EULER CHARACTERISTIC
VERSUS
HOMOTOPY CARDINALITY

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Decategorification

Sets  Categories
Elements  Objects
Equations between  Isomorphisms between
  elements  objects
Functions  Functors
Equations between  Natural isomorphisms between
  functors  functors

Categorification

Decategorification takes a category &
produces the set of isomorphism classes
of objects;

Categorification is our attempt to undo this!

What if we categorify all of mathematics?
Let's start with N.
The obvious categorification of \( \mathbb{N} \) is \( \text{FinSet} \), the category of finite sets & functions between them:

\[
\begin{array}{ccc}
\mathbb{N} & \overset{\text{Decategorification}}{\longrightarrow} & \text{FinSet} \\
+ & : & \text{coproduct, aka disjoint union} \\
x & : & \text{product, aka Cartesian product} \\
0 & : & \text{initial object, aka empty set} \\
1 & : & \text{terminal object, aka 1-element set} \\
1 = 1 + \cdots + 1 & n = 1 + \cdots + 1 & \text{aka } n\text{-element set}
\end{array}
\]

The laws of arithmetic become natural isomorphisms:

\[A \times (B+C) \cong A \times B + A \times C\]
Some results in arithmetic are easy
to categorify:

\[
\binom{n}{k} \leftrightarrow \frac{n!}{k! \times (n-k)!}
\]

\[\uparrow\]

set of

k-element subsets

of \(n\)

group of permutations

of \(n\) mod stabilizer

of a k-element subset

\[n! := \text{Aut}(n)\]

Others are surprisingly hard...
DIVISION BY 3:

Given sets $A$, $B$ and isomorphism

$$f: 3 \times A \cong 3 \times B,$$

construct isomorphism

$$g: A \cong B.$$

1901: Bernstein claimed he could do it —
never said how.

1927: Lindenbaum & Tarski claimed they
could do it — never said how.

1949: Tarski claimed he'd forgotten how
Lindenbaum did it — invented new method.

1989: Conway & Doyle explain how to do it:
see http://math.ucr.edu/home/baez/week147.html

Try it! As a warmup, try division by 2 —
much easier.
WHAT ABOUT SUBTRACTION?

Can we categorify $\mathbb{Z}$?

Schanuel noticed an obstruction to the existence of "negative sets":

If $A + B \cong 0$ in some category,
then $A \cong B \cong 0$.

Nonetheless he proposed the Euler characteristic as a generalization of cardinality allowing negative integer values:

$\chi(X) = \dim H^0(X, \mathbb{Q}) - \dim H^1(X, \mathbb{Q}) + \cdots$

It's defined on finite CW complexes, homotopy invariant, & gets along with $+, \& , \times$.

But Schanuel preferred another variant....
EULER MEASURE
(Hadwiger,..., Schanuel)

Let Poly be the algebra of polyhedral subsets of \( \mathbb{R}^n \), generated by half-spaces \( \{ l(x) \geq c \} \) via finite unions, intersections & complements. There's a unique function

\[ \chi : \text{Poly} \rightarrow \mathbb{Z} \]

such that

1) \( \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \)

2) \( \chi(A) = (-1)^k \) if \( A \) homeomorphic to \( \mathbb{R}^k \)

Here \( \mathbb{R} \) or \( (0,1) \) plays the role of "the set with \(-1\) elements". \( \chi \) is invariant under homeomorphism but not homotopy, & agrees with usual Euler characteristic on compact sets.
EXAMPLES:

\[ \chi (\bullet) = 1 \quad \text{since} \quad \bullet \cong \mathbb{R}^0 \]

\[ \chi (\circ) = -1 \quad \text{since} \quad \circ \cong \mathbb{R}^1 \]

\[ \chi (\cdots) = \chi (\bullet) + \chi (\circ) \]
\[ = 1 + (-1) = 0 \]

\[ \chi (\cdot) = \chi (\circ) + \chi (\bullet) \]
\[ = 0 + 1 = 1 \]

\[ \chi (\Delta) = \chi (\circ) + \chi (\cdot) + \chi (\bullet) \]
\[ = 0 + 0 + 0 \]
\[ = 0 \]

So: Euler characteristic can be computed by "chopping up & adding up"!
Schanuel made a category Poly whose objects are polyhedral sets (in any $\mathbb{R}^n$) & whose morphisms are functions whose graphs are polyhedral sets, & proposed this as a categorification of $\mathbb{Z}$:

\[ \chi(A + B) = \chi(A) + \chi(B) \]
\[ \chi(A_+ c B) = \chi(A) + \chi(B) - \chi(C) \]

given pushout

\[
\begin{array}{ccc}
C & \hookrightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & A_+ c B
\end{array}
\]

\[ \chi(A \times B) = \chi(A) \times \chi(B) \]

Prop proved many things about this, e.g. categorifying equations like $\binom{-2}{3} = -4$. 
WHAT ABOUT DIVISION?

Can we categorify \( \mathbb{Q}^+ \)?

Division is easier to understand than subtraction!

\[
4/2 = 2
\]

Action of \( \mathbb{Z}_2 \) on 4 has 2 orbits

\[
5/2 = 2^{\frac{1}{2}}
\]

Action of \( \mathbb{Z}_2 \) on 5 has \( 2^{\frac{1}{2}} \) orbits?!

Given a group \( G \) acting on a set \( S \), let the weak quotient \( S//G \) be the groupoid with \( S \) as objects & a morphism \( g : s \to s' \) whenever \( g(s) = s' \). Define the cardinality of a groupoid \( X \) by

\[
|X| = \sum_{\text{iso classes}} \frac{1}{|\text{Aut}(x)|}
\]

Then:

\[
|S//G| = |S|/|G|
\]
Carrying on in this vein, we can define the cardinality of an \( n \)-groupoid: a gadget with objects

\[
\bullet \quad \rightarrow \\
\text{morphisms}
\]

2-morphisms

\[
\bullet \quad \rightarrow \\
\circ \quad \rightarrow
\]

and so on, composable in various ways & all invertible (at least up to equivalence).

This sounds complicated, but \( n \)-groupoids are just another way of talking about homotopy \( n \)-types: the nerve of an \( n \)-groupoid is a simplicial set with \( T_i = 0 \) for \( i > n \). So, let's define a cardinality for homotopy types!
HOMOTOPY CARDINALITY

Define the homotopy cardinality of a space $X$ to be

$$|X| = \frac{1}{|\pi_1(X)|} \cdot |\pi_2(X)| \cdot \frac{1}{|\pi_3(X)|} \cdots$$

if $X$ is connected &

$$\sum_i |X_i| = \sum_i |X_i|$$

more generally. It's defined on $\text{FinTop}$, the category of spaces w. finite homotopy groups, finitely many nonzero. It has:

$$|A + B| = |A| + |B|$$

$$|A \times B| = |A| \times |B|$$

$$|A \times_c B| = \frac{|A| \times |B|}{|C|} \quad \text{given homotopy pullback}$$

$$A \times_c B \longrightarrow B$$

$$\downarrow \quad \downarrow \text{fibration}$$

$$A \longrightarrow C$$
Given a fibration

\[ F \to E \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ B \quad \leftarrow \quad \text{connected} \]

we have \( |E| = |F| \times |B| \).

Looping is like reciprocal of a connected space:

\[ |\Omega X| = \frac{1}{|X|} \]

while classifying space is like reciprocal of a topological group:

\[ |BG| = \frac{1}{|G|} \]

More generally, the homotopy quotient

\[ X \// G := X \times_G EG \]

satisfies

\[ |X \// G| = |X| / |G| \]
CAN WE UNIFY EULER $\chi$
AND HOMOTOPY $| |$?

$\mathbb{N} \rightarrow \mathbb{Z}$

$\downarrow \downarrow$

$\mathbb{Q}^+ \rightarrow \mathbb{Q}$

categorifies to

$\downarrow \downarrow$

$\text{FinSet} \rightarrow \text{FinCW}$?

$\downarrow \downarrow$

$\text{FinTop} \rightarrow ???

$\chi$ and $| |$ are two faces of same concept,
but rarely seen together unless you stretch them:

Example: If $G$ is a finite group, $|BG| = \frac{1}{|G|}$.

What's $\chi(G)$? Count cells in simplicial
collection:

$\chi(G) = 1 - (|G| - 1) + (|G| - 1)^2 - \cdots$

* $\xrightarrow{g}$

0-cell nondeq.

1-cells nondeq.

2-cells

$\frac{1}{1+(|G|-1)}$

$= \frac{1}{|G|}$
Example: If $X$ is a compact surface of genus $g$, $\chi(X) = 2-2g$.
What's $|X|$? Only $\pi_1(X)$ is nontrivial, so $|X| = \frac{1}{|\pi_1(X)|}$.

To compute $|\pi_1(X)|$, take the usual presentation of $\pi_1(X)$ and let $a_n$ be the number of elements of length $n$:

$$|\pi_1(X)| = \sum_{n \geq 0} a_n$$

$$= \lim_{t \to 1} \sum_{n \geq 0} a_n t^n \quad \text{(Abel summation)}$$

$$= \frac{1}{2-2g}$$

Shown by Floyd/Plotnick & Grigorchuk.

MORAL: $\chi$ AND $|X|$ RELATED
       BY ANALYTIC CONTINUATION!