A function \( f : (a, b) \to \mathbb{R} \) is **monotonically increasing** if for all \( x, y \in (a, b) \),
\[
x \leq y \ \text{implies} \ f(x) \leq f(y).
\]
A function \( f : (a, b) \to \mathbb{R} \) is **differentiable** if it is differentiable at \( x \) for all \( x \in (a, b) \).

1. Prove that if \( f : (a, b) \to \mathbb{R} \) is differentiable and monotonically increasing, then \( f'(x) \geq 0 \) for all \( x \in (a, b) \).

2. Prove that if \( f : (a, b) \to \mathbb{R} \) is differentiable and \( f'(x) \geq 0 \) for all \( x \in (a, b) \), then \( f \) is monotonically increasing.

Theorem 5.13 of Rudin contains two versions of l’Hôpital’s rule. They’re both important, so make sure to learn them, but I want you to prove this one:

**Theorem (L’Hôpital’s Rule.)** Suppose that \( f, g : [a, b] \to \mathbb{R} \) are differentiable at all points \( x \in (a, b) \), \( f' \) and \( g' \) are continuous on \( (a, b) \), and \( g'(x) \neq 0 \) for all \( x \in (a, b) \). Suppose also that for some \( p \in (a, b) \) we have

\[
\lim_{x \to p} f(x) = 0
\]

and

\[
\lim_{x \to p} g(x) = 0
\]

Then the limit

\[
\lim_{x \to p} \frac{f(x)}{g(x)}
\]

exists and equals

\[
\lim_{x \to p} \frac{f'(x)}{g'(x)}.
\]

3. Prove this theorem.

Hint: First prove that for any \( x \in (a, b) \) we have

\[
f(x) = (x - p)(f'(p) + \epsilon(x))
\]

where \( \epsilon : (a, b) \to \mathbb{R} \) is a function obeying

\[
\lim_{x \to p} \epsilon(x) = 0
\]

Similarly, prove that

\[
g(x) = (x - p)(g'(p) + \delta(x))
\]

where \( \lim_{x \to p} \delta(x) = 0 \). Then use these to prove the theorem.