Gauge Theory on a Graph:

A graph $\mathcal{G}$ is a finite set of edges $E$ and finite set $V$ of vertices and maps:

$s, t : E \rightarrow V$

\[ s(e), \quad e, \quad t(e) \]

Fixing a group $G$, a connection $\&$ is a map

$A : E \rightarrow G \text{ (group elt)}$

Assigning to each edge $e$ a holonomy $A_e \in G$.

Recall: $SO(3)$ rotations in 3d, $R$ linear

$R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ s.t

$Rv \cdot Rw = v \cdot w \text{ (preserves dot product)}$

$\det R = 1$

$SO(3,1)$: transformations of spacetime

$L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $Lv \cdot Lw = v \cdot w$

$V \cdot W = V_1W_1 + V_2W_2 + V_3W_3 - V_4W_4 \in \det L = 1$
A gauge transform on \( G \) is a map

\[ g : V \rightarrow G \]  
(from vertices to group)

- Space, Spacetime are curved

\[ \text{ex)} \]

\[ \text{start here.} \]

\[ \text{carry arrow in/out} \]
\[ \text{rotating,} \]
\[ \text{end back up} \]
\[ \text{at N pole w/} \]
\[ \text{arrow rotated.} \]

\( \text{gauge transformation} \) - takes into account

we may make a different choice of bases.

\[ \mathbf{f}_1 \rightarrow \mathbf{e}_2 \]

\[ \mathbf{f}_2 \rightarrow \mathbf{e}_1 \]

we must translate from \( \mathbf{f}_1 \) to \( \mathbf{f}_2 \)

at every point.

we use a gauge transformation.

\( \text{gauge transformation} \) - assigns to each pt
in space same rotation
(gap elt)

\( \text{gauge theory} \) - worrying about choice of
based on tang. space.
Gauge transformations act on connections via

\[(gA)_e = g_{t(e)} A_e g^{-1}_{s(e)}\]

Choice 1

"source" \( s(e) \)

\[A_e \in G \]

"target" \( t(e) \)

Choice 2

Horizontal arrows are change of coords.

So, since diagram commutes:

\[(gA)_e = g_{t(e)} A_e g^{-1}_{s(e)}\]
groups act on themselves: left/right muet. or conjugation

Let $A = \{\text{connections on } \mathbb{F} \}$

(all funts from $E \to G$)

$g = \{\text{gauge transformations on } \mathbb{F} \}$

$g$ is a group:

$(gh)v = guv; v \in V, g, h \in g$

and $1v = 1 \in g$

$g$ acts on $A$: we have a map

$g \times A \to A$ given by

$(gA)_v = g_{t(e)}A_v g_{s(e)}^{-1}$ st

$1A = A$

$g(h(A)) = (gh)(A)$

Check:

$(1A)_e = 1e A \overset{1^{-1}}{=} A_e$
connections tell how curved space is
we're going to say 2 connections are equivalent if you can get from one to another by a gauge transformation.

\[ a \rightarrow a/g \]

If \( G \) is a compact topological group

1) \( \text{SO}(3) \) compact
2) \( \text{SO}(3,1) \) not compact

we want to understand \( L^2(\mathbb{R}) \), \( L^2(\mathbb{R}/g) \)

\( \text{gauge invariant} \)

it has a unique Borel measure called normalized Haar measure \( \mu \), s.t.

1. \( \int_G 1 \, d\mu(g) = 1 \)

2. \( \int_G f(g) \, d\mu(g) = \int_G f(hg) \, d\mu(g) \quad \forall h \in G \)

\( \text{(left invariance of Haar measure)} \)

\( \text{(group acting by left mulit)} \)
\[ \int_{G} f(g) \, du(g) = \int_{G} f(gh) \, du(g) \quad \forall h \in G. \]

(right invariance)

Ex) When \( G = \mathbb{R} \)
not compact, can't get property (i) (can't get normalization)

We have \( \int_{\mathbb{R}} f(x+n) \, dx = \int_{\mathbb{R}} f(x) \, dx \quad \forall h \in \mathbb{R} \)

so(3, 1) has a measure satisfying 2, 3 but not 1 (same as \( \mathbb{R} \))

If \( G \) is compact, we define

\[ L^2(G) = \left\{ \psi : G \to \mathbb{C} \mid \int_G |\psi(g)|^2 \, du(g) < \infty \right\} \]

This is defined using Haar measure.

Now -

\[ \mathcal{D} = G^E \quad \text{(product of copies of } G, \text{ one for each edge)} \]

gets a measure:

\[ \mu \times \ldots \times \mu \quad \text{one for each edge} \]

Let \(|E| = n\) \quad (n copies)
such that if $f : \mathcal{A} \rightarrow \mathcal{C}$

$$\int f d(u_1 \times \cdots \times u) = \int f(g_1, \ldots, g_n) d\mu(g_1) \ldots d\mu(g_n)$$

This is the measure we'll use to define $L^2(\mathcal{A})$

$$L^2(\mathcal{A}) = \{ Y : \mathcal{A} \rightarrow \mathcal{C} \mid \int |Y(A)|^2 d(u_1 \times \cdots \times u) < \infty \}$$

**Ex**)

\[ \bullet \rightarrow \bullet \]  \[ \mathcal{A} = G \]

$Y : \mathcal{A} \rightarrow \mathcal{C}$ is $Y : G \rightarrow \mathcal{C}$

$$\int |Y(g)|^2 d\mu(g) \text{ where } X \subseteq G$$

$x \uparrow$ tells us the probability that a particle will be rotated a certain amount

This is the probability that if you parallel transport a particle along this edge it gets "rotated" by some amount $\theta \in \mathbb{K}$.  

\[ n = \pi \]
$g$ acts on $\Delta$ so $g$ acts on $L^2(\Delta)$ via:

$$(g\psi)(A) = \psi(g^{-1}A) \quad \forall \psi \in L^2(\Delta)$$

We put $g^{-1}$ here (as opposed to $g$) because:

if we put $g$ here, we wouldn't have an action of $g$ on $L^2(\Delta)$.

Check:

1. $1\psi = \psi$

$$(gh)\psi = g(h\psi)$$

2. $1\psi(A) = \psi(1A) = \psi(A) \Rightarrow 1\psi = \psi$.

$$(gh)\psi(A) = \psi((gh)A) = \psi(g(h(A)))$$

\[= (g\psi)(h(A))\]

So, we need to have $g^{-1}$ to end up with the right thing:

$$(gh)\psi(A) = \psi(gh^{-1}A) = \psi(h^{-1}g^{-1}(A))$$

$$(A)_{gh} = (A)_{g^{-1}} = (A)_{h^{-1}} = \psi(h^{-1}(g^{-1}(A)))$$

$$(A)_{g^{-1}} = (A)_{h^{-1}} = \psi(h^{-1}(g^{-1}(A)))$$

$$= g(h(\psi))(A)$$
so \((gh)\psi = g(h(\psi))\)

**Note:** If our group is abelian, we don't need the inverse.

**Recall:** if \(\psi, \phi \in L^2(\Omega)\)

\[
\langle \psi, \phi \rangle = \int_{\Omega} \overline{\psi(A)} \phi(A) \mu_{\text{copies of } \mu \text{ for each edge}}
\]

**Check:** if \(g \in g\) (gauge transformation)

\[
\langle g\psi, g\phi \rangle = \langle \psi, \phi \rangle
\]

\[
\langle g\psi, g\phi \rangle = \int_{\Omega} (\overline{g\psi}(A)) g\phi(A) \mu_{\text{each edge}}
\]

\[
= \int_{\Omega} \overline{\psi}(g^{-1}A) \phi(g^{-1}A) \mu_{\text{each edge}}
\]

\[
= \int_{\Omega} \overline{\phi}(g^{-1}A) \mu_{\text{each edge}}
\]

\[
= \int_{\Omega} \overline{\phi}(g^{-1}A_{1}, \ldots, g^{-1}A_{n}) \mu(A_{1}) \ldots \mu(A_{n})
\]
\[ = \int \phi(A_1, \ldots, A_n) \, du(A_1) \ldots du(A_n) \]

\[ = \langle \psi, \phi \rangle \]

So we say \( g \) has a unitary action (or unitary representation — action on a vector space)
on \( L^2(\mathbb{R}) \).

- gauge symmetries — change description of same situation.

Let \( \mathbb{A}/g = \{ \text{equivalence classes } [A], A \in \mathbb{A} \} \)

\[ [A] = [gA] \]

Ex) gauge fixing vs. moding out by gauge transformations

\( \mathbb{R}^2 \) is acted on by \( \mathbb{R}^2 \)

\[ a \]

\[ \underline{g} \]  

\[ \text{gauge fixing — pick one } \mathbb{R}\]

\[ \text{from each equiv. class (on each line)} \]

\[ \mathbb{A}/g = \text{lines } \sim \mathbb{R} \]
This will be called the "spin network basis."

What? an explicit orthonormal basis on $L^2(\mathcal{A})$.

gauge invariant states, while unit vectors $|\gamma, y\rangle$ describe invariant under gauge transform matrices.

Unit vectors $y \in L^2(\mathcal{A})$ describe a measure on $\mathcal{A}/g$.

We can push a measure on $\mathcal{A}$ forward to a measure on $\mathcal{A}/g$.

$A \mapsto [A]_{\gamma}$

$p \mapsto \alpha$
If $G = SU(2)$ and $\gamma$ is trivalent (3 edges meet at every vertex),

This orthonormal basis is in 1-1 correspondence w/ spin networks.

ia) labellings of edges by spins $s^i$ at each vertex. The spins on the 3 edges that touch that vertex are an admissible triple:

\[
\begin{align*}
\mathbf{j} \cdot \mathbf{j} &\leq (\mathbf{j} + 1)^2 \\
\mathbf{j} + \mathbf{j} + \mathbf{j} &\in \mathbb{Z}
\end{align*}
\]

(A inequality)

\[\text{sketch: } -u^2 + v^2 + x^2 \Rightarrow \text{Hyperbola} \]