Categorified Gauge Theory

John C. Baez

joint work with:
Toby Bartels,
Alissa Crans,
Aaron Lauda,
& Urs Schreiber

in honor of
Larry Breen’s 60th birthday

Institute Galilée
December 15, 2004

More details at:
http://math.ucr.edu/home/baez/breen/
Gauge Theory

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:

since composition of paths then corresponds to multiplication:

while reversing the direction of a path corresponds to taking inverses:

and the associative law makes the holonomy along a triple composite unambiguous:

In short: the topology dictates the algebra!

The electromagnetic field is described by a connection where the group is U(1). Other forces are described using other groups.
To really let the topology dictate the algebra, we should replace the Lie group by a ‘smooth groupoid’: a groupoid in some convenient category of smooth spaces. Mackaay and Picken have noted that for any manifold $M$ there is a smooth groupoid $\mathcal{P}_1(M)$, the path groupoid, for which:

- objects are points $x \in M$,
- morphisms are thin homotopy classes of smooth paths $\gamma : [0, 1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$.

For any Lie group $G$, a principal $G$-bundle $P \to M$ gives a smooth groupoid $\text{Trans}(P)$, the transport groupoid, for which:

- objects are torsors $P_x$ for $x \in M$,
- morphisms are torsor morphisms.

Via parallel transport, any connection on $P$ gives a smooth functor called its holonomy:

$$\text{hol} : \mathcal{P}_1(M) \to \text{Trans}(P)$$

A trivialization of $P$ makes $\text{Trans}(P)$ equivalent to $G$, so it gives:

$$\text{hol} : \mathcal{P}_1(M) \to G$$
Next let’s study how 1-dimensional ‘strings’ transform as we move them along 2-dimensional surfaces. Naively we might wish our holonomy to assign a group element to each surface like this:

\[ g \]

We can compose surfaces of this sort vertically:

\[ \begin{array}{c}
\bullet \\
g \\
\bullet
\end{array} \]

and horizontally:

\[ \begin{array}{c}
\bullet \\
g \\
\bullet \\
\bigcirc \\
g' \\
\bullet
\end{array} \]

Suppose both of these correspond to multiplication in some Lie group $G$. To obtain a well-defined holonomy for this surface regardless of whether we do vertical or horizontal composition first:

\[ \begin{array}{c}
\bullet \\
g_1 \\
\bullet \\
g_2 \\
\bigcirc \\
g'_1 \\
\bullet \\
g'_2 \\
\bullet
\end{array} \]

we must have

\[ (g_1 g_2)(g'_1 g'_2) = (g_1 g'_1)(g_2 g'_2). \]

This forces $G$ to be abelian!

Pursuing this approach, we ultimately get the theory of connections on ‘abelian gerbes’. If $G = U(1)$, such a connection looks locally like a 2-form — and it shows up naturally in string theory, satisfying equations very much like those of electromagnetism!
To go beyond this and get nonabelian higher gauge fields, we should let the topology dictate the algebra, and consider a connection that gives holonomies both for paths and for surfaces.

So, let’s replace the path groupoid by some 2-groupoid where:

- objects are points of $M$: $\bullet_x$

- morphisms are certain paths in $M$: $\bullet \xrightarrow{\gamma} \bullet$

- 2-morphisms are certain equivalence classes of paths of paths in $M$: $\bullet \xrightarrow{\gamma f} \bullet$

A 2-groupoid allows composition of morphisms:

- vertical composition of 2-morphisms:

- and horizontal composition of morphisms:

satisfying various laws, including one that makes this unambiguous:
More precisely, define the **path 2-groupoid** \( \mathcal{P}_2(M) \) to be the smooth 2-groupoid in which:

- objects are points \( x \in M \),
- morphisms are smooth paths \( \gamma: [0, 1] \to M \) with \( \gamma(t) \) constant near \( t = 0, 1 \),
- 2-morphisms are thin homotopy classes of smooth maps \( f: [0, 1]^2 \to M \) with \( f(s, t) \) independent of \( s \) near \( s = 0, 1 \) and constant near \( t = 0, 1 \).

We might hope for something like this:

For any Lie 2-group \( \mathcal{G} \), a principal \( \mathcal{G} \)-2-bundle \( P \to M \) gives a smooth 2-groupoid \( \text{Trans}(P) \) where:

- objects are 2-torsors \( P_x \),
- morphisms are 2-torsor morphisms, \( f: P_x \to P_y \)
- 2-morphisms are 2-torsor 2-morphisms \( \theta: f \Rightarrow g \).

Via parallel transport, a 2-connection on \( P \) gives a smooth 2-functor called its **holonomy**:

\[
\text{hol}: \mathcal{P}_2(M) \to \text{Trans}(P).
\]

A trivialization of \( P \) makes \( \text{Trans}(P) \) equivalent to \( \mathcal{G} \) so it gives

\[
\text{hol}: \mathcal{P}_2(M) \to \mathcal{G}.
\]

*Can we make this precise? Is it true?*
Internalization

The crucial trick is ‘internalization’. Ehresmann and Lawvere showed how to ‘internalize’ concepts by defining them in terms of commutative diagrams:

A small category, say $C$, has a set of objects $\text{Ob}(C)$, a set of morphisms $\text{Mor}(C)$, source and target functions

$$s, t : \text{Ob}(C) \to \text{Mor}(C),$$

a composition function

$$\circ : \text{Mor}(C) \times \text{Mor}(C) \to \text{Mor}(C)$$

and an identity-assigning function

$$\text{id} : \text{Ob}(C) \to \text{Mor}(C)$$

making these diagrams commute.

and letting these diagrams live within some category $K$:

A category in $K$, say $C$, has an object $\text{Ob}(C) \in K$, an object $\text{Mor}(C) \in K$, source and target morphisms

$$s, t : \text{Ob}(C) \to \text{Mor}(C),$$

a composition morphism

$$\circ : \text{Mor}(C) \times \text{Mor}(C) \to \text{Mor}(C)$$

and an identity-assigning morphism

$$\text{id} : \text{Ob}(C) \to \text{Mor}(C)$$

making these diagrams commute.

Similarly we can define functors in $K$ and natural transformations in $K$, obtaining a 2-category $K\text{Cat}$. We can also define groups in $K$ and homomorphisms in $K$, obtaining a category $K\text{Grp}$. 
Smooth Categories, 2-Groups, and Lie 2-Groups

We can categorify concepts from differential geometry with the help of internalization:

- A **smooth category** is a category in Diff.
- A **strict 2-group** (or **categorical group**) is a category in Grp.
- A **strict Lie 2-group** is a category in LieGrp.

A strict 2-group is the same as a strict monoidal category such that:

- for every object $x$ there exists an object $y$ with $x \otimes y = 1$, $y \otimes x = 1$;
- for every morphism $f$ there exists a morphism $g$ with $fg = 1$, $gf = 1$.

More generally, a **2-group** (or **gr-category**) is a weak monoidal category such that:

- for every object $x$ there is a specified object $x^{-1}$ equipped with isomorphisms

  $$i_x : 1 \to x \otimes x^{-1}, \quad e_x : x^{-1} \otimes x \to 1$$

  forming an adjunction;
- for every morphism $f$ there exists a morphism $g$ with $fg = 1$, $gf = 1$.

We can also define general **Lie 2-groups** the same way, working in DiffCat rather than Cat.
Examples of 2-Groups

1) Any abelian group $A$ gives a strict 2-group with one object and $A$ as the automorphisms of this object. Lie 2-groups of this kind will be structure 2-groups of 2-bundles having an *abelian gerbe* of sections.

2) Any category $C$ gives a 2-group $\text{Aut}(C)$ whose objects are equivalences $f : C \to C$ and whose morphisms are natural isomorphisms between these.

3) A group $H$ is a category with one object and all morphisms invertible. In this case, 2) gives a strict 2-group $\text{Aut}(H)$ whose objects are automorphisms of $H$ and whose morphisms from $f$ to $f'$ are elements $k \in H$ with $f'(h) = kf(h)k^{-1}$.

4) Any Lie group $H$ gives a strict Lie 2-group $\text{Aut}(H)$ defined as in 3) but with everything smooth. Lie 2-groups of this sort will be structure 2-groups of 2-bundles having a *nonabelian gerbe* of sections.

... and many ‘more concrete’ examples, some listed in my paper with Aaron Lauda.
Toby Bartels has developed a theory of ‘2-bundles’. We can think of a manifold $M$ as a smooth category with only identity morphisms. A 2-bundle over $M$ consists of:

- a smooth category $P$ (the **total space**),
- a smooth category $F$ (the **standard fiber**),
- a smooth functor $p: P \to M$ (the **projection**),

such that each point $x \in M$ has an open neighborhood $U$ for which there exists a smooth equivalence:

$$f: p^{-1}U \to U \times F$$

such that this diagram commutes:

$$
\begin{array}{ccc}
    p^{-1}U & \xrightarrow{f} & U \times F \\
    \downarrow p|_{p^{-1}U} & & \downarrow \\
    U & & 
\end{array}
$$

The equivalence $f$ is called a **local trivialization**.
If $F$ is a smooth category, $\mathcal{G} = \text{Aut}(F)$ is a smooth 2-group. Given a 2-bundle $P \to M$ with standard fiber $F$, and choosing local trivializations over open sets $U_i$ covering $M$, we obtain:

- smooth maps
  \[ g_{ij}: U_i \cap U_j \to \text{Ob}(\mathcal{G}) \]

- smooth maps
  \[ h_{ijk}: U_i \cap U_j \cap U_k \to \text{Mor}(\mathcal{G}) \]
  with
  \[ h_{ijk}(x): g_{ij}(x)g_{jk}(x) \to g_{ik}(x) \]

- smooth maps
  \[ k_i: U_i \to \text{Mor}(\mathcal{G}) \]
  with
  \[ k_i(x): g_{ii}(x) \to 1 \in \mathcal{G}. \]

Furthermore:

- $h$ satisfies an equation on quadruple intersections $U_i \cap U_j \cap U_k \cap U_l$:

\[ (\text{the associative law}) \]
• $k$ satisfies two equations on double intersections $U_i \cap U_j$:

![Diagram](image)

(the **left unit law**) and

![Diagram](image)

(the **right unit law**).

More generally, for any smooth 2-group $\mathcal{G}$ we say a 2-bundle $P \to M$ has $\mathcal{G}$ as its **structure 2-group** when $g_{ij}, h_{ijk}, k_i$ factor through an action $\mathcal{G} \to \text{Aut}(F)$.

In particular, if $\mathcal{G}$ acts on $F = \mathcal{G}$ by left multiplication, $P$ is a **principal $\mathcal{G}$-2-bundle**. Its fibers are then $\mathcal{G}$-2-torsors in a suitable sense.

Any 2-bundle has a stack of sections. A principal $\mathcal{G}$-2-bundle with $\mathcal{G} = \text{Aut}(H)$ for some Lie group $H$ has a nonabelian gerbe of sections!
2-Connections on Principal 2-Bundles

So far Urs Schreiber and I have only handled 2-connections on principal 2-bundles where the structure 2-group $\mathcal{G}$ is strict.

A strict Lie 2-group $\mathcal{G}$ is determined by:

- the Lie group $G$ consisting of all objects of $\mathcal{G}$,
- the Lie group $H$ consisting of all morphisms of $\mathcal{G}$ with source 1,
- the homomorphism $t: H \to G$ sending each morphism in $H$ to its target,
- the action $\alpha$ of $G$ on $H$ defined using conjugation in $\text{Mor}(\mathcal{G})$ via

$$\alpha(g)h = 1_g h 1_g^{-1}$$

The system $(G, H, t, \alpha)$ satisfies equations making it a crossed module. Conversely, any crossed module of Lie groups gives a strict Lie 2-group.
Let $\mathcal{G}$ be a strict Lie 2-group, let $(G, H, t, \alpha)$ be its crossed module, and let $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ be the corresponding differential crossed module.

If $P \to M$ is a principal 2-bundle with structure group $\mathcal{G}$ built using a cover $U_i$ of $M$, we can describe a 2-connection on $P$ in terms of:

- a $\mathfrak{g}$-valued 1-form $A_i$ on each open set $U_i$,
- an $\mathfrak{h}$-valued 2-form $B_i$ on each open set $U_i$,

together with some extra data and equations for double and triple intersections — following the ideas of Breen and Messing.

If $P$ is trivial all this reduces to:

- a $\mathfrak{g}$-valued 1-form $A$ on $M$,
- an $\mathfrak{h}$-valued 2-form $B$ on $M$.

Let’s restrict attention to this case and ponder the existence of a holonomy 2-functor

$$F: \mathcal{P}_2(M) \to \mathcal{G}$$

built using parallel transport.
Parallel Transport

Recall: $\mathcal{G}$ is a strict Lie 2-group with crossed module $(G, H, t, \alpha)$. A 2-connection on a trivial principal $\mathcal{G}$-2-bundle over $M$ consists of:

- a $\mathfrak{g}$-valued 1-form $A$ on $M$,
- an $\mathfrak{h}$-valued 2-form $B$ on $M$.

This data determines a smooth **holonomy** 2-functor

$$\text{hol}: \mathcal{P}_2(M) \to \mathcal{G}$$

if and only if the **fake curvature** vanishes:

$$F_A - dt(B) = 0,$$

where $F_A$ is the usual curvature of $A$, namely the $\mathfrak{g}$-valued 2-form

$$F_A = dA + A \wedge A.$$

The fake curvature vanishing ensures that parallel transport along a path of paths is *invariant under thin homotopies* — in particular, invariant under reparametrization! This implies that $\text{hol}(f)$ is well-defined for any 2-morphism $f: \gamma \to \gamma'$ in the the path 2-groupoid.

Vanishing fake curvature is also needed to obtain

$$\text{hol}(f): \text{hol}(\gamma) \to \text{hol}(\gamma').$$

All this generalizes to nontrivial principal $\mathcal{G}$-2-bundles: we obtain a holonomy 2-functor

$$\text{hol}: \mathcal{P}_2(M) \to \text{Trans}(P)$$

if and only if the fake curvature vanishes.