Math 9A Final Exam Spring 2006

---NO CALCULATORS---NO POTTY BREAKS
Each problem is worth 20 points, evenly divided, unless otherwise indicated. Good luck.

1. Find the limit or state DNE if it does not exist.
   i) \[ \lim_{{h \to 0}} \frac{{(x + h)^2 - x^2}}{h} = \lim_{{h \to 0}} \frac{{2(x + h)}}{1} = 2x \]
   ii) \[ \lim_{{x \to 0}} \frac{{2x - \frac{x}{2}}}{x} = \lim_{{x \to 0}} \frac{{x}}{x} = 1 \]
   iii) \[ \lim_{{x \to 1}} \frac{{1 - \sqrt{x}}}{1 - x} = \lim_{{x \to 1}} \frac{{-1}}{2} = -\frac{1}{2} \]
   iv) \[ \lim_{{x \to 0}} \frac{{\sin 5x}}{\sin 2x} = \frac{{5 \csc 5x}}{2 \csc 2x} = \frac{5}{2} \]
v) \[ \lim_{x \to 0} \frac{\tan x}{x + \sin x} = \frac{1}{2} \]

2. Define a function \( f(x) \) by

\[
f(x) = \begin{cases} 
0, & x \leq -1 \\
1/x, & -1 < x < 1 \\
0, & x = 1 \\
1, & 1 < x 
\end{cases}
\]

i) (8 pts) Graph it.

ii) (1 pt each) Fill in the blanks

\[
\begin{align*}
\lim_{x \to -1^-} f(x) &= 0 \\
\lim_{x \to -1^+} f(x) &= 0 \\
\lim_{x \to 0} f(x) &= \pm \infty \\
\lim_{x \to -\infty} f(x) &= -\infty \\
\lim_{x \to +\infty} f(x) &= \infty \\
\lim_{x \to 0^+} f(x) &= 1 \\
\lim_{x \to 0^-} f(x) &= 1
\end{align*}
\]

iii) (1 pt each) Answer true (T) or false (F): "The function \( f(x) \) is continuous at

\[
-1: \quad F, \\
0: \quad T, \\
1: \quad F
\]

3. Find the limit or state DNE if it does not exist.

i) \[ \lim_{x \to \infty} \frac{2x^2 + 3}{5x^2 + 7} = \frac{2}{5} \]
ii) 
\[
\lim_{\theta \to \infty} \frac{\cos \theta - 1}{\theta} = -2 \leq \frac{\cos \theta - 1}{\theta} \leq \frac{2}{\theta}, \quad -\infty < \theta \to 0 \quad \text{and} \quad \frac{2}{\theta} \to 0.
\]
So \(\frac{\cos \theta - 1}{\theta} \to 0\) as \(\theta \to 0\).

iii) 
\[
\lim_{x \to 0} \frac{x^2 - 1}{x^2} = \lim_{x \to 0} \frac{1}{x^2} = -\infty \quad \text{or} \quad \text{DNE}.
\]

iv) 
\[
\lim_{x \to 0} x \sin \frac{1}{x} = -|x| \leq \frac{1}{x} \leq |x|.
\]
So \(x \sin \frac{1}{x} \to 0\) as \(x \to 0\).

4. Using one of our theorems, show that the equation
\[
\int_{-1}^{1} x^3 - 15x^2 + 1 = 0
\]
has at least one solution in each of the 3 intervals (-4,0), (0,1), (1,4).

\[
\begin{array}{c|c|c}
\text{Interval} & f(x) & \text{Sign Change} \\
\hline
(-4,0) & -3 & \text{Concave Down} \\
(0,1) & 1 & \text{Concave Up} \\
(1,4) & -13 & \text{Concave Down}
\end{array}
\]

So by the Intermediate Value Theorem, it has a zero in each interval.

5. Find the derivative of the function.

i) 
\[
y = 3 - 0.7x^3 + 0.3x^7
\]
\[
y' = -0.7 \cdot 3 \cdot x^2 + 0.3 \cdot 7 \cdot x^6 = (-2.1) \cdot x^2 + (2.1) x^6 = (2.1) x^2 (x^4 - 1)
\]
2) \[ y = \theta^{2} + \sec \theta + 1 \]
\[
\frac{dy}{d\theta} = 2\theta + \sec \theta \cdot \tan \theta
\]

iii) \[
s = \frac{1}{\sqrt{t-1}} \Rightarrow \frac{1}{t^{1/2} \cdot t^{-1/2}} = (t^{1/2} \cdot t^{-1/2})^{-1}
\]
\[
\frac{ds}{dt} = -\left( t^{1/2} \cdot t^{-1/2} \right)^{2} \cdot t^{-1/2} = \frac{-1}{2\sqrt{t} \cdot (t^{-1/2})^{2}}
\]

iv) \[ y = 5 \tan(x^{2}) \]
\[
\frac{dy}{dx} = 5 \cdot \sec^{2}(x^{2}) \cdot 2x = 10x \cdot \sec^{2}(x^{2})
\]

6. For the function \[ g(x) = 2x^{2} + 1 \]
the derivative \[ g'(x) = 4x \] . Show how this follows directly from the definition of derivative as a certain limit.
\[
g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{(2(x+h)^{2} + 1) - (2x^{2} + 1)}{h}
\]
\[
= \lim_{h \to 0} \frac{4x + 2h}{1} = 4x
\]

7. i) Find \( \frac{dy}{dx} \) where \[ \sqrt{xy} = 1 \]
\[
\frac{1}{2} (x y^{-1/2} - x^{1/2} y + 0) = 0 \quad (\text{or} \quad xy = 1 \Rightarrow x = \frac{1}{y})
\]
\[
\frac{d}{dx} (x^{2} + y^{2}) = 0
\]
\[
\frac{d}{dx} (\frac{-y}{x}) = \frac{-\frac{dy}{dx}}{x}
\]
\[
\frac{d}{dx} = \frac{-y}{x}
\]
ii) Find $dr/ds$ where
\[ r \cos 2s + \sin^2 s = \pi \]
\[ r(- \sin^2 x) \frac{d}{dx} \left( \frac{d}{dx} \right) 2 \sin s \cos s = 0 \]
\[ \cos 2s \frac{dr}{ds} = 2 \left( r \sin 2s - \sin^2 s \right) \cos s \]
\[ \frac{dr}{ds} = 2 \frac{r \sin 2s - \sin^2 s \cos s}{\cos 2s} \]

8. Consider the parametrized curve
\[ x = \frac{1}{2} \tan t, \quad y = \frac{1}{2} \sec t \]
i) Give an equation for the tangent line at $t = \pi/3$.
\[ \frac{dx}{dt} \bigg|_{\pi/3} = \frac{1}{2} \sec^2 t \bigg|_{\pi/3} = \frac{1}{2} \frac{1}{\cos^2 \pi/3} = \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{2} \]
\[ \frac{dy}{dt} \bigg|_{\pi/3} = \frac{1}{2} \sec t \tan t \bigg|_{\pi/3} = \frac{1}{2} \frac{\sin \pi/3}{\cos \pi/3} = \frac{\sqrt{3}}{2} \]
so
\[ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = 1 \]
\[ \tan \alpha = \frac{\sin \alpha}{\cos \alpha} \]
\[ x_0 = \frac{1}{2} \sec \pi/3 = \frac{\sqrt{3}}{2}, \quad y_0 = \frac{1}{2} \sec \pi/3 = \frac{\sqrt{3}}{2} \]
\[ (y - y_0) = m(x - x_0) \]
\[ y - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \left( x - \frac{\sqrt{3}}{2} \right) \]
\[ y - 1 = \frac{\sqrt{3}}{2} \left( x - \frac{\sqrt{3}}{2} \right) + \frac{\sqrt{3}}{2} \]
\[ y = \frac{1}{2} x + 1 \]

ii) Find the value of $d^2y/dx^2$ at the same point.
\[ y' = \frac{dy}{dx} = \frac{\sin t}{\cos t} \]
\[ y'' = \frac{d^2y}{dx^2} = \frac{\cos t \tan t}{\cos^2 t} = 2 \frac{\cos t}{\cos^2 t} = 2 \cos^3 t \]
At $t = \pi/3$, we have
\[ 2 \cos^3 \left( \frac{\pi}{3} \right) = 2 \left( \frac{1}{2} \right)^3 = \frac{1}{4} \]
9. The volume of a cube is increasing at the rate of 1200 cubic centimeters per minute at the instant each edge is 20 cm. At what rate are the length of the edges changing at that instant?

\[ V = x^3 \]

\[ \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \]

\[ 1200 = 3(20)^2 \frac{dx}{dt} \]

\[ 1200 = 1200 \frac{dx}{dt} \]

\[ \frac{dx}{dt} = 1 \text{ (cm/min)} \]

10. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel disks and the vertex of the inside cone is at the center of the larger cone’s base. What values of \( r \) and \( h \) give the smaller cone the largest possible volume?

By similar triangles

\[ \frac{r}{12-h} = \frac{6}{12} = \frac{1}{2} \]

\[ 2r = 12-h \]

\[ h = 12-2r \]

\[ V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 (12-2r) \]

\[ \frac{dV}{dt} = \frac{1}{3} \pi \left[ 2r(12-2r) + r^2 (-2) \right] = \frac{1}{3} \pi \left[ 24r - 4r^2 - 2r^2 \right] = \frac{1}{3} \pi \left[ 24r - 6r^2 \right] \]

So critical points are \( r = 0, r = 4 \), other end pt is \( r = 6 \)

\( V = 0 \) \( \text{ occurs here} \)

So \( r = 4, h = 4 \)

(Only need check the end points and the critical points.)
11. Consider the function \( y = -2x^3 + 6x^2 - 3 \)

i) (5 pts) Find the interval(s) on which the function is increasing and on which the function is decreasing.

\[
y' = -6x^2 + 12x = -6x(x-2)
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
\text{Sign of } y' & \text{-} & \text{-} & \text{-} & \text{0} & \text{+} & \text{+} & \text{+} & \text{1} & \text{-} & \text{-} & \text{-} & \text{-} \\
\hline
y' & < & < & < & 0 & > & > & > & 1 & < & < & < \\
\hline
\end{array}
\]

\( y' \) is a decreasing \( \rightarrow \) \( \text{min} \) \( \rightarrow \) a decreasing.

ii) (5 pts) Find the interval(s) on which the function is concave up and on which it is concave down.

\[
y'' = -12x + 12 = -12(x-1)
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
\text{Sign of } y'' & \text{-} & \text{-} & \text{-} & \text{1} & \text{-} & \text{-} & \text{-} & \text{-} & \text{-} & \text{-} & \text{-} & \text{-} & \text{-} \\
\hline
y'' & < & < & < & 1 & < & < & < & < & < & < & < & < \\
\hline
\end{array}
\]

\( y'' \) is conc. up \( \rightarrow \) conc. down.

iii) (10 pts) Sketch the graph, labelling the points (both \( x \) and \( y \) coordinates) of any local minima, local maxima and inflection points.

![Graph with points labeled]

- \((1,5)\) (local max)
- \((0, -3)\) (local min)
- \((4, 11)\) (inflection pt.)
12. Calculate the derivatives of
i)(5 pts) 
\[ f(x) = \frac{x}{x + 1} \]
\[ \frac{df}{dx} = \frac{(x + 1) \cdot 1 - x \cdot 1}{(x + 1)^2} = \frac{x + 1 - x}{(x + 1)^2} = \frac{1}{(x + 1)^2} \]

ii)(5 pts) and 
\[ g(x) = \frac{-1}{x + 1} \]
\[ \frac{dg}{dx} = \frac{(x + 1) \cdot 0 - (-1) \cdot 1}{(x + 1)^2} = \frac{1}{(x + 1)^2} \]

iii)(10 pts) In view of the results of i) and ii) what can you conclude about the difference between \( f(x) \) and \( g(x) \)? (Hint: It follows from a Corollary to the Mean Value Theorem.)

Since the derivatives are the same, they differ by a constant. (Cor. 2) (Indirect) \( f(x) - g(x) = 1 \)