1. Suppose you have a spring in $\mathbb{R}^n$ with fixed endpoints, tracing out a curve
\[ q : [s_0, s_1] \to \mathbb{R}^n, \quad q(s_0) = a, q(s_1) = b. \]
If the spring is in a potential $V : \mathbb{R}^n \to \mathbb{R}$, what curve will the spring trace out when it’s in equilibrium?

Since the total spring energy is
\[ E = \int_{s_0}^{s_1} \left[ \frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right] ds, \]
we set $\delta E = 0$ and investigate the implications.

\[
\delta E = \delta \left( \int_{s_0}^{s_1} \left[ \frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right] ds \right) \\
= \frac{\partial}{\partial \varepsilon} \left( \int_{s_0}^{s_1} \left[ \frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right] ds \right) \bigg|_{\varepsilon=0} \quad \text{def of } \delta \\
= \int_{s_0}^{s_1} \frac{k}{2} \frac{\partial}{\partial \varepsilon} \left[ \dot{q}(s) \cdot \dot{q}(s) \right] + \frac{\partial}{\partial \varepsilon} \left[ V(q(s)) \right] ds \bigg|_{\varepsilon=0} \quad \text{linearity} \\
= \int_{s_0}^{s_1} k \dot{q}_\varepsilon(s) \frac{\partial}{\partial \varepsilon} \left[ q_\varepsilon(s) \right] + \nabla V(q_\varepsilon(s)) \frac{\partial}{\partial \varepsilon} \left[ q_\varepsilon(s) \right] ds \bigg|_{\varepsilon=0} \quad \text{chain rule} \\
= \int_{s_0}^{s_1} -k \left( \frac{\partial}{\partial \varepsilon} \dot{q}_\varepsilon(s) \right) + \nabla V(q_\varepsilon(s)) \frac{\partial}{\partial \varepsilon} \left[ q_\varepsilon(s) \right] ds \bigg|_{\varepsilon=0} \quad \text{IBP} \\
= \int_{s_0}^{s_1} -k \ddot{q}(s) + \nabla V(q(s)) \frac{\partial}{\partial \varepsilon} \left[ q_\varepsilon(s) \right] ds \bigg|_{\varepsilon=0} \quad \text{factoring} \\
= \int_{s_0}^{s_1} -k \ddot{q}(s) + \nabla V(q(s)) \frac{\partial}{\partial \varepsilon} \left[ q_\varepsilon(s) \right] ds \bigg|_{\varepsilon=0} \quad \text{letting } \varepsilon = 0.
\]

So if this is 0 for all allowable variations $\delta q$, we must have an integrand of 0, i.e.,
\[ k \ddot{q}(s) = \nabla V(q(s)). \]

2. Suppose the spring is in a constant downwards gravitational field in $\mathbb{R}^3$, so that
\[ V(x, y, z) = mgz, \]
where $m$ is the mass density of the spring and $g$ is the acceleration of gravity (9.8 m/s$^2$). What sort of curve does the spring trace out, in equilibrium?

Apply the answer from (1), $\nabla V = k \ddot{q}(s)$, and compute
\[ \nabla V(x, y, z) = \nabla(mgz) = [0, 0, mg] \]
to obtain the system

\[
\begin{align*}
\ddot{q}_1(s) &= 0 \\
\ddot{q}_2(s) &= 0 \\
\ddot{q}_3(s) &= \frac{ma}{k}.
\end{align*}
\]

All equations may be solved directly by successive integrations; the first two yield linear functions, and the third gives a polynomial in \(z\):

\[
\begin{align*}
q_1(s) &= (b_1 - a_1)s + a_1 \\
q_2(s) &= (b_2 - a_2)s + a_2 \\
q_3(s) &= \frac{ma}{2k}s^2 + \left(b_3 - a_3 - \frac{ma}{2k}\right)s + a_3,
\end{align*}
\]

where the values of the constants are deduced by comparison to the components of \(q(s_0) = a, q(s_1) = b\).

Thus the spring traces out a parabola lying in the vertical plane whose intersection with the \(xy\)-plane is the straight line from \((a_1, a_2)\) to \((b_1, b_2)\).

3. Using the energy, as given previously, replace the parameter \(s\) by \(it\) and show that up to a constant, the energy of the static string becomes the action for a particle moving in a potential.

\[
E = \int_{s_0}^{s_1} \left[ \frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right] ds
\]

\[
= \int_{s_0}^{s_1} \left[ \frac{k}{2} \frac{\partial}{\partial s} q(s) \cdot \frac{\partial}{\partial s} q(s) + V(q(s)) \right] ds
\]

\[
= \int_{t_0}^{t_1} \left[ \frac{k}{2} \frac{\partial}{\partial t} q(it) \cdot \frac{\partial}{\partial t} q(it) + V(q(it)) \right] d(it)
\]

\[
= \int_{t_0}^{t_1} \left[ \frac{k}{2} i\dot{q}(it) \cdot i\dot{q}(it) + V(q(it)) \right] i dt
\]

\[
= \int_{t_0}^{t_1} \left[ -\frac{k}{2} \dot{q}(it) \cdot \dot{q}(it) + V(q(it)) \right] i dt
\]

\[
= -i \int_{t_0}^{t_1} \left[ \frac{k}{2} \dot{q}(it) \cdot \dot{q}(it) - V(q(it)) \right] dt
\]

\[
= -i S(q)
\]
4. The analogy between statics and dynamics.

<table>
<thead>
<tr>
<th>Principle of Least Energy</th>
<th>Principle of Least Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>spring</td>
<td>particle</td>
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<td>energy</td>
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<td>“tension” energy</td>
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<td>potential energy</td>
<td>potential energy</td>
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<tr>
<td>spring constant $k$</td>
<td>mass $m$</td>
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</tbody>
</table>

5. What particular dynamics problem (pun intended) is the statics problem in 2 analogous to? How is the solution to the statics problem related to the solution of this dynamics problem?

The problem is: “What curve does a particle trace out when it moves through a (gravitational) potential, minimizing action?”

The answer is again a parabola; the negative sign introduced during the rotation into imaginary time has the effect of flipping the parabola, so it opens downwards, as is appropriate for the path of a projectile.

6. What does Newton’s law $F = ma$ become if we formally replace $t$ by $s = it$?

Since

$$F = ma$$
$$-\nabla V = m \frac{\partial^2}{\partial t^2} q(it)$$
$$-\nabla V = -m \ddot{q}(it)$$
$$\nabla V = m \ddot{q}(it)$$
$$F = -m \ddot{q}(it)$$
$$F = -ma,$$

we see that in an imaginary world, Newton’s law is reversed.