Why $n$-Categories?

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$k = 0$

$n = 0$

$k = 1$

$n = 1$

$k = 2$

$n = 2$

many figures by Aaron Lauda
Every Interesting Equation is a Lie!

\[ x = x \] true, but boring

\[ x = y \] potentially interesting—
but says two *different*
things are *the same*!

Any interesting equation is really a summary of an interesting *process*. For example:

\[
\begin{align*}
2 + 3 & \quad \parallel \\
 & \quad 5
\end{align*}
\]

is short for:

\[
\begin{array}{c}
2 \quad 3 \\
\mid \\
5 \\
\end{array}
\]

\[
\begin{array}{c}
2 \quad \langle \quad 3 \\
\mid \\
\downarrow \\
5 \\
\mid \\
\downarrow \\
X \\
\end{array}
\]
Codimension 1:
Composition, Associator,...

We add by putting 0-dimensional rocks in a 1-dimensional line:

\[
\begin{array}{c}
2 + 3 \\
\hline \hline
\hline
\hline
5
\end{array}
\]

Proving associativity takes time:

\[
\begin{array}{c}
(2 + 3) + 1 \\
\hline \hline
\hline
\hline
2 + (3 + 1)
\end{array}
\]

We call this proof the **associator**: note 1-dimensional ‘worldlines’ in 2-dimensional ‘spacetime’, hence again codimension 2-1 = 1.
The associator satisfies the **pentagon identity**: 

\[(A+B) + C + D = (A+B) + (C+D)\]

But the process of proving this traces out a 2d surface in 3 dimensions: the **pentagonator**!

And so on....
Higher Associative Laws:
a Simplicial Viewpoint

The hierarchy of ‘higher associative laws’ can also be formalized using simplices:

If every way of filling the triangular ‘horn’ factors through $F$, we may call it a process of composing $A$ and $B$, and call $C$ a composite. This applies to addition of sets:

and many other examples, especially composition of paths in a topological space.
Considering higher-dimensional horns, we get this hierarchy:

Object:

\[ \bullet \]

Morphism:

\[
\begin{array}{c}
A \\
\rightarrow \\
A
\end{array}
\]

Composition:

\[
\begin{array}{c}
A \\
\downarrow \\
B \\
\rightarrow \\
AB
\end{array}
\]

Associator:

\[
\begin{array}{c}
(AB)C \\
\Rightarrow \\
A(BC)
\end{array}
\]

... and so on forever: the Stasheff associahedra!

(But there’s a subtlety in higher dimensions, appearing already in the next associahedron: the *commutative* law gets involved!)
Codimension 2: Braiding, Yang–Baxterator,...

If space is at least 2-dimensional, we can prove the commutative law:

\[ A + B = B + A \]

by sliding two piles of rocks around each other. But the proof takes time:

Note 1-dimensional ‘worldlines’ in 3-dimensional ‘spacetime’: the braiding. The braiding in turn satisfies the Yang–Baxter equation:

\[
\begin{align*}
\begin{array}{c}
A \quad B \quad C \\
C \quad B \quad A
\end{array} & = \\
\begin{array}{c}
A \quad B \quad C \\
C \quad B \quad A
\end{array}
\end{align*}
\]
The process of proving the Yang–Baxter equation traces out a 2d surface in 4 dimensions, the Yang–Baxterator:

This in turn satisfies the Zamolodchikov tetrahedron equation:

but the proof of this traces out a 3d surface in 5 dimensions... and so on!
Higher Commutative Laws: a Cubical Viewpoint

The hierarchy of ‘higher commutative laws’ can also be formalized using cubes.

Braiding:

Yang-Baxterator:

Similarly, the Zamolodchikov tetrahedron equation relates the ‘front’ and ‘back’ of a 4-cube, each of which is built from 4 Yang–Baxterator 3-cubes... and so on!
Codimension 3: Syllepsis,...

If space is 2-dimensional, there are two fundamentally different proofs that $A + B = B + A$: the braiding versus the inverse braiding:

since these are nonisotopic braids in 3d spacetime. But if space is at least 3-dimensional, one proof can be continuously deformed to the other:

since all braids in 4d spacetime are isotopic! This process traces out a 2d surface in 5 dimensions: the syllepsis.
The syllepsis satisfies a law of its own... but the proof of this traces out a 3d surface in 6 dimensions... and so on!

And so on for higher codimensions!

For example, in codimension 4 we get an isotopy between the syllepsis and the ‘inverse syllepsis’, which are 2d surfaces in 6 dimensions. This isotopy traces out a 3d surface in 7 dimensions, and satisfies a law whose proof traces out a 4d surface in 8 dimensions, etc....

In short: a hierarchy of ‘higher braidings’, one for each codimension $k \geq 2$, each satisfying a hierarchy of laws.

Warning: this is a drastically simplified version of the story!
Why $n$-Categories?

We’ve seen how beautiful but overwhelmingly complex structures arise when we treat every equation as a summary of a process. I’ve only begun to describe these structures! To keep track of them, we need $n$-categories - and not just a definition, but a detailed theory of them.

In particular:

Let a $k$-tuply monoidal $n$-category be an $(n+k)$-category that is trivial below dimension $k$ - viewed as an $n$-category with $k$ ways to multiply objects.

Everything I’ve said so far should be summarized by some theorem relating $k$-tuply monoidal $n$-categories to ‘$n$-braids in codimension $k$’. A bit more precisely...
The Braid Hypothesis: The free $k$-tuply monoidal $n$-category on one object is $n\text{Braid}_k$, where:

- objects are finite collections of points in $\mathbb{R}^k$, i.e. elements of

$$X_k = \bigcup_{j=0}^{\infty} \left\{ (x_1, \ldots, x_j) : x_i \in \mathbb{R}^k, x_i \text{ distinct} \right\} / S_j$$

- morphisms are paths in $X_k$,
- 2-morphisms are paths of paths in $X_k$,
- etc...
- $n$-morphisms are homotopy classes of paths of paths of paths... in $X_k$. 

THE PERIODIC TABLE

We expect $k$-tuple monoidal $n$-categories go like this:

<table>
<thead>
<tr>
<th></th>
<th>$n = 0$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 0$</td>
<td>sets</td>
<td>categories</td>
<td>2-categories</td>
</tr>
<tr>
<td>$k = 1$</td>
<td>monoids</td>
<td>monoidal categories</td>
<td>monoidal 2-categories</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>commutative monoids</td>
<td>braided monoidal categories</td>
<td>braided monoidal 2-categories</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>“”</td>
<td>symmetric monoidal categories</td>
<td>sylleptic monoidal 2-categories</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>“”</td>
<td>“”</td>
<td>symmetric monoidal 2-categories</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>“”</td>
<td>“”</td>
<td>“”</td>
</tr>
<tr>
<td>$k = 6$</td>
<td>“”</td>
<td>“”</td>
<td>“”</td>
</tr>
</tbody>
</table>

- $n$ acts like the *dimension*.
- $k$ acts like the *codimension*. 
The Braid Hypothesis: Examples

The free monoid on one generator is 0Braid\(_1\), the **natural numbers**: isotopy classes of collections of points on the line.

The free braided monoidal category on one generator is 1Braid\(_2\), the **braid groupoid**: collections of points in the plane and isotopy classes of braids in 3d going between these.

The free symmetric monoidal category on one generator is 1Braid\(_3\), the **groupoid of finite sets**: collections of points in \(\mathbb{R}^3\) and isotopy classes of braids in 4d going between these. *We live here!*

The free braided monoidal 2-category on one generator is 2Braid\(_4\), the **2-braid 2-groupoid**: collections of points in the plane, braids in 3d between these, and isotopy classes of 2-braids in 4d between these.

All these are just different *views* of a single concept: ‘the true natural numbers’.
How To Understand $n$-Categories

Topology lights the way, since every space $X$ has a ‘fundamental $\omega$-groupoid’, $\Pi_\infty(X)$. In the simplicial framework:

\[
\text{it's the simplicial set whose } j\text{-cells are just maps } F: \Delta^j \to X.
\]

Technically this is a Kan complex: every horn has a filler!

The Homotopy Hypothesis (baby version): equivalence classes of $\omega$-groupoids are the same as homotopy types: homotopy equivalence classes of locally nice spaces (e.g. CW complexes).

$n$Braid$_k$ corresponds to the homotopy type where we take $X_k$, the space of finite collections of points in $\mathbb{R}^k$, and ‘kill homotopy groups above $\pi_n$’ by attaching cells.
This is Just the Beginning...

... though not of my talk, you’ll be glad to know.

More interesting than \(n\)-groupoids are ‘\(n\)-categories with duals’, where all \(j\)-morphisms have, not weak inverses, but ‘duals’ or ‘adjoints’:

\[
\begin{array}{ccc}
A & \rightarrow & A^* \\
\downarrow & & \downarrow \\
A^* & \rightarrow & \bullet
\end{array}
\]

satisfying weakened ‘zigzag identities’:

\[
\begin{array}{ccc}
A & \Rightarrow & A \\
\downarrow & & \downarrow \\
A & \Rightarrow & A^* \\
\end{array}
\]

which satisfy laws of their own... and so on.

This allows a form of ‘subtraction’, so it gives us some new views of ‘the true integers’...
The Tangle Hypothesis: The free $k$-tuply monoidal $n$-category with duals on one generator is $n\text{Tang}_k$: top-dimensional morphisms are $n$-dimensional framed tangles in $n + k$ dimensions.
Algebraic Structures and the Free Such Structures on One Generator

<table>
<thead>
<tr>
<th>sets</th>
<th>1</th>
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<tbody>
<tr>
<td>monoids</td>
<td>( \mathbb{N} )</td>
</tr>
<tr>
<td>groups</td>
<td>( \mathbb{Z} )</td>
</tr>
<tr>
<td>( k )-tuply monoidal ( n )-categories</td>
<td>( n \text{Braid}<em>k \cong \Pi</em>{n+k}(X_k) )</td>
</tr>
<tr>
<td>( k )-tuply monoidal ( \omega )-categories</td>
<td>( X_k )</td>
</tr>
<tr>
<td>( k )-tuply groupal ( n )-groupoids</td>
<td>( \Pi_{n+k}(S^k) )</td>
</tr>
<tr>
<td>( k )-tuply groupal ( \omega )-groupoids</td>
<td>( S^k )</td>
</tr>
<tr>
<td>strict ( k )-tuply groupal ( \omega )-groupoids</td>
<td>( K(\mathbb{Z}, k) )</td>
</tr>
<tr>
<td>( k )-tuply monoidal ( n )-categories with duals</td>
<td>( n \text{Tang}_k )</td>
</tr>
</tbody>
</table>

Thom-Pontryagin map:

\[ n \text{Tang}_k \to \Pi_{n+k}(S^k) \]

From homotopy to homology:

\[ \Pi_{n+k}(S^k) \to K(\mathbb{Z}, k) \]