_0^i = V^i \oplus V_0^0, i \geq 1$ (which might be called an augmented OSS). Given these data, a V-compatible MOS structure B consists by definition of a filtered vector space

$$B^0 \subset \cdots \cdots \subset B^n = B,$$

together with a set of symbol maps

$$\sigma_B^i : B^i \to B^i/B^0 \to V^i \otimes_{\mathbb{C}} B^{i-1}$$

satisfying the obvious 'comodule' rule, which amounts to commutativity of

$$
\begin{array}{ccc}
B^i/B^0 \xrightarrow{\ \sigma_B^i\ } & V^i \otimes B^{i-1} \xrightarrow{\ id \otimes \sigma_B^{i-1}\ } & V^i \otimes V^{i-1} \otimes B^{i-2} \\
\sigma^i \otimes id \downarrow & & \downarrow \\
S^2(V^{i-1}) \otimes B^{i-1} \longrightarrow & V^i \otimes V^{i-1} \otimes B^{i-1},
\end{array}
$$

σ^i = symbol map for V^i. A *morphism* of MOS structures is defined in the obvious way.

Example Let E be an R_n-module and set

$$B^i(E) = B_{R_n}^i(E) = V_0^i \otimes_{R_n} E, \ i = 1, \cdots n, \ B(E) = B^n(E)$$

(note indeed that V_0^i, besides being an OS structure, is also an R_n-module by the obvious rule $(r.f)(x) = f(rx)$), with symbol map induced from the comultiplication map

$$V_0^i \to V^i \to V^i \otimes V_0^i,$$

dual to $m \otimes_{\mathbb{C}} R \to m$, by tensoring with E. The assignment

$$E \longmapsto B(E)$$

is a covariant functor from R-modules to V-compatible MOS structures.

Now it is important to be able to go the other way. To this end define, for an MOS structure B,

$$C(B) = C^n(B) = \mathrm{H}om_{MOS}(V_0^n, B).$$

It is easy to check that the R_n-module structure on V_0 induces one on $C(B)$, and that C yields a functor from MOS structures to R-modules.

We summarise some relevant properties of these functors as follows.

Proposition 1.1. *(i) B is right exact and C is left exact ;*
(ii) there are natural maps

$$E \to C(B(E)), \qquad B(C(D)) \to D,$$

which are isomorphisms whenever E is free (resp. D is 'cofree', i.e. a sum of copies of V_0).

Proof. (i) is clear from the usual exactness properties of Hom and $\otimes$. As for (ii), the maps are defined by

$$e \longmapsto (v \longmapsto v \otimes e) \in Hom_{MOS}(V_0, V_0 \otimes E),$$

$$v \otimes \phi \longmapsto \phi(v), \ \phi \in Hom_{MOS}(V_0, D).$$

It is easy to check that these are well-defined etc. To complete the proof it suffices to prove that

$$Hom_{MOS}(V_0, V_0) \simeq R.$$

more precisely that the map $R \longmapsto Hom_{MOS}(V_0, V_0), r \longmapsto rI$, is a isomorphism. It is easy to see that this map is injective. For surjectivity we argue by induction. Take $\phi \in Hom_{MOS}(V_0, V_0)$. By induction there exists $r \in R_n$ inducing the same map as ϕ on V_0^{n-1}; using compatibility with comultiplication it is easy to see that r and ϕ also induce the same map on $V = V_0/V_0^0$, and consequently $\phi - rI$ is effectively a map $V_0^n/V_0^{n-1} \to V_0^0$, i.e. given by an element of $Hom_{\mathbb{C}}(m_n^{n*}, \mathbb{C}) = m_n^n$, say s. Then $\phi = (r+s)I$. $\qquad\square$

Remark 1.2 Predictably the category of $V-$ MOS also admits internal Hom and tensor product. For example if B_1, B_2 are $V-$ MOS we may define an MOS $C = B_1 \otimes_V B_2$ inductively by $C^0 = B_1^0 \otimes B_2^0$ (all tensor products over $\mathbb{C}$ unless otherwise mentioned) and setting C^i to be the preimage of $V^i \otimes C^{i-1}$ by the natural map

$$B_1^i \otimes B_2^i \to V^i \otimes V^i \otimes B^{i-1} \otimes B^{i-1}$$

(where inductively $V^i \otimes C^{i-1}$ sits naturally in $V^i \otimes V^i \otimes B^{i-1} \otimes B^{i-1}$ via (comultiplication)$\otimes$(inclusion)). Similarly we may define $sym_V^j(B)$ and $sym_V(B)$, the 'symmetric coalgebra' (or OSS) on B, which naturally admits an OS structure coming from the natural maps

$$sym_V^i(B) \to \sum_j sym_V^j(B) \otimes_V sym_V^{i-j}(B).$$

2. Subset spaces and some complexes on them

Recall that , for a (separated) topological space X, the very symmetric product $X < m >$ introduced in [R1] parametrises nonempty subsets of X of cardinality $\leq m$. We now introduce an analogous space parametrising subsets with a distinguished sub-subset. Define

$$X < m, i >= \{(S, T) : T \subset S\} \subset X < m + i > \times X < i >,$$

with the induced topology as (closed) subset; thus $X < m, i >$ is just the graph of the tautological or incidence correspondence between $X < m + i >$ and $X < i >$. Note the natural continuous surjective map

$$\pi_{m,i} : X < m > \times X < i >\to X < m, i >,$$

$$(S', S'') \longmapsto (S' \cup S'', S'').$$

For $i = 1$ is easy to see that via this map $X < m, 1 >$ may be identified topologically with the quotient of $X < m > \times X$ by the relation identifying $(S', x) \in X < m-1 > \times X \subset X < m > \times X$ with $\pi_{m,1}(S', x) = (S' \cup \{x\}, x) \in X < m - 1, 1 >\subset X < m > \times X$.

For sheaves A, B on X-say of modules over some ring which will typically be $\mathbb{C}$-we (abusively) denote by $\lambda^i A \boxtimes B$, as sheaf on $X < i, 1 >$, the direct image $\pi_{i,1*}(\lambda^i A \boxtimes B)$.

Now let g be a sheaf of Lie algebras on X. In [R1] we associated to g a Jacobi complex $J_m(g)$ on $X < m >$, an OS structure on $V^m(g) = \mathbb{H}^0(J_m(g))$, and consequently a (commutative associative) artin local algebra structure on $R_m(g) = \mathbb{C} \oplus \mathbb{H}^0(J_m(g))^*$. Note that this construction carries over essentially verbatim to the case where g is a differential graded Lie algebra (DGLA) sheaf (i.e. a 'Lie object' is the category of complexes of sheaves on X-as opposed to ordinary sheaves); in the DGLA case J_m becomes a double complex, but we shall generally identify it and other multiple complexes with the associated simple complex. See the Appendix for an interpretation of $R_m(g)$ in this case.

Now let E be a g-module. We shall associate to E a complex on $X < m, 1 >$, called the modular Jacobi complex and denoted by $J_m(g, E)$. The terms are defined by $J_m^i(g, E) = \lambda^i(g) \boxtimes E$ on $X < i, 1 >\subset X < m, 1 >$, $0 \leq i \leq m$ (where natually $X < 0, 1 >$ is the diagonal $X \subset X \times X = X < 1 > \times X$), and the differential $\partial_i : \lambda^i(g) \boxtimes E \to \lambda^{i-1}(g) \boxtimes E$ is given by the standard formula from Lie algebra homology:

$$\partial_i(v_1 \times \cdots \times v_i \times e) = \frac{1}{i!} \sum_{\sigma \in S_i} (\operatorname{sgn} \sigma)[v_{\sigma(j)}, v_{\sigma(k)}]v_{\sigma(1)} \times \cdots \times \hat{v}_{\sigma(j)} \times \cdots \times \hat{v}_{\sigma(k)} \times \cdots \times v_{\sigma(i)} \times e$$

$$+ \frac{1}{i} \sum_{j=1}^{i} (-1)^j v_1 \times \cdots \times \hat{v}_j \times \cdots \times v_i \times v_j(e).$$

Now we have a natural map

$$\phi : X < m > \times X < m - 1, 1 >\to X < 2m, 1 >$$

$$(S, (S', x)) \longmapsto (S \cup S', x)$$

and an evident map of complexes

$$(J_m/J_1)(g, E) \to \phi_*(J_m(g) \boxtimes J_{m-1}(g, E))$$

given by the natural sheaf maps, analogous to the map $\wedge^i(V) \to \oplus(\wedge^{i-k}(V) \otimes \wedge^k(V))$ for a vector space V

$$\lambda^i(g) \boxtimes E \to \overset{i}{\underset{k=0}{\oplus}} \lambda^{i-k}(g) \boxtimes (\lambda^k(g) \boxtimes E)$$

(it is easy to check that this is compatible with differentials). Now let $p : X < m, 1 > \to X$ be the natural map. Then by the above the sheaf $V^m(E) := \mathbb{R}^0 p_*(J_m(g, E))$ on X is endowed with an MOS structure with respect to the (constant) OS structure $\mathbb{H}^0(J_m(g))$, whence a sheaf of $R_m(g)$-modules $C^m(V^m(E))$ which we denote by $M_m(g, E)$. Note that the assignment $E \to M_m(g, E)$ is a covariant functor from g-modules to $R_m(g)$-modules.

3. Universal variation of Hodge structure

Let X be a compact complex manifold with tangent sheaf T so that $H^0(T) = 0$. In [R1] we constructed the universal formal deformation

$$\hat{X}^u / \hat{R}^u = \lim_m X_m^u / R_m^u \,,$$

where $R_m^u = R_m(T)$ is the algebra associated to the OS structure $\mathbb{H}^0(J_m(T))$. More generally if g is a sheaf of $\mathbb{C}$-Lie algebra over a topological space X with $H^0(g) = 0$and E is a sheaf of faithful g-modules, we have constructed the universal $m-$th order g-deformation E_m of E over $R_m(g)$, the algebra associated to $\mathbb{H}^0(J_m(g))$. A first point to be made is the the MOS viewpoint permits a more 'conceptual' interpretation of this construction (I am grateful to the referee for his insistence that this be explained in detail).

Theorem 3.1. *E_m is canonically isomorphic to $M_m(g, E)$.*

Proof. Let $(E^\cdot, \partial)$ be a soft resolution of E and $(g^\cdot, \delta)$ be a soft resolution of g where $g^\cdot$ is a dgla acting on $E^\cdot$ compatibly with the g-action on E. Applying suitable Schur functors as in [FH], $g^\cdot$ induces a soft resolution of $\lambda^i(g)$ which we denote by $(g^\cdot_{-i}, \delta^\cdot_{-i})$, whence also a soft resolution of $J_m(g, E)$, which may be used to compute $p_*(J_m(g, E))$, yielding a complex $(K^\cdot, d^\cdot)$ on X where

$$K^r = \sum_{i,j} \Gamma(g^i_{i+j+r}) \otimes E^i, i \geq -m.$$

Now note that, because $H^0(g) = 0$, we have $H^j(\lambda^i(g)) = 0, j < i$, hence $K^\cdot$ is acyclic in negative degrees. More precisely, we may 'cancel off' the negative part of $K^\cdot$ step-by-step as follows. First d^{-m} is a map

$$\Gamma(g^0_{-m}) \otimes E^0 \to \Gamma(g^1_{-m}) \otimes E^0 \oplus \Gamma(g^0_{-m}) \otimes E^1 \oplus \Gamma(g^0_{-m+1}) \otimes E^0$$

whose first component $\delta^0_{-m} \otimes id_{E^0}$ is already injective. Consequently $K^\cdot$ is quasi-isomorphic to a complex $K_{(1)}$ where

$$K^{-m}_{(1)} = 0, K^{-m+1}_{(1)} = (\Gamma(g^1_{-m})/B^1(g^\cdot_{-m})) \otimes E^0 \oplus \Gamma(g^0_{-m}) \otimes E^1 \oplus \Gamma(g^0_{-m+1}) \otimes E^0,$$

and $K^i_{(1)} = K^i, i > -m+1$. Here $B^\cdot$ denotes coboundaries. To be precise $\Gamma(g^1_{-m})/B^1(g^\cdot_{-m})$ is to be thought of as a subspace of $\Gamma(g^1_{-m})$ complementary to $B^1(g^\cdot_{-m})$. Now the map induced by $\delta^1_{-m} \otimes id_{E^0}$ on $(\Gamma(g^0_{-m})/B^1(g^\cdot_{-m})) \otimes E^0$ is still injective if $m \geq 2$

(again thanks to $H^0(g) = 0$), as are $\delta^0_{-m} \otimes id_{E^1}$ and $\delta^0_{-m+1} \otimes idE^0$, hence $K^{\cdot}_{(1)}$ is in turn quasi-isomorphic to a complex $K^{\cdot}_{(2)}$ in degrees $\geq -m+2$ with

$$K^{-m+2}_{(2)} = (\Gamma(g^2_{-m})/B^2(g_{-m})) \otimes E^0 \oplus (\Gamma(g^1_{-m})/B^1(G^{\cdot}_{-m})) \otimes E^1$$

$$\oplus (\Gamma(g^1_{-m+1})/B^1(g^{\cdot}_{-m+1})) \otimes E^0 \oplus ...$$

Continuing in this manner, we obtain after m steps a complex $(K_{(m)}, d_{(m)})$ in nonnegative degrees where $K^0_{(m)}$ is of the form $\sum_{i \geq 0} G_i \otimes E^i$ for certain vector spaces G^i. Now it is easy to see as above that for $i > 0$ the map $\delta \otimes id_{E^i}$ which is a component of $d^0_{(m)}$ is injective. For $i = 0$, G_0 will be a certain quotient of $\sum_0^m sym^j(\Gamma(g^1))$ and among the components of d^0_m on $G^0 \otimes E^0$, one lands in $\Gamma(g^2) \otimes \sum_1^m sym^{j-1}(\Gamma(g^1)) \otimes E^0$ (map induced by (δ plus graded bracket)$\otimes id_{E^0}$). By definition, the kernel of this map is precisely $V^m_0 \otimes E^0 = (\mathbb{C} \oplus \mathbb{H}^0(J_m(g))) \otimes E^0$ and consequently $K_{(m)}$ is quasi isomorphic to a complex $K_{(m+1)}$ in nonnegative degrees with $K^0_{(m+1)} = V^m_0 \otimes E^0$ and, say, $K^1_{(m+1)} = \sum H^i \otimes E^i$ for certain vector spaces H^i. By similar considerations $K_{(m+1)}$ is quasi- isomorphic to a complex $(L^{\cdot}, \Delta^{\cdot})$ which starts

$$V^m_0 \otimes E^0 \rightarrow V^m_0 \otimes E^1 \oplus L^1_0 \rightarrow ...$$

and where Δ^0 which is induced by $id \otimes \partial^0 +$ (map induced by g-action) clearly goes into $V^m_0 \otimes E^1$. Thus $\mathbb{R}^0 p_* J_m(g, E)$ is simply given by the kernel of Δ^0. In light of Proposition 1.1 and its proof, $M_m(g, E)$ coincides with the kernel of a map

$$R_m(g) \otimes E^0 \rightarrow R_m(g) \otimes E^1$$

given by $id \otimes \partial^0 +$ (map induced by action), which is precisely the definition of E_m. $\hspace{2cm} \square$

For later use we record a corollary of the construction (which follows easily from the fact that cocycles are locally coboundaries)

Corollary 3.2. $V^m(E)$ *is locally isomorphic to* $V^m_0 \otimes E$.

Our purpose now is to extend this construction to the De Rham complex of X and consequently to obtain, for X Kählerian, a construction of the universal variation of Hodge structure associated to X, which is a (cohomology) vector bundle over $\text{Spec}(\hat{R}^u)$ together with a (Hodge) filtration and a (Gauss-Manin) trivialization or flat structure.

Consider the De Rham complex of X

$$\Omega^{\cdot} : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \rightarrow \cdots \rightarrow \Omega^n_X.$$

As is well known, T acts on this via Lie derivative $v \times \omega \longmapsto L_v(\omega)$, an action which commutes with exterior derivative and is a derivation with respect to wedge products. Consequently, we have an associated modular Jacobi (bi)complex $J_m(T, \Omega^{\cdot})$ on $X < m, 1 >$ which we call the Jacobi-De Rham complex of X. This gives rise to a V^m- compatible MOS structure $\mathbb{R}^0 p_* J_m(T, \Omega^{\cdot})$ (i.e a complex of sheaves of such), whence a complex of $R^u_m = R_m(T)$-modules

$$\Omega^{\cdot}_m := M_m(T, \Omega^{\cdot}) : M_m(T, \mathcal{O}) = \mathcal{O}_m \rightarrow M_m(T, \Omega^1) \rightarrow \cdots.$$

As remarked above, $\mathcal{O}_m$ coincide with $\mathcal{O}_{X_m}$, the structure sheaf of the m-universal deformation X^u_m/R^u_m. This identification may be extended as follows.

Theorem 3.3. *In the above situation, $\Omega_m^{\cdot}$ is canonically isomorphic to the relative De Rham complex $\Omega_{X_m/R_m}^{\cdot}$.*

Proof. For $\cdot = 0$ we're OK, so consider the case $\cdot = 1$. The T-linear derivation $d : \mathcal{O} \to \Omega^1$ gives rise by functoriality to an R_m^u-linear derivation $\mathcal{O}_m \to \Omega_m^1$, whence an $\mathcal{O}_m$-linear map

$$\phi_m^1 : \Omega_{X_m/R_m}^1 \to \Omega_m^1 .$$

Clearly $\phi_m^1 \otimes_{R_m} \mathbb{C} = \phi_0^1$ is the identity. Moreover it is easy to see that both sheaves are locally $\mathcal{O}_m$-free: for Ω_{X_m/R_m}^1 this is obvious and for Ω_m^1 it follows from the fact that it is R_m-flat by construction (indeed a local basis for Ω^1 yields a map $n\mathcal{O}_m \to \Omega_m^1$ which is clearly surjective and whose kernel yields 0 when tensored with $\mathbb{C}$, hence is 0 by Nakayama). Consequently, a suitable $n \times n$ matrix representing ϕ_m^1 will have the form $I_n +$ (nilpotent), hence is invertible, so ϕ_m^1 is an isomorphism. Now in general, for $\cdot = i$ the derivation property of the Lie derivative action of T on $\Omega_X^{\cdot}$ makes $\Omega_m^{\cdot}$ an algebra under wedge products, whence an $\mathcal{O}_m$-linear map

$$\phi_m^i = \wedge^i(\phi_m^1) : \wedge_{\mathcal{O}_m}^i (\Omega_{X_m/R_m}^1) \to \Omega_m^i.$$

As both sheaves are locally $\mathcal{O}_m$-free and ϕ_0^i is the identity, it follows as above that ϕ_m^i is an isomorphism. Compatibility of $\phi_m^{\cdot}$ with d -i.e, $d \circ \phi_m^i = \phi_m^{i+1} \circ d$- follows easily from the fact that this holds for $i = 0$ (by construction) and the multiplicativity of $\phi_m^{\cdot}$. This completes the proof. $\square$

Given Theorem 3.2 it is clear in principle that the De Rham cohomology of X_m/R_m is readable from the cohomology of the Jacobi-DeRham complex and we shall now make this explicit. Consider the free R_m-module

$$H_{DR}^r(X_m/R_m) = \mathbb{H}^r(X, \Omega_{X_m/R_m}^{\cdot}) .$$

This is endowed with a Hodge filtration $F^{\cdot}$, which is induced by the stupid filtration on the complex $\Omega_{X_m/R_m}^{\cdot}$, as well as a (Gauss-Manin) isomorphism $H_{DR}^r(X_m/R_m) \simeq H^r(X, R_m) \simeq H^r(X, \mathbb{C}) \otimes R_m$ induced by the (quasi-isomorphic) inclusion $R_m \to \Omega_{X_m/R_m}^{\cdot}$. When X is Kählerian we have by Deligne [D] that $F^i/F^{i+1} \cong H^{n-i}(\Omega_{X_m/R_m}^i)$, a free R_m-module. The filtered module $(H_{DR}^r(X_m/R_m), F^{\cdot})$ may be called the m-universal variation of Hodge structure (m-UVHS) associated to X. Our goal is to give an explicit formula for it, together with the Gauss-Manin isomorphism, in terms of X itself.

Theorem 3.4. *Let X_m/R_m be the m-universal deformation of a compact kähler manifold X with $H^0(T) = 0$. Then: (i) the Leray spectral sequence*

$$E_2^{p,q} = \mathbb{H}^p(X, R^q p_* J_m(T, \Omega^{\cdot})) \Rightarrow \mathbb{H}^{p+q}(X < m, 1 >, J_m(T, \Omega^{\cdot}))$$

degenerates at E_2;
   (ii) we have R_m-linear, Hodge filtration-preserving isomorphisms

$$E_2^{r,0} = E_\infty^{r,0} = \mathbb{H}^r(X, B^m \Omega_{X_m/R_m}^{\cdot}) = B^m \mathbb{H}^r(X, \Omega_{X_m/R_m}^{\cdot}) \qquad (3.1)$$

$$\begin{aligned} \mathbb{H}^r(X, \Omega_{X_m/R_m}^{\cdot}) &\cong C^m \mathbb{H}^r(X, \mathbb{R}^0 p_* J_m(T, \Omega^{\cdot})) \qquad (3.2) \\ &\subset C^m \mathbb{H}^r(X < m, 1 >, J_m(T, \Omega^{\cdot})) ; \end{aligned}$$

   (iii) the Jacobi-Hodge-De Rham spectral sequence

$$E_1^{p,q} = \mathbb{H}^p(X, \mathbb{R}^0 p_* J_m(T, \Omega^q)) \Rightarrow \mathbb{H}^{p+q}(X, \mathbb{R}^0 p_* J_m(T, \Omega^{\cdot}))$$

degenerates at E_1 ;
     (iv) the Gauss-Manin isomorphism

$$GM_m : H^r_{DR}(X_m/R_m) \to H^r_{DR}(X) \otimes R_m$$

is adjoint to a map $GM_m : B^m H^r_{DR}(X_m/R_m) \to H^r_{DR}(X) \otimes [\mathbb{C} \oplus \mathbb{H}^0(J_m(T))]$ induced by a map of complexes

$$M_m : J_m(T, \Omega^{\cdot}) \to J_m(T, \Omega^{\cdot}_{triv}),$$

$\Omega^{\cdot}_{triv} = \Omega^{\cdot}$ with trivial T-action, which respects the Hodge filtration up to a shift of m, i.e. $M_m(F^i) \subset F^{i-m}$, hence $GM_m(F^i) \subset F^{i-m}$.

Proof. To begin with, the inclusion

$$\mathbb{C}_X \to \Omega^{\cdot}$$

is , by the Poincaré lemma, a T-linear isomorphism (with the trivial action of T on $\mathbb{C}_X$). Now recall the finite-to-one map

$$\pi : X < m > \times X \to X < m, 1 >$$

(cf. section 2) and let X be embedded diagonally $X \hookrightarrow X \times X = X < 1 > \times X \subset X < m > \times X$. Then it follows that

$$J_m(T, \Omega^{\cdot}) \underset{qis}{\sim} J_m(T, \mathbb{C}) \underset{qis}{\sim} \pi_*(p_1^* J_m(T) \oplus p_2^* \mathbb{C}_X).$$

This clearly implies the degeneration assertion of (i). As for (ii), we have proven in Theorem 3.1 that

$$\Omega^{\cdot}_{X_m/R_m} \simeq C^m(\mathbb{R}^0 p_* J_m(T, \Omega^{\cdot})).$$

whence a map

$$\mathbb{R}^0 p_* J_m(T, \Omega^{\cdot}) \to B^m \Omega^{\cdot}_{X_m/R_m} \simeq B^m C^m \mathbb{R}^0 p_* J_m(T, \Omega^{\cdot}).$$

It is easy to see that this is an isomorphism for $m = 0$, and both sides are V^m-coflat, locally a sum of copies of $V_0^m \otimes \mathcal{O}_X$, hence this is an isomorphism. On the other hand $\Omega^{\cdot}_{X_m/R_m}$ being R_m-flat we have

$$\mathbb{H}^r(X, \mathbb{R}^0 p_* J_m(T, \Omega^{\cdot})) = \mathbb{H}^r(X, B^m \Omega^{\cdot}_{X_m/R_m}) = B^m \mathbb{H}^r(X, \Omega^{\cdot}_{X_m/R_m}).$$

This proves (3.1). Applying C^m to both side and using R_m-freeness of $\mathbb{H}^r(X, \Omega^{\cdot}_{X_m/R_m})$, (3.2) follows too. Given (ii), (iii) follows easily from degeneration of the usual (relative) Hodge-De Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X_m/R_m}) \Rightarrow H^{p+q}_{DR}(X_m/R_m).$$

Finally for (iv)-which is really the main point-we shall construct an explicit isomorphism based on interior multiplication. For local vector fields $v_1, \cdots, v_k$ we let

$$i_{v_1 \cdots v_k} : \Omega^i \to \Omega^{i-k}$$

denote interior multiplication by $v_1 \times \cdots \times v_k$ (which coincides with $i_{v_1} \circ \cdots \circ i_{v_k}$). Recall Cartan's formula for the Lie derivative action of T on $\Omega^{\cdot}$:

$$L_v(\omega) = i_v(d(\omega)) + d(i_v(\omega)).$$

Now let us define a map M, preserving total degree

$$\begin{aligned} M = M^{\cdots} &: J_m(T, \Omega^{\cdot}) \quad \to J_m(T, \Omega^{\cdot}_{triv}) & (3.3)\\ M^{i,j,k} &: \lambda^j T \boxtimes \Omega^i \to \lambda^{j-k} T \boxtimes \Omega^{i-k}, k \geq 0, \end{aligned}$$

$$M^{i,j,k}(v_1\times\cdots\times v_j\times\omega)=\pm\sum(-1)^{r_1+\cdots+r_k}v_1\times\cdots\times\hat{v}_{r_1}\times\cdots\times\hat{v}_{r_k}\times\cdots\times v_j\times i_{v_{r_1}\cdots v_{r_k}}(\omega)\ .$$

In other words M is the map whose restriction on $\lambda^j T\boxtimes\Omega^i$ is given by $\oplus_{k\geq0}M^{i,j,k}$; note that each $M^{i,j,0}$ is the 'identity'. The Cartan formula shows that M is a morphism of complexes, whence a map

$$
\begin{aligned}
{}^tGM_m:\mathbb{H}^r(X,\mathbb{R}^0p_*J_m(T,\Omega^{\cdot}))\ &\to\ \mathbb{H}^r(X,\mathbb{R}^0p_*J_m(T,\Omega^{\cdot}_{triv}))\\
&=\ \mathbb{H}^r(\Omega^{\cdot}_X)\otimes[\mathbb{C}\oplus\mathbb{H}^0(J_m(T))].
\end{aligned}
$$

In view of the commutative triangle

$$
\begin{array}{ccc}
& J_m(T,\mathbb{C}_X) & \\
\wr\swarrow & & \searrow\wr \\
J_m(T,\Omega^{\cdot})\quad\xrightarrow{\ \ M\ \ } & & \quad J_m(T,\Omega^{\cdot}_{triv})
\end{array}
$$

${}^tGM_m$ is an isomorphism, and it is obvious that it decreases Hodge level by at most m. Applying the C^m functor now suffices to conclude. Note that by construction the inverse of our GM isomorphism is induced by the quasi-isomorphism $R_m\to\Omega^{\cdot}_{X_m/R_m}$; the same is obviously true of the 'usual' Gauss-Manin in any way it is defined (e.g. [K]). Thus our GM coincides with the usual.

$\square$

The following discussion concerning cohomology and obstructions was suggested by some recent work of Clemens [C]. Consider the exact sequence

$$0\to m_m^m\otimes\Omega^p\to\Omega_m^p\to\Omega_{m-1}^p\to0.$$

By [D] (assuming X Kählerian) this induces an exact sequence

$$0\to m_m^m\otimes H^q(\Omega^p)\to H^q(\Omega_m^p)\to H^q(\Omega_{m-1}^p)\to0,$$

i.e. the coboundary map $\partial_m:H^q(\Omega_{m-1}^p)\to m_m^m\otimes H^{q+1}(\Omega^p)$ vanishes, and $H^q(\Omega_m^p)$ is R_m^u-free. It is easy to see that the vanishing of ∂_{m-1} implies a priori that ∂_m factors through a map $H^q(\Omega^p)\to m_m^m\otimes H^{q+1}(\Omega^p)$. 'Dually', let $K^m\subset sym^mH^1(T)$ be the kernel of the m-th order obstruction map $ob_m:sym^mH^1(T)\to\mathbb{H}^1(J_{m-1}(T))$ (cf. [R2],2.3). Alternatively, K^m may also be defined inductively as the kernel of an obstruction map $ob_m:K^{m-1}.H^1(T)\to H^2(T)$, where $K^{m-1}.H^1(T)$ denotes the intersection of $sym^mH^1(T)$ and $K^{m-1}\otimes H^1(T)$ in $\otimes^mH^1(T)$. Then the natural map

$$K^m\otimes H^q(\Omega^p)\to H^{q+1}(\Omega^p),$$

which represents the obstruction to lifting (p,q)-cohomology over the universal m-th order deformation, vanishes. Equivalently, the tautological map $J_m(T,\Omega^p)\to\lambda^m(T)\boxtimes\Omega^p[m]$ induces a map $V^m(\Omega^p)\to sym^mH^1(T)\otimes\Omega^p$ whose image coincides with $K^m\otimes\Omega^p$ (cf. Corollary 3.2). Then the induced map on cohomology

$$H^q(V^m(\Omega^p))\to K^m\otimes H^q(\Omega^p)$$

is surjective, i.e the coboundary or obstruction map

$$K^m\otimes H^q(\Omega^p)\to H^{q+1}(V^{m-1}(\Omega^p))$$

vanishes (and moreover $H^q(V^m(\Omega^p))$ is cofree). This follows from [D] together with a trivial 'Universal coefficient theorem' which says that $H^q(B^m(\Omega_m^p))=B^m(H^q(\Omega_m^p))$ provided $H^{q-1}(\Omega_m^p)$ is free; alternatively one can simply mimic Deligne's semicontinuity argument for $V^m(\Omega^p)$ in place of Ω_m^p.

Now recall the calculus fact (already used in [R3]):

$$i([x,y])(\omega) = L_x(i(y)\omega) - L_y(i(x)\omega) - d(i(x \wedge y)\omega) + i(x \wedge y)(d\omega).(*)$$

Define a map

$$i : J_m(T) \boxtimes \Omega^p \to J_{m-1}(T, \Omega^{p-1})[1]$$

by the formula

$$(v_1, ..., v_k, \omega) \mapsto \sum (-1)^j (v_1, ...\hat{v}_j, ..., v_k, i(v_j)\omega).$$

Then (*) shows, assuming X Kähler, that i is a 'weak' morphism of complexes, in the sense that the appropriate diagrams commute in the derived category. This is good enough to induce a map on cohomology

$$V^m \otimes H^q(\Omega^p) \to H^q(V^{m-1}(\Omega^{p-1}))$$

and similarly for $m - 1$, hence a commutative diagram

$$
\begin{array}{ccc}
sym^m H^1(T) \otimes H^q(\Omega^p) & \longrightarrow & sym^{m-1} H^1(T) \otimes H^{q+1}(\Omega^{p-1}) \\
\downarrow & & \downarrow \\
\mathbb{H}^1(J_{m-1}(T)) \otimes H^q(\Omega^p) & \longrightarrow & H^{q+2}(V^{m-2}(\Omega^{p-1}))
\end{array}
$$

which induces

$$
\begin{array}{ccc}
K^{m-1}.H^1(T) \otimes H^q(\Omega^p) & \longrightarrow & K^{m-1} \otimes H^{q+1}(\Omega^{p-1}) \\
\downarrow & & \downarrow \\
H^2(T) \otimes H^q(\Omega^p) & \longrightarrow & H^{q+2}(\Omega^{p-1})
\end{array}
$$

As we have seen, the right vertical map vanishes, and we conclude the following which was first proven by Clemens [C] by another method:

Corollary 3.5. *(Clemens) For X Kählerian, the map $H^2(T) \to \mathrm{Hom}(H^q(\Omega^p), H^{q+2}(\Omega^{p-1}))$ vanishes on the image of the m-th order obstruction map in $H^2(T)$, for all m.*

Remark. A similar result, also due to Clemens, holds in the 'relative' situation, with X replaced by a pair $(Y \subset X)$ of manifolds, T by $T_{X/Y}$ (see Appendix) and $\Omega_X^{\cdot}$ by (the mapping cone of) $\Omega_X^{\cdot} \to \Omega_Y^{\cdot}$. The above proof still applies.

It is instructive to rewrite some of the conclusion of Theorem 3.4 in traditional geometric language. Thus $H_m^r = H_{DR}^r(X_m/R_m)$ may be viewed as a geometric vector bundle-viz. $Spec(Sym(H_m^{r*}))$-over $Spec(R_m) =: U_m$ and the Gauss-Manin isomorphism yields a commutative triangle:

$$
\begin{array}{ccc}
H_m^r & \overset{\simeq}{} & U_m \times H^r \\
& \searrow \quad \swarrow & \\
& U_m &
\end{array}
\tag{3.4}
$$

where $H^r = H_0^r = H_{DR}^r(X)$. The geometric vector bundles H_m^r and $U_m \times H^r$ correspond to OS structures over V_m which are the 'symmetric coalgebra' over V^m (see Remark 1.2) on the MOS structures $H^r(\mathbb{R}^0 p_* J_m(T, \Omega^{\cdot}))$ and $H^r(\mathbb{R}^0 p_* J_m(T, \Omega_{triv}))$, respectively, the latter being evidently isomorphic to $V^m \otimes S^{\cdot}(H)$, where $S^{\cdot}(H)$, the

symmmetric coalgebra (= algebra) on H over $\mathbb{C}$ is endowed with the evident OS structure (e.g. via its duality with $S^{\cdot}(H^*)$). Now the Hodge subbundle $F_m^i \subset H_m^r$, together with the trivialization (3.4), give rise to a morphism to a Grassmannian

$$p_m : U_m \to \mathrm{Grass}(\dim F_0^i, H^r) \; ,$$

which is none other than the m-th order germ of the period map associated to (X, H^r, T^i). Thus Theorem 3.2 yields, in principle, an explicit formula for p_m. In the next section we shall make this concrete in the case of Calabi-Yau manifolds.

4. The Calabi-Yau case

In this section we fix a Calabi-Yau manifold X, i.e. an n-dimensional compact Kähler manifold admitting a nowhere-vanishing n-form Φ (which is then unique up to a constant), and such that $H^0(T) = 0$. We call such a pair (X, Φ) a *measured* Calabi-Yau manifold (MCYM). A (local) isomorphism f between MCYM's $(X, \Phi), (X', \Phi')$ is supposed to preserve the n-form, i.e. $f^*\Phi' = \Phi$. Thus the Lie algebra sheaf of infinitesimal automorphisms of a MCYM (X, Φ) may be identified as the subset $\hat{T} \subset T$ of divergence-free vector fields v, i.e. those with $L_v(\Phi) = 0$. The inclusion of Lie algebras $\hat{T} \subset T$ corresponds to an inclusion of rings

$$R_m = R_m(T) \subset \hat{R}_m = R_m(\hat{T}).$$

Indeed it is easy to see that $\hat{R}_m = R_m[t]/(\underline{m}, t)^{m+1}$. The universal deformation over $\hat{R}_m$ is just $X_m^n \times_{R_m} \hat{R}_m$. The advantage of $\hat{R}_m$ is that the pullback of the cohomology bundle $H_m^n = H_{DR}^n(X_m^u/R_m)$ over $\hat{U}_m = \mathrm{Spec}(\hat{R}_m)$ admits a 'tautological' section: namely that corresponding to the map

$$J_m(\Phi) : J_m(\hat{T}, \mathbb{C}[-n]) \to J_m(\hat{T}, \Omega^{\cdot})$$

induced by the $\hat{T}$-linear (!) map $\mathbb{C}[-n] \xrightarrow{\Phi} \Omega^{\cdot}$. To be precise, there is an inclusion (of a direct summand)

$$\pi_* p_1^* J_m(\hat{T})[-n] \hookrightarrow J_m(\hat{T}, \mathbb{C}[-n])$$

(indeed $\pi_* p_1^* J_m(\hat{T})[-n]$ is identical with the part of $J_m(\hat{T}, \mathbb{C}[-n])$ in negative degrees and this part forms a sub, as well as quotient, complex because the differential in degree -1, given by the action, vanishes). This inclusion induces

$$[\Phi] : \mathbb{H}^0(J_m(\hat{T})) \to \mathbb{H}^n(\mathbb{R}^0 p_* J_m(\hat{T}, \Omega^{\cdot})) \; ,$$

which corresponds to Φ as a cross-section of the geometric vector bundle corresponding to $H_m^n \otimes \hat{R}_m$. On the ring level this corresponds to the $\hat{R}_m$-algebra homomorphism

$$S_{\hat{R}_m}^{\cdot}((H_m^n)^* \otimes \hat{R}_m) \to \hat{R}_m$$

given by the composite (where $S^{\cdot} = S_{\mathbb{C}}^{\cdot}$)

$$S^{\cdot}((H_m^n)^*) \to S^{\cdot}(\hat{R}_m) \to \hat{R}_m$$

where the first map is $S^{\cdot}[\Phi]^*$ and the second is $\hat{R}_m$-multiplication. Now we may follow this by the map induced by the Gauss- Manin isomorphism $H_m^n \simeq H^n \otimes R_m$ (3.4), then project to the H^n factor. The map thus obtained

$$\hat{p}_m : \hat{S}_m \to H^n =: H$$

is the m-th order period map associated to the MCYM (X, Φ). This is related to the usual period map p_m by the diagram

$$
\begin{array}{ccc}
\hat{U}_m & \xrightarrow{\hat{p_m}} & H\backslash\{0\} \\
\downarrow & & \downarrow \\
U_m & \xrightarrow{p_m} & \mathbb{P}(H).
\end{array}
$$

Now $\hat{p}_m$ corresponds to a homomorphism $\hat{p}_m^* : S^{\cdot}(H^*) \to \hat{R}_m$, which is obviously determined by its restriction $\hat{p}_m^{1*}$ on H^*, the linear functions on H. Tracing through the construction of the Gauss-Manin isomorphism in Theorem 3.2, we conclude the following formula for the dual $\hat{p}_m^1$ of $\hat{p}_m^{1*}$:

Theorem 4.1. *The map $\hat{p}_m^1 : \mathbb{H}^0(J_m(\hat{T})) \to H_{DR}^n(X)$ is given by $\mathbb{H}^0(j_m)$ where $j_m : J_m(\hat{T}) \to \Omega^{\cdot}[n]$ is defined by $j_m(v_1 \times \cdots \times v_k) = i_{v_1 \cdots v_k}(\Phi)$.*

Proof. Our map is given as the composite of three maps. First, embedding $J_m(\hat{T})$ in the negative-degree portion of the n-th column (viz. $J_m(\hat{T}, \Omega^n)$) of the double complex $J_m(\hat{T}, \Omega^{\cdot})$ through multiplication by Φ. Second, applying our explicit Gauss-Manin (interior multiplication) operator M constructed in the proof of Theorem 3.2. Third, projecting to the zeroth row, which is just given by $\Omega^{\cdot}$. From our formula for M (3.3), is is plain that this map is given by $\mathbb{H}^0(j_m)$. $\square$

We shall now use this description of the period map to describe a set of defining equations for its image (Schottky relations). For convenience, set $\hat{T}^m = \mathbb{H}^0(J_m(\hat{T}))$, $\underline{m}_m = \hat{T}^{m*}$, the maximal ideal of $\hat{R}_m$. To mirror the adic filtration on $\hat{R}_m$ we define a slight modification $\hat{F}$ of the Hodge filtration on H (which we identify with its dual H^* by Poincaré duality) by

$$
\begin{aligned}
\hat{F}^i H &= \quad H \quad & i = 1 \\
&= \quad F^i H \quad & i \neq 1.
\end{aligned}
$$

$\hat{F}$ naturally induces a filtration on the symmetric algebra $S(H)$, ($=$ unique filtration such that the level of $ab = $ (level of a) $+$ (level of b)) and we denote by S_m its m-th quotient $S(H)/\hat{F}^{m+1}S(H)$. Note that e.g. by Theorem 4.1, $\hat{p}_m^{1*}$ on H takes $\hat{F}$ into the adic filtration on $\hat{R}_m$. As $\hat{F}$ was extended 'multiplicatively' to $S^{\cdot}(H)$, $\hat{p}_m^*$ also takes $\hat{F}$ into the adic filtration, yielding a map $S_m \to \hat{R}_m$. It is also clear that $\hat{p}_m^*$ induces an isomorphism

$$
gr_{\hat{F}}^1 S(H) = H/F^2 H \simeq gr^1(\hat{R}_m) = \hat{T}^{1*} . \tag{4.0}
$$

It follows firstly that $\hat{p}_m^*$ is surjective('local Torelli') but also, more significantly that for any $a \in \hat{F}^i H$ we may choose $b \in \oplus_{j=i}^n S^j H$ so that

$$
\hat{p}_n^*(a) = \hat{p}_n^*(b) \in \underline{m}_n^i \subset \hat{R}_n . \tag{4.1}
$$

Indeed writing $b = \sum_i^n b_j$ we may firstly choose $b_i \in S^i(H)$ by (4.0) so that $\hat{p}_n^*(a - b_i) \in \underline{m}^{i+1}$, then choose $b_{i+1} \in S^{i+1}(H)$ so that $\hat{p}_n^*(a - b_i - b_{i+1}) \in \underline{m}^{i+1}$, etc. We let Y_n denote the set of elements $a - b$ of this kind as a ranges over (a basis of) $F^i H$, $i = 2, \cdots, n$. These are our 'Schottky relations' of order n; they are essentially a refined version of the well-known Yukawa forms

$$
\eta^i : S^i(H^{n-1,1}) \to H^{n-i,i}
$$

in that a defines a linear form on $H^{n-i,i}$ and $a \circ \eta^i$ is 'given' by (the degree-i part of) b as linear form on $S^i(H^{n-1,1})$. Next we define our Schottky relations of higher order. For each $m \geq n$ we define by induction a lift Y_m of Y_n,

$$Y_m \subset I_m = ker(S_m \to \hat{R}_m) \ ,$$

as follows. For any $y \in I_m$ let $y' \in S_{m+1}$ be an arbitrary lift of y. Thus $\hat{p}^*_{m+1}(y') \in \hat{\underline{m}}^{m+1}_{m+1}$ may be expressed as a polynomial of degree $m+1$ in $\hat{T}^{1*}$, which may be represented by an element $z \in S^{m+1}(H)$ and we clearly have $p^*_{m+1}(z) = p^*_{m+1}(y\prime)$. Now set

$$y'' = y' - z \in I_{m+1} \ ,$$
$$Y_{m+1} = \{y'' : y \in Y_m\} \ .$$

Theorem 4.2. *For $m \geq n$, p^*_m induces an isomorphism*

$$S_m / < Y_m > \simeq \hat{R}_m \ .$$

Proof. We know the induced map is well defined, surjective and takes $\hat{F}$ into the adic filtration. To prove injectivity it will suffice to show p^*_m induces an isomorphism, for each i,

$$gr^i(\hat{p}^*_m) : gr^i(S_m / < Y_m >) \to gr^i(\hat{R}) = S^i(\hat{T}^{1*}) \ .$$

Given $a \in \hat{F}^i S_m$, it is clear that we may find $b \in S^i(H)$ with $a - b \in < Y_m >$. But note the exact diagram

$$\hat{F}^{i+1} S_m \ \longrightarrow \ S^i(H) \ \longrightarrow \ S^i(H/F^2 H)$$
$$\downarrow \wr$$
$$S^i(\hat{T}^{1*}) \ .$$

Thus if $gr^i(\hat{p}^*_m)(a) = 0$ then $gr^i(\hat{p}^*_m(b)) = 0$. So $b \in \hat{F}^{i+1}(S_m)$. Hence $a = b = 0$ in $gr^i(S_m / < Y_m >)$. $\square$

With more specifically transcendental considerations, we shall next give more explicit versions of the above results. These involve a certain 'Green-Green' pairing #, obtained by combining the Yukawa pairing $*$ with the Green's operator G on X. We now proceed to define #.

Let $\Phi^{-1} \in H^0(\wedge^n T)$ denote the section dual to the n-form Φ, i.e.

$$i_{\Phi^{-1}}(\Phi) = 1 \ .$$

For forms $\alpha_i \in A^{n_i}(X), i = 1, 2$, $\alpha_1 \otimes \alpha_2$ may be considered as a tensor of degree $n_1 + n_2$, whence a tensor $i_{\Phi^{-1}}(\alpha_1 \otimes \alpha_2)$ of degree $n_1 + n_2 - n$ and the Yukawa product $\alpha_1 * \alpha_2$ is by definition the alternation (skew-symmetrization) of $i_{\Phi^{-1}}(\alpha_1 \otimes \alpha_2)$. Note that

$$A^{i_1,j_1} * A^{i_2,j_2} \subseteq A^{i_1+i_2-n,j_1+j_2} \ ,$$

$$\bar{\partial}(\alpha_1 * \alpha_2) = \bar{\partial}\alpha_1 * \alpha_2 \pm \alpha_1 * \bar{\partial}\alpha_2 \ ,$$

and in particular $\alpha_1 * \alpha_2$ is $\bar{\partial}$-closed provided α_1 and α_2 are . Also note that $*$ is defined by local and holomorphic data, and makes sense for the appropriate sheaves (both holomorphic and C^∞).

Now fixing a Kähler metric on X (e.g. the Ricci-flat one), let $G = G_{\bar{\partial}} = G_\partial = \frac{1}{2}G_d$ be the Green's operator on X (cf. [GH]). Thus

$$(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})G(c) = c$$

whenever c is $d-$ or $\partial-$ or $\bar{\partial}$-exact. In fact in this case we have by the Hodge identities

$$c = \sqrt{-1}\bar{\partial}\partial\Lambda G(c)$$

where Λ is the usual 'dual Lefschetz' operator. We define

$$a\#b = \sqrt{-1}\partial\Lambda G\partial(a * b) .$$

It is immediate that

$$A^{i_1,j_1}\#A^{i_2,j_2} \subseteq A^{i_1+i_2-n+1,\, j_1+j_2-1}$$

$$\bar{\partial}(a\#b) = \partial(a * b),\ \partial(a\#b) = 0.$$

Now let us identify the tangent sheaf T with Ω^{n-1} via interior multiplication, by Φ, and note that the subsheaf $\hat{T} \subset T$ thus corresponds to the subsheaf of closed forms $\hat{\Omega}^{n-1} \subset \Omega^{n-1}$. The following identity on $\hat{T}$ is easy to prove but crucial

$$[a,b] = \partial(a * b) . \tag{4.2}$$

Here a, b can be holomorphic sections of $\hat{T}$ or $\hat{T}$-valued $(0,k)$ forms, i.e. ∂-closed $(n-1,k)$-forms. The case $k = 1$, i.e. $a, b \in A^{n-1,1}$, is essentially equivalent to the 'Tian-Todorov Lemma'; note that in this case

$$a * b = i_{a\wedge b}(\Phi) .$$

[We prove (4.2) for $k = 0$ as the general case is similar. Write locally $\Phi = dz_1 \wedge ... \wedge dz_n, e_i = \partial/\partial z_i, a = \sum f_i e_i, b = \sum g_j e_j$ where

$$\sum \partial f_i/\partial z_i = \sum \partial g_j/\partial z_j = 0 (*)$$

by divergence- freeness. Then

$$a * b = \sum_{i \neq j}(-1)^{i+j} f_i g_j dz_1...\hat{dz_i}...\hat{dz_j}...dz_n,$$

so $\partial(a * b)$ corresponds via $\Phi-$ multiplication to the vector field (all summations over $i \neq j$)

$$\sum \partial f_i/\partial z_i g_j e_j - \sum \partial f_i/\partial z_j g_j e_i$$

$$-\partial g_j/\partial z_j f_i e_i + \sum \partial g_j/\partial z_i f_i e_j$$

which by (*) equals

$$-\sum_j \partial f_j/\partial z_j g_j e_j - \sum_{i \neq j} \partial f_i/\partial z_j g_j e_i$$

$$+\sum_i \partial g_i/\partial z_i f_i e_i - \sum_{i \neq j} \partial g_j/\partial z_i f_i e_j$$

$$= [\sum f_i e_i, \sum g_j e_j] = [a,b]$$

$$\square]$$

Now recall the m-th order period map

$$\hat{p}_m^1 : \hat{T}^m \to H$$

which we describe in Theorem 4.1 as $\mathbb{H}^0(j_m)$. Note that this description implies in particular that the k-th graded

$$gr^k(\hat{p}_m^1) : S^k\hat{T}^1 \to H^{n-k,k}(X)$$

is induced by projection $\hat{T}^1 = H^{n,0} + H^{n-1,1} \to H^{n-1,1}$ and Yukawa multiplication $(H^{n-1,1})^k \overset{*}{\to} H^{n-k,k}$. We refine this observation as follows.

Theorem 4.3. *There is a natural splitting*

$$\hat{T}^m = \overset{m}{\underset{1}{\oplus}} S^k \hat{T}^1 \ ,$$

with respect to which the k-th component $\hat{p}_m^{1,k}$, i.e. the k-th derivative of the period map, is given by

$$\hat{p}_m^{1,k}(a_1, \cdots, a_k) = \frac{1}{k!} \sum_{\pi \in \sum_k} \sum_{j=max(n-k+1,1)}^{k} (-1)^{j-1} (a_{\pi(1)} \# \cdots \# a_{\pi(j)} * a_{\pi(j+1)} * \cdots * a_{\pi(k)}) \ (4.3)$$

where the RHS is viewed as a cocycle in $(A^{..}, \partial, \bar{\partial})$.

Proof. The main thing will be to construct this splitting. The splittings we shall construct for various m will be mutually compatible, so there is no loss of generality in assuming $k = m \geq 2$. Let $a_1, ..., a_m \in \hat{T}^1 = H^{n,0} + H^{n,1}$. Note Φ identifies $H^{n,0} = \mathbb{C}$. We shall construct a lifting b of $a_1 \cdots a_m \in S^m((T)^1)$ to $\hat{T}^m$. Let a_i^1 be the $(n-1,1)$ component of a_i. Then we may write $a_1 \cdots a_m = a_1^1 \cdots a_m^1 + a'$ where $a' \in \oplus_{j<m} S^i(\hat{T})$, so the list of a' is defined by induction. Thus we may assume $a_i = a_i^1$ are of type $(n-1,1)$ and harmonic, hence both ∂- and $\bar{\partial}$-closed. Now the cohomology of $J_m(T)$ and $J_m(\hat{T})$ may be computed by formally applying suitable Schur functors to Dolbeault complexes resolving T and $\hat{T}$ as in the proof of Theorem 3.1, giving rise in the case of T to a double complex $(B^{..}, \delta, \bar{\partial})$, where δ is the differential inherited from that of $J_m(T)$. Now we begin with

$$b^{-m,m} = a_1 \cdots a_m \in B^{-m,m} = S^m(A^{n-1,1}) \ ,$$

whose horizontal differential is

$$\delta(b^{-m,m}) = \frac{1}{m!} \sum_{\pi \in \Sigma_m} [a_{\pi(1)}, a_{\pi(2)}] a_{\pi(3)} \cdots a_{\pi(m)} \ .$$

As the a_i are ∂-closed, we have

$$\delta(b^{-m,m}) \quad = \quad \frac{1}{m!} \sum_{\pi \in \Sigma_m} \partial(a_{\pi(1)} * a_{\pi(2)}) a_{\pi(3)} \cdots a_{\pi(m)}$$

$$= \quad \frac{1}{m!} \sum_{\pi} \bar{\partial}(a_{\pi(1)} \# a_{\pi(2)}) a_{\pi(3)} \cdots a_{\pi(m)}$$

Thus a natural choice for the component of b in $B^{-m+1,m-1}$ is

$$b^{-m+1,m-1} = \frac{1}{m!} \sum (a_{\pi(1)} \# a_{\pi(2)}) a_{\pi(3)} \cdots a_{\pi(m)} \ .$$

With this we indeed have

$$\bar{\partial}(b^{-m+1,m-1}) = \delta(b^{-m,m}) \ .$$

Next we must find a $b^{-m+2,m-2}$ with

$$\bar{\partial}(b^{-m+2,m-2}) = \delta(b^{-m+1,m-1}) \ .$$

As above , we may set

$$b^{-m+2,m-2} = \frac{1}{m!} \sum_{\pi \in \Sigma_m} ((a_{\pi(1)} \# a_{\pi(2)}) \# a_{\pi(3)}) \cdots a_{\pi(m)} \ .$$

Continuing in this way we obtain a hypercocycle $b^{..}$ for $B^{..}$, lifting $a_1 \cdots a_m$, yielding the required splitting.

Now by theorem (4.1), it follows that (identifying as always T with Ω^{n-1}):

$$
\begin{aligned}
\hat{p}_m^1(b) &= \sum_{j=1}^m i_{b^{-m+j,m-j}}(\Phi) \\
&= \frac{1}{m!} \sum_{\bar{n}} \sum_{j=1}^m i_{(a_{\pi(1)}\# \cdots \# a_{\pi(j)}) \wedge a_{\pi(j+1)} \wedge \cdots \wedge a_{\pi(m)}}(\Phi) \\
&= \frac{1}{m!} \sum_{\pi} \sum_{m-j+1 \leq n} (a_{\pi(1)}\# \cdots \# a_{\pi(j)}) * a_{\pi(j+1)} * \cdots * a_{\pi(m)} ,
\end{aligned}
$$

proving (4.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using (4.3), we can give a more explicit construction of the Schottky relations Y_m. Let

$$\eta^{i*} = \gamma^i : A^{i,n-i} = A^{n-i,i*} \to S^i(A^{n-1,1})^* = S^i(A^{1,n-1})$$

be the 'Yukawa comultiplication', and likewise

$$\rho^j : A^{1,n-1} \to S^j(A^{1,n-1})$$

be the Green-Green comultiplication, dual to the $\#$ multiplication; we may extend ρ^j as a derivation to a map

$$\rho^{i,j} : S^i(A^{1,n-1}) \to S^{i+j-1}(A^{1,n-1}) .$$

(i.e. acting on one factor at a time). Then define

$$\nu^{i,j} : A^{i,n-i} \to S^{i+j-1}(H^{1,n-1})$$

as the harmonic projection of $\rho^{i,j} \circ \gamma^i$. (Note that $H^{1,n-1} = (H^{n-1,1})^*$). It then follows directly from (4.3) and the above recipe for Y_m that

Corollary 4.4. *A complete set of m-th order Schottky relations for X is given by*

$$\{a + \sum_{j=1}^m (-1)^j \nu^{i,j}(a) : a \in H^{i,n-i}.i = 2, \cdots, n\}$$

Remark 4.5. Unfortunately the Green's operator G and its relatives, having to do with explicit realizations of the cohomology, are notoriously difficult to compute explictly, except in some special cases, e.g. curves. On the other hand it is possible to work with Čech cohomology where an explicit analogue of G has been constructed in the work of Toledo and Tong [TT]. This looks like a promising way to write down period maps fairly explicitly in general cases. We hope to return to this elsewhere. See [L] for some explicit calculations in the curve case.

APPENDIX A. THE BASIC CONSTRUCTION, DGLA CASE

Lie algebras are important in deformation theory because the symmetries of objects to be deformed generally form a continuous group whose tangent space (={elements infinitely near identity}) is a Lie algebra and it is these symmetries that are used in regluing pieces of the original object to form the deformation. Deformation problems of this type may be called 'free' or 'unconstrained'. There are however important deformation problems which are on the contrary 'constrained' or 'semitrivialized' in that some 'part' of the object to be deformed is to remain

undeformed or, more accurately to deform in a trivialized-rather than just trivial-manner. The most familiar example of a semitrivialized problem is the Hilbert scheme, i.e. deformations of a submanifold $Y \subset X$, fixing X. The (full) Lie algebra associated to the embedding $Y \subset X$ is $T_{Y/X}$, the sheaf of vector fields on X tangent to Y along Y, and the corresponding (unconstrained) deformation problem is that of deformations of the pair (X, Y). The viewpoint we adopt here is that deforming Y in a fixed X amounts to deforming the pair (X, Y) and trivializing the X part. In the general case this viewpoint leads to a (special kind of) differential graded Lie algebra (DGLA), essentially consisting in a Lie algebra g, a g-module h and a derivation $g \to h$; roughly g will do the deforming and h the trivialising. In the Hilbert scheme case $h = T_X$, of course, but it is important to observe that the Lie algebra structure of T_X itself plays no role, only the $T_{X/Y}$-module structure. A succinct way to describe the situation in this case in terms of the normal sheaf $N = T_X/T_{X/Y}$ is that $N[-1]$ forms a 'Lie algebra in the derived category' controlling the deformations of Y in the fixed X.

Now recall that by definition a $(0, 1)$-DGLA consists of a Lie algebra g, a g-module h plus a Lie derivation $d : g \to h$; as we shall consider no other kind of DGLA, we shall for convenience drop the (0,1) tag. Note that the universal enveloping algebra $U(g \to h)$ is a differential graded associative algebra, with elements of h having degree 1 and the differential being the natural extension of d as internal derivation of degree $+1$, Note that the Lie 'inclusion' $g \to (g \to h)$ induces a homomorphism $U(g) \to U(g \to h)$ and in view of the rule

$$b.a = a.b - a(b) \ , \ a \in U(g), \ b \in S^{\cdot}(h) \ ,$$

$a(b)$ being the action of a on b, it is easy to see that as $U(g)$-modules, we have

$$U(g \to h) \simeq U(g) \otimes S^{\cdot}(h).$$

Note that to a DGLA sheaf $g \to^d h$ on X we may associate Jacobi bicomplexes $J_m(g \to h)$ whose jth row is of the form $\pi_{m-j,j*}J_{m-j}(g, \sigma^j h)$ where $\pi_{m-j,j} : X < m - j, j > \to X < m >$ is the natural (forgetful) map, zeroth row is just $J_m(g)$ and whose kth column is an Eagon-Northcott type complex of the form

$$\lambda^k g \to ...\lambda^j g \boxtimes \sigma^{k-j} h \to ...\sigma^k h;$$

it is the derivation property of d that ensures that these fit together to form a bicomplex. As in the Lie algebra case, $\mathbb{H}^0(J_m(g \to h))$ is an OS structure with a natural OS morphism $\mathbb{H}^0(J_m(g \to h)) \to \mathbb{H}^0(J_m(g))$, inducing a ring homomorphism $R_m(g) \to R_m(g \to h)$.

Remark A.1. For cohomological purposes we may replace $g \to h$ by a Čech bicomplex, thus turning $J_m(g \to h)$ into a multicomplex $\{\wedge^a C^b(g) \otimes S^c C^d(h)\}$ (where $\wedge$ and S are to be understood in the graded sense). The contribution to $\mathbb{H}^0$ from the 'pure h' part $a = b = 0$ may be identified with the kernel of the differential $\tilde{d}$ induced by d on the enveloping algebra of the DGLA $C^{\cdot}(g) \to C^{\cdot}(h)$. Moreover, in cases of interest to us, we have that $H^0(g) \to H^0(h)$ is injective and in this case it is easy to check that there is no other contribution to $\mathbb{H}^0(J)$, so in fact

$$\mathbb{H}^0(J_m(g \to h)) = \text{coker}(\tilde{d}). \qquad (A.1)$$

Now our purpose is to interpret $R_m(g \to h)$ deformation theoretically. To this end let E be a g-module over X, R_m an artin local $\mathbb{C}$-algebra of exponent m, and E_m a flat g-deformation of E over R_m, i.e. an R_m-module locally isomorphic to

$E \otimes R_m$ with gluing maps in $\mathrm{Exp}(g) \otimes R_m = G_m$. As in [R2], E_m comes from a Kodaira-Spencer homomorphism $\alpha_m : R_m(g) \to R_m$, i.e. $E_m \simeq \alpha_m^* E_m^u$ where E_m^u is the m-universal object. Now h itself being a g-module, α_m similarly gives rise to a g-deformation $h_m = \alpha_m^* h_m^u$ where $h_m^u = M_m(g, h) = \mathrm{Hom}_{MOS}(V_0^m, \mathbb{R}^0 p_*(J_m(g, h))$ (Theorem 3.1)(note objects like h_m^u appear there without the 'u' superscript); note that $m_m h_m^u = \mathrm{Hom}_{MOS}(V_m, \mathbb{R}^0 p_*(J_m(g, h))$. Now the canonical map $J_m(g) \to \pi_{m,1*} J_m(g, h)[1]$, which is nothing but the map from the zeroth to the first row of $J_m(g \to h)$, gives rise to a canonical cohomology class $B_m(h) \in C^m \mathbb{H}^1(J_m(g, h)) = H^1(h_m^u)$ (in fact this clearly lifts to $H^1(m_m h_m^u)$ and we shall use $B_m(h)$ to denote the lift as well), whence a class

$$B(\alpha_m, h) = \alpha_m^*(B_m(h)) \in H^1(m_m h_m^u).$$

Note incidentally that g itself is a $g-$module via the $1/2$ ad action and for this action $g \to^{id} g$ is a DGLA (the $1/2$ factor is needed to make id a derivation),whence a canonical element $B(\alpha_m, g) \in H^1(m_m g_m)$. For instance if $g = T_X$ then $g_m = T_{X_m/R_m}$, the relative tangent sheaf.

By a $(g \to h)$-deformation of E we mean the data (α_m, A) consisting of a g-deformation of E given by α_m plus a 'trivialization' of the corresponding canonical cohomology class, given e.g. by a Čech cochain $A = (v_\alpha) \in \check{C}^\circ(h_m)$, $\delta A = B(\alpha_m, h)$. These objects can all be represented concretely as in [R2]. Recall to begin with that α_m corresponds to a morphic hypercocycle $v_m = \epsilon(u_m) \in \mathbb{H}^0(J_m(g)) \otimes m_m$ where $u. = u_m \in \check{C}^1(g) \otimes m_m$ satisfies the Čech integrability condition

$$\delta(u_m) = -1/2[u_m, u_m]; \qquad\qquad (A.2)$$

a deformation corresponding to α_m, such as E_m (or g_m or h_m) is essentially *defined* by the condition that its Čech complex $(\check{C}^\cdot(E_m), \delta_m)$ be isomorphic to $(\check{C}^\cdot(E) \otimes R_m, \delta_0 \otimes 1 + u_m)$ (viewing u_m as an operator of degree $+1$ on $\check{C}^\cdot(E_m) \otimes R_m$). Then $B(\alpha_m, g)$ is represented by (u_m) (which by (A.2) is indeed a cocycle for g_m and in fact for $m_m g_m$, which is a flat R_{m-1}-module); similarly, $B(\alpha_m, h)$ is represented by $d(u_m)$. Put another way, the operator $\delta_0 \otimes 1$ on $\check{C}^\cdot(E) \otimes R_m$ pulled over to $\check{C}^\cdot(E_m)$ becomes $\delta - u_m$, so u_m is indeed a cocycle for $m_m g_m$. Consequently A may be represented by

$$a. \in \check{C}^0(h) \otimes m_m,$$

$$\delta(a.) + u.(a.) = d(u.)$$

i.e. $a_\alpha - a_\beta + 1/2 u_{\alpha\beta}(a_\alpha + a_\beta) = d(u_{\alpha\beta})$. The following then is the DGLA analogue of the main result of [R2].

Theorem A.2. *Assume $H^0(g) \to H^0(h)$ is injective. Then for any m, there exists a $(g \to h)$ deformation $(\alpha_m^{uc}, A_m^{uc})$ over $R_m^{uc} = R_m(g \to h)$ which is constrained universal in that for any $(g \to h)$ deformation (α_m, A_m) over R_m, there is a factorization of α_m*

$$R_m^u \overset{\alpha_m^{uc}}{\to} R_m^{uc} \overset{\beta_m}{\to} R_m$$

such that $A_m = \beta_m^* A_m^{uc}$.

Proof. Analogous, mutatis mutandis, to that in [R2] (compare Theorem 3.1 above). To begin with, α_m^{uc} simply corresponds to the natural map $q_m : J_m(g \to h) \to J_m(g)$, while A_m^{uc} comes from the extra data needed to lift an element of $\mathbb{H}^0(J_m(g))$ to one in $\mathbb{H}^0$ of the double complex formed from the last 2 rows of $J_m(g \to h)$. Now

given an arbitrary $(g \to h)$-deformation (α_m, A_m) over R_m, α_m corresponds to a cochain $u_m \in C^1(g) \otimes m_m$ and A_m to $a \in C^0(h) \otimes m_m$ as above. Then

$$(u_m, \frac{1}{2}u_m^2, \cdots, \frac{1}{m!}u_m^m; a, u_m \times a, \cdots, \frac{1}{(m-1)!}u_m^{m-1} \times a; \frac{1}{2}a^2, u_m \times \frac{1}{2}a^2; \cdots; \cdots; \frac{1}{m!}a^m)$$

$$(A.2)$$

yields a morphic element of $m_m \otimes \mathbb{H}^0(J_m(g \to h))$ corresponding to β_m as required.

$\square$

Examples

1. Let X be a compact complex complex manifold and $Y \subset X$ a submanifold with normal bundle N. We have a DGLA

$$(T_{X/Y} \to T_X) \sim N[-1] .$$

As noted before ([R2],section 5), $T_{X/Y}$ controls deformations $(Y_m \subset X_m/R_m)$ of the pair $Y \subset X$ and given such a deformation with Kodaira-Spencer homomorphism $\alpha_m : R_m(T_{X/Y}) \to R_m$, evidently $B(\alpha_m, T_X)$ is just the class corresponding to the induced deformation X_m of X alone, so that the data A is simply an identification of X_m with the trivial deformation $X \times \mathrm{Spec}(R_m)$.

Remark A.3. The cokernel of $\tilde{d}$ on $U(N[-1])$ is a certain quotient of $S^{\cdot}(T_X)$ which is essentially the sheaf $D_{Y \to X}^{\cdot}$ of 'normal differential operators' along Y, considered by Burchard, Clemens et al [BC]. Note the natural map

$$\mathbb{H}^0(J_m(N[-1])) = \mathrm{coker}(\tilde{d}, \mathbb{H}^0) \quad \to \quad \mathbb{H}^0(\mathrm{coker}(\tilde{d}, U(N[-1])))$$
$$\to \quad \mathbb{H}^0(D_{Y \to X}^{\cdot}).$$

2. (cf.[R3]) Let X be a compact complex Kähler manifold with tangent sheaf T and $\eta \in H^{p,q}(X)$. We have a map

$$T \to \Omega^{p-1}[q] \qquad\qquad (A.3)$$

given by interior multiplication by η. Representing η by a closed form, note the formula

$$i_{[x,y]}(\eta) = L_x(i_y(\eta)) - L_y(i_x(\eta)) - d(i_{x \wedge y}(\eta))$$

for vector fields x,y. Replacing Ω^{p-1} by, e.g. its Dolbeault resolution we have by the usual Kähler machinery

$$d(i_{x \wedge y}(\eta)) = \bar{\partial}(j_{x \wedge y}(\eta))$$

for a suitable well-defined $j_{x \wedge y}(\eta) \in A^{p-1,q-1}$, therefore this term vanishes in the derived category and $\mathcal{L}_\eta = (T \to \Omega^{p-1}[q])$ forms a DGLA. As shown in [R3], $\mathcal{L}_\eta$ controls precisely the deformations of X in which (the GM-constant lift of) η maintains Hodge level p. In the situation of example 1, we may take $\eta = [Y] \in H^{p,p}(X)$, $p = \mathrm{codim}(Y)$ and then we have an exact diagram

$$
\begin{array}{ccccc}
N[-1] & \longrightarrow & T_{X/Y} & \longrightarrow & T \longrightarrow \\
\downarrow & & \downarrow & & \| \\
\Omega^{p-1}[p-1] & \longrightarrow & \mathcal{L} & \longrightarrow & T \longrightarrow .
\end{array}
$$

See [R3] for geometric application of this.

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