A Spring in Imaginary Time

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1. If we have a spring with fixed ends tracing a curve \( q \) in \( \mathbb{R}^n \) whose energy is \( E \) as given, we find that taking the variation of \( E \) gives:

\[
\delta E = \int_{s_0}^{s_1} \left( \frac{k}{2} \dot{q}(s) \cdot \ddot{q}(s) + V(q(s)) \right) ds
\]

The boundary terms from the integration by parts disappear since we only consider variations which fix the endpoints of the curve traced by the spring (i.e. \( \delta q = 0 \) at \( s_0 \) and \( s_1 \)). Then if we have that \( \delta E = 0 \) for all variations \( \delta q \), then the equation \( q \) must satisfy is

\[
-k\ddot{q}(s) + \nabla V(q(s)) = 0
\]

or

\[
k\ddot{q}(s) = \nabla V(q(s))
\]

2. If \( V = mgz \) in \( \mathbb{R}^3 \), we have that \( \nabla V = (0, 0, mg) \), so that \( \ddot{q}(s) = (0, 0, mg) \). That is, the curve has a constant positive acceleration \( \frac{mg}{k} \) in the \( z \) direction with respect to the parameter \( s \), and constant velocity in the \( x \) and \( y \) directions with respect to \( s \). So we can also think of the curve as having constant acceleration in the \( z \) direction with respect to distance in the \( (x, y) \) direction of the particle’s horizontal velocity. So the curve is a parabola with local maxima at the endpoints.

3. Replacing \( s \) by \( t \), we get

\[
E = \int_{s_0}^{s_1} \left( \frac{k}{2} \dot{q}(t) \cdot \ddot{q}(t) + V(q(t)) \right) dt
\]

Indeed, this is just \(-i\) multiplied by the action along a path of a particle moving in a potential \( (K - V) \), where \( K = \frac{k}{2} \|q\|^2 \) with \( k \) playing the role of the mass \( m \).

4. We have the analogy:
5. The statics problem in (2) corresponds to the dynamics problem of a particle moving in a potential with constant gradient. The solution to that problem has the particle moving in a parabola with local minima at the endpoints—the acceleration is in the direction opposite to that observed in the statics problem of the spring.

6. Formally replacing $t$ by $t$ in Newton’s equation $F = ma$, where $a(t)$ is $\ddot{x}(t)$, the second derivative of position with respect to $t$:

$$F = m \frac{d^2}{dt^2} x(t)$$
$$= m \ddot{x}(t)$$
$$= -m \ddot{x}(t)$$

(This equation $F = -m \ddot{x}$ is reminiscent of Hooke’s law for springs, except that “acceleration” $\ddot{x}$ plays the role of displacement.)