10/15/02

Imp: sequence of #’s:

\[ 1, \infty, 5, 6, 3, 3, 3, 3, \ldots \]

= # of regular polytopes in various dimensions.

regular polytopes in 0-d:

\[ 1, \infty, 5, 6, 3, 3, 3, \ldots \]

regular polygons

regular polyhedra (all faces are same)

tetrahedron

3 triangles per vertex

octahedron

4 triangles meet @ vertex

5 triangles meet @ vertex

20 sides - icosahedron

(polytopes in 2-d) convex

trying to do 6 triangles @ a vertex:

tiling of plane
move up to squares:

try w/ more squares:

move up to pentagons:

can't put 3 hexagons together

cube — the earth since you can build up things (stack weel) out of cubes

icosahedron — "roundness" in a sense

5 Platonic solids!

• Plato — relationship bet. 4 elts & these 5 solids.
  hmm... predict a 5th element.
continuing explaining the sequence...

The 3 infinite sequence:

\[ \text{n- simplices:} \]

vertices are \( n+1 \) points in \( \mathbb{R}^n \)
all equidistant from each other.

to get from put pt in center and pull up until sides are equal.

4-simplex has 5 tetrahedral faces & 5 vertices

\[ \text{n-cube:} \]

\([0,1]^n \subseteq \mathbb{R}^n\]

4-cube has 8 cubical faces & 16 vertices.

4th dim. move vertices from front face, back face in 4th dim.
Poincaré duality:

- dual is itself.

- dual is octahedron (and vice versa)

icosahedron $\leftrightarrow$ dodecahedron

- 20 faces
- 12 vertices

- 12 faces
- 20 vertices

Duality takes

- vertices $\rightarrow$ faces
- edges $\rightarrow$ edges
- faces $\rightarrow$ vertices

* The regular polygons are all self-dual as are the regular polytopes in 0-d, 1-d.
* Every dimensional simplex is self-dual!

But the cubes aren't self-dual.

- (diamond)

- cube $\leftrightarrow$ octahedron (super diamond!)
# Vertices is an $n$-dim` cross-polytope

**Vertices are:**

$(\pm 1, 0)$, $(0, \pm 1)$

Cube

Octahedron

$(\pm 1, 0, 0)$

$(0, \pm 1, 0)$

$(0,0, \pm 1)$

Distance bet.

Any 2 of these

Pts is $\sqrt{2}$

Vertices of $n$-dim` cross-polytopes are

$(0,..., \pm 1,...,0)$

The 4-dim` cross-polytope

has 16 tetrahedral faces $\mathcal{Q}_1$

8 vertices.

Note: In dimensions $3$ & $4$ weird things happen

in all branches of mathematics.
The 4-d regular polytopes:

The 4-simplex, 4-cube, 4-d cross-polytope, and 3 more!

We get these remaining 3 by looking at the symmetry groups of the Platonic solids, (so 3 of 4 dim'l stories are related.)

tetrahedron:

![Tetrahedron Diagram]

The rotational symmetry group of any Platonic solid is a finite subgroup of $SO(3)$.

Must be a subgroup of $S_4$ (permutations of 4 vertices).

so can fix 1 vertex 1 2 3 4 1 4 2 3 $\triangleright$ cyclically permute

or

These give us only the even permutations so we get $A_4$

$|A_4| = 12$.

We could have figured out how many elts (12) there were:

we have 4 choices for favorite vertex, thee which of 3 faces get mapped where.
size of $|A_4|=12$ : $4 \times 3$ choices of vertex

choices of vertex faces per vertex

choices of faces meet @ any vertex

The cube has a symmetry group of size $8 \times 3 = 24$.

Maybe it's $S_4$?

green axes connect opposite corners... top-bottom diagonal.

The cube has 4 diagonals and they can be permuted arbitrarily so the group is $S_4$.

Note: $A_4 \leq S_4$ so there should be a relationship between tetrahedron and cube.

The tetrahedron is in the cube: every other vertex.

Not every perm. of $S_4$ preserves this green tetrahedron, but this gives $A_4 \hookrightarrow S_4$. 
icosahedron: \( 12 \times 5 = 60 \)
dodecahedron: \( 20 \times 3 = 60 \)

The symmetry gap has size 60.

If we're lucky, it'll be \( A_5 \).

Is it \( A_5 \)? Find 5 things in dodecahedron that get permuted.

The Greek's favorite \( \phi \) is the golden ratio.

Euclidean algorithm shows goes on forever (must be irrational)

\( \phi : 1 : \phi - 1 \)

\[ \frac{\phi}{1} = \frac{1}{\phi - 1} \]

\[ \frac{1}{\phi} = \phi - 1 \]
\[ l = \phi^2 - \phi \]
\[ \phi^2 - \phi - 1 = 0 \quad \phi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 + \sqrt{5}}{2} = 1.6180339... \]

related to pentagon:

3 intersecting golden ratios

yx plane \((\pm \phi, \pm 1, 0)\)

xz plane \((\pm 1, 0, \pm \phi)\)

yz plane \((0, \pm \phi, \pm 1)\)

this has 12 vertices these 3 intersecting planes they're the vertices of the icosahedron

so golden rectangles are hiding inside icosahedron.
A cross is a configuration of 3 golden rectangles formed by the vertices of an icosahedron. There are 5 of these.

Every edge is in exactly one of these.

Each cross has 6 edges of an icosahedron in it, and an icosahedron has 30 edges. So there are 5 of these crosses in the icosahedron.

*5 ways to stick a cube in dodecahedron.

Check: you can get any even permutation of these by a symmetry.

So symmetry group of dodecahedron/icosahedron is $A_5$. 
10/17/02  **Platonic Solids (cont'd)**

- we live in 4-d Minkowski spacetime
- Lie gps - gps of continuous symmetries

There are 6 regular polytopes in 4-dimensions:

1) 4-d Simplex  \(\sim\) tetrahedron

2) 4-cube  \(\sim\) cube

3) 4d cross-polytope  \(\sim\) octahedron

We've talked about these; we have 3 more.
(One like icosahedron, one like dodecahedron)

We get these other 3 from symmetry groups of 3d platonic solids:

\[
\begin{align*}
A_4 & \text{ - tetrahedron (even perm)} & \# \text{elts} = 12 \\
S_4 & \text{ - symm. gp of cube/octahedron} & 24 \\
A_5 & \text{ - symm. gp of dodecahedron/icosahedron} & 60
\end{align*}
\]

Let's start w/ \(A_4\).

Pullback square:

\[
\begin{array}{ccc}
\tilde{A}_4 & \longrightarrow & \text{SU}(2) \\
\downarrow \quad & & \downarrow \text{onto map} \\
A_4 & \longrightarrow & \text{SO}(3)
\end{array}
\]

\(\tilde{A}_4 \subset \text{SU}(2)\)
each elt of $A_4$ has 2 elts from $\tilde{A}_4$ mapping to it.

$|\tilde{A}_4| = 24$

$\tilde{A}_4 \subset \text{unit quaternions} \rightarrow SU(2) = S^3 \cong \mathbb{R}^4$

$\downarrow$

$\tilde{A}_4 \subset A_4 \rightarrow SO(3)$

$S^2$ take convex hull

---

So - $\tilde{A}_4$ is the vertices of a 4d regular polytope.

Note: Double cover of $SO(7)$ isn't a sphere. only true w/ $S^3, S^2$. 
\[ 1 = \text{id} \]

\[ \text{su}(2) = S^3 \]

22 other pts located on this 3 sphere.
\[ \{1, -1\}, \quad \text{id} = 1 \]

- this elt \(^x\), gets sent to rotation by \(2\Theta\) in \(SO(3)\).

- rotating tetrahedron \(1/3\) around is smallest rotation.
  So- there are 8 points in \( \hat{A}_4 \) where \(\Theta = \pi/3\)
  that map to rotations by \(\pi/3\).

  (can rotate tetrahedron \(1/3\) of way left or right, and
  can do for each of 4 vertices)

\[ \uparrow \quad \text{or} \quad \downarrow \]

These 8 points form a cube \( S^2 \subseteq S^3 \).

\[ \in \quad S^3 \]

This is a local nbhd of id. All nbhds of other pts. look like this.
We've got a solid whose 2-d faces are triangles & 3-d faces are octahedra.

This polytope has 24 vertices, 96 edges

96 triangular faces

24 octahedral faces

We have 6 octahedra meet at each vertex (top, bottom, front, back, left, right)

naively you'd meet 6 \cdot \# of vertices

but we've over counted!

\[ 24 \times 6 = 144 \text{ octahedra} \]

\[ \frac{\text{vertices}}{\text{octahedra per vertex}} \]

But each octahedron has 6 vertices, so we divide by 6.

Thus -- it's self-dual (same \# vertices & faces)

Duality -- switch vertices for top dim'le faces.

Triangles: 8 \Delta's per vertex

\[ 12 \times 24 \]

\[ \frac{\text{triangles}}{\text{vertices per vertex}} \]
But, each triangle has 3 vertices.

\[ 24 \times 12 \div 3 = 24 \times 4 = 96 \]

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Triangles</th>
<th># Vertices per vertex</th>
<th># Vertices per triangle</th>
</tr>
</thead>
</table>

\[ 24 \times 8 \div 2 = 96 \text{ edges} \]

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Edges</th>
<th>Each edge has 2 vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>per vertex</td>
<td></td>
<td>per vertex</td>
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</tbody>
</table>

What we've described above is the 24-cell, which is self-dual and has no analogue in 3-dim. Comes from symmetries of tetrahedron!

Symmetries of cube:

\[
\text{Symmetry group } S_4 
\rightarrow
\text{SU(2)} \leq S^3
\]

48 elts

\[ \text{2-1 onto} \]

\[
\text{Symmetry group } S_4
\rightarrow
\text{SO(3)}
\]

There are NOT the vertices of a (Platonic solid) regular polytope. Puzzle - what solid do you get?
This new shape won't be self-dual, so we'll take its dual.

\[ \overset{\sim}{A}_5 \leftarrow \overset{\sim}{A}_5 \rightarrow SU(2) = S^3 \]

A_5 form the vertices of a regular polytope.

Want rotations of icosahedron have smallest angle? It has 5-fold symmetry. So, rotate \( \overset{\sim}{A}_5 \).

So - \( \overset{\sim}{A}_5 \) will have 12 vertices with \( \theta = \frac{\pi}{5} \) corresponding to rotations by \( \frac{2\pi}{5} \) that are symmetries of icosahedron.

* Here - we don't have clockwise/counterclockwise rotation.
For icosahedron—every vertex has 2 pts. directly opposite.
Not true for tetrahedron—could rotate 0 or 180 since no vertex opposite of each.

These 12 pts are arranged in shape of icosahedron.
This 4d solid has 3d faces shaped like tetrahedra.

\[
\text{Stick in each triangle a tetrahedron.}
\]

How many tetrahedra?

\[
\frac{120 \times 29}{4} = 600
\]

We call this the 600-cell.

We can take the dual of this—the 120-cell.
This is the last of the 4-d regular polytopes.
The 120-cell has 600 vertices and 120 3d faces shaped like dodecahedra.

Poincare' dual—put a vertex in each tetrahedra:
The Platonic solids also have rotation/reflection symmetry groups:

Finite subgroups of $O(3)$:

- $S_4$ — we can permute vertices of tet, any way
- $S_4 \times \mathbb{Z}_2$
- $A_5 \times \mathbb{Z}_2$ coming from

\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

is a symm. of all but tetrahedron

(send each pt $x$ to $-x$)

Plato had a theory of subatomic particles: need triangular faces

barycentric subdivision of something is same as its dual.

tetrahedron, octahedron, icosahedron.

But cube is different:

"subatomic particle"

everything made up from these!