First we’ll compute
\[ \int_0^a x^3 \, dx \]
the ‘hard way’, directly using the definition of the Riemann integral. Then we’ll compute it using
the Fundamental Theorem of Calculus. In all 5 problems, \( f: [0, a] \to \mathbb{R} \) will be the function
with \( f(x) = x^3 \).

1. Prove using mathematical induction that
\[ 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2. \]

(By the way, this is incredibly cool because we also have
\[ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \]
So, we get the amazing formula
\[ 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2. \]
But we don’t need this today!)

2. Let \( P = \{x_0, \ldots, x_n\} \) be the partition of the interval \([0, a]\) with
\[ x_i = \frac{i}{n} a. \]
a) Compute \( U(P, f) \).
b) Compute \( L(P, f) \).

3. Prove that this function \( f: [0, a] \to \mathbb{R} \) is Riemann integrable using the Fundamental Criterion
for Riemann Integrability.

4. Compute either the lower Riemann integral
\[ \int_0^af(x) \, dx \]
or the upper Riemann integral
\[ \int_0^af(x) \, dx \]
for this function $f$.

(Problem 3 implies that the lower and upper Riemann integrals are equal, and both equal the Riemann integral $\int_0^a x^3 \, dx$.)

5. Rigorously prove that there is a function $F : [0, a] \to \mathbb{R}$ with $F' = f$.

(You know a function $F$ with this property, but prove it has this property, using theorems we’ve shown in class.)

6. Use Problem 5 and the Fundamental Theorem of Calculus, Theorem 6.21, to compute

$$\int_0^a x^3 \, dx.$$