Example: Let $G = \mathbb{R}^4$ (spacetime translations).

Notice:

$$\mathbb{R}^4 \cong SO_0(3,1) \times \mathbb{R}^4 = P$$ (Poincare' group)

If we have two groups, say $G$ and $H$ and we have a rep $(\rho, \nu)$ of $G$ and a rep $(\rho', \nu')$ of $H$, we can get a rep $(\rho \otimes \rho', \nu \otimes \nu')$ of $G \times H$:

$$(\rho \otimes \rho')(g, h) = \rho(g) \otimes \rho'(h) \quad g \in G, \ h \in H$$

(We've already talked about how to start w/ 2 reps $\rho, \rho'$ of $G$ so that $\rho \otimes \rho'$ is a rep of $G_2$, also.)

This is related to, but different from, the trick for taking 2 reps $(\rho, \nu), (\rho', \nu')$ of $G$ and getting the rep $(\rho \otimes \rho', \nu \otimes \nu')$ of $G$:

$$(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g).$$

Knowing reps of $\mathbb{R}$ means we know reps of products of $\mathbb{R}$ w/ itself.

**Thm:** If $(\rho, \nu)$ is a (unitary) irrep of $G$ and $(\rho', \nu')$ is a (unitary) irrep of $H$ then $(\rho \otimes \rho', \nu \otimes \nu')$ is a (unitary) irrep of $G \times H$.

Moreover—every (unitary) irrep of $G \times H$ is equivalent to one of this sort.
Note: We saw all irreps of IR were $e^{kt}$, $k \in \mathbb{C}$ now write as $e^{ikt}$, $k \in \mathbb{IR}$

Corollary: Every irrep of $\mathbb{IR}^4$ is of the form

$$\rho_k \otimes \rho_k \otimes \rho_k \otimes \rho_k \quad \text{where} \quad K = (k_0, k_1, k_2, k_3) \in \mathbb{IR}^4$$

and $\rho_k$ is the irrep of IR given by

$$\rho_k(t) = e^{ikt} \quad \text{for} \quad t \in \mathbb{IR}, k \in \mathbb{C}$$

The unitary irreps of $\mathbb{IR}^4$ are of the form

$$\rho_k \otimes \rho_k \otimes \rho_k \otimes \rho_k \quad \text{where}$$

$$K = (k_0, k_1, k_2, k_3), \quad (k_0, \ldots, k_3) \in \mathbb{IR}^4$$

---

We call $(k_0, k_1, k_2, k_3) \in \mathbb{IR}^4$ the "energy-momentum vector" $E_k$ of this rep.

There will be a 1-dim'l space of intertwiners iff

$$k + k' + k'' = k''' + k''''$$

conservation of energy-momentum
$SL(2, \mathbb{C})$ acts as itself on $\mathbb{C}^2$, also the conjugate of this action—give us $(1/2, 0), (0, 1/2)$.

Next examples: not compact?

$$
\begin{array}{ccc}
SU(2) & \longrightarrow & SL(2, \mathbb{C}) & \longrightarrow & \mathbb{P} \\
\downarrow^{2^{-1}} & & \downarrow^{2^{-1}} & & \downarrow^{2^{-1}} \\
SO(3) & \longrightarrow & SO_0(3,1) & \longrightarrow & SO_0(3,1) \times \mathbb{R}^4 = \mathbb{P}
\end{array}
$$

Irreps of $SU(2)$ are classified by "spins":

$$j = 0, 1/2, 1, 3/2, \ldots$$

These are related to angular momentum. Finite-dim'l irreps of $SL(2, \mathbb{C})$ aren't unitary (except for 1-dim'l trivial rep—always unitary) and are classified by pairs of spins $(j, k)$—left-handed/right-handed spin.

\[
\begin{cases}
(1/2, 0) \text{ is the tautologous rep of } SL(2, \mathbb{C}) \text{ on } V = \mathbb{C}^2, \\
(0, 1/2) \text{ is the conjugate rep of } SL(2, \mathbb{C}) \text{ on } \overline{V}.
\end{cases}
\]

Neutrinos \text{ and } \bar{\text{antineutrinos}}.
fusion: \( e^- + p \rightarrow n + v_e \)  
\( \text{electron} + \text{proton} \rightarrow \text{neutron} + v_e \) 
\((-) \quad (+) \quad (\text{neutral}) \quad \nu_{\text{neutral}} \)

\( n \rightarrow p + e^- + \bar{v}_e \) 
\( \text{antineutrino} \)

Unitary irreps of \( \hat{P} \) are complicated — most of them are classified by:

- \( m \in \mathbb{R}, m > 0 \), mass
- \( \text{spin} \quad j = 0, \frac{1}{2}, 1, \ldots \)

But there are also unitary irreps with \( m = 0 \) and spin either \((j,0)\) or \((0,j)\) — massless particles have a handedness to their spin.

There are also weirder unitary irreps, e.g., "tachyons" with mass being imaginary.

In quantum electrodynamics we have 2 kinds of particles — i.e., reps of \( \hat{P} \):

- \( \text{photons} \quad m = 0, \quad \text{spin} \ (1,0) \oplus (0,1) \) 
  (approximately)

- \( \text{electron/positrons} \quad m = 0.511 \text{ MeV}/c^2 \quad \text{spin} = \frac{1}{2} \)
We draw these reps as:

\[
\begin{array}{c}
\text{electron} \\
\downarrow
\end{array}
\quad
\begin{array}{c}
\text{photon} \\
\downarrow
\end{array}
\]

\text{Note:}
\text{antiphoton} = \text{photon (self-dual)}

and the basic intertwiner is:

\[
\begin{array}{c}
\downarrow
\end{array}
\]

i.e., goes from electron/position rep \& photon rep to electron/position rep.

\[
? = \downarrow \downarrow + \begin{array}{c}
\text{virtual photon (internal edge)}
\end{array} + \begin{array}{c}
\text{transfer a photon}
\end{array}
\]

\[
\downarrow \downarrow \quad \downarrow \downarrow
\]

\text{Note: Braiding commutes w/ everything, so}
\[
\begin{array}{c}
\text{switch then photon}
\end{array}
\]
\[
\begin{array}{c}
\text{don't know who emits the photon, who absorbs it!}
\end{array}
\]

\[
\begin{array}{c}
\text{photon then switch}
\end{array}
\]

\[
\begin{array}{c}
\text{switch}
\end{array}
\]
The simply-connected ones above (1st four) give well-defined intertwiners, the others "diverge" so we need to invent clever "renormalization" tricks to extract intertwiners from them.
unfortunately—the infinite sum also (probably) diverges.

So—one basic problem is: see if any interacting QFT in 4 dimensions makes rigorous sense!
Irreducible reps of $SU(2)$ and $SL(2, \mathbb{C})$.

$SU(2)$ and $SL(2, \mathbb{C})$ both have a 2-d "defining" rep on $\mathbb{C}^2$. It's irreducible since there aren't any 1-d subspaces of $\mathbb{C}^2$ invariant under all $SU(2)$. Let's write

$$V = \mathbb{C}^2$$

w/ this rep of $SU(2)$, $SL(2, \mathbb{C})$ on it.

What about $V^*$, $\nabla$? $V^*$ is equivalent to $V$ as a rep of $SL(2, \mathbb{C})$ and thus $SU(2) \leq SL(2, \mathbb{C})$ why?

Want an intertwiner:

$$i: V \overset{\sim}{\longrightarrow} V^*$$

(2 ways to take vectors, get a linear functional.)

1) inner product space
2) use symplectic structure.

We can get this by defining

$$\omega: V \otimes V \longrightarrow \mathbb{C}$$

$$v \otimes w \longmapsto \omega(v \otimes w) = \omega(v, w)$$
which is

1) nondegenerate: \( w(v, w) = 0 \quad \forall \quad w \in V \)
   \( \Rightarrow v = 0 \).

2) invariant under \( SL(2, \mathbb{C}) \):

\[
\begin{align*}
  w(gv, gw) &= w(v, w) \quad \text{for } g \in SL(2, \mathbb{C}).
\end{align*}
\]

\( w \) gives a map

\[
\begin{align*}
i : V &\longrightarrow V^* \\
v &\longmapsto w(v, \cdot)
\end{align*}
\]

so since \( w \) is nondegenerate, \( i \) is 1-1.

moreover: since \( \dim V = \dim V^* = 2 \), \( i \) is onto.

\((V = \mathbb{C}^2)\).

The fact that \( i \) is an intertwiner is equivalent to
the invariance of \( w \).

Note: invariance of \( w \) says \( w \) is an intertwiner
from \( V \otimes V \) to \( \mathbb{C} \) (as a trivial rep of \( SL(2, \mathbb{C}) \)).

\[
\begin{align*}
  V \otimes V &\quad \text{input} \quad \text{output} \\
  \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
  V &\quad \text{output}
\end{align*}
\]
\( w \) has 2 inputs, \( i \) has 1 input, 1 output.
So take \( w \) and turn an input into an output:

\[
\begin{array}{c}
\Rightarrow \\
\downarrow \\
\Rightarrow
\end{array}
\]

\[ V \xrightarrow{w} V \]

algebraically:

\[
\begin{array}{c}
\Rightarrow \\
\downarrow \\
\Rightarrow
\end{array}
\]

\[ V \xrightarrow{SS} V \otimes \mathbb{C} \]

\[ \downarrow 1_v \otimes i_v \text{ (unit)} \]

\[ V \otimes V \otimes V^* \]

\[ \downarrow w \otimes 1_{V^*} \]

\[ \mathbb{C} \otimes V^* \]

\[ \downarrow SS \]

\[ V^* \]

Since \( w \) is an intertwiner and we get \( i \) from \( w \) using diagram tricks, it's an intertwiner.

**Moral**: If \((\rho, V)\) is a finite-dime rep of \( G \) and we have a nondegenerate invariant \( w : V \otimes V \to \mathbb{C} \) then \((\rho^*, V^*) \cong (\rho, V)\).
(determinant) - tells how volume is changed
of transf.

\[ \det = 1 \text{ preserves volume} \]

But - what's \( w \) in our case?

\[ w : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C} \]

Area:

\[ \begin{array}{c}
\uparrow y \\
\downarrow x
\end{array} \]

Area - \( dx \, dy = dx \wedge dy \)

so a transformation that preserves \( dx, dy \) preserves area

2-form - eats 2 vectors, gives a number
and that's what \( w \) is!

Note - \( g \in SL(2, \mathbb{C}) \) means \( g : \mathbb{C}^2 \to \mathbb{C}^2 \) has \( \det(g) = 1 \) which means \( g \) preserves volumes, or in 2 dimensions, preserves areas

i.e. preserves the area 2-form \( dx \wedge dy \) where \( x, y \) are coordinates on \( \mathbb{C}^2 \).

\( dx, dy \) are dual basis of \( (\mathbb{C}^2)^* \) so \( dx \wedge dy \) is a 2-form i.e. skew-symmetric bilinear map from \( \mathbb{C}^2 \times \mathbb{C}^2 \) to \( \mathbb{C} \), so we let

\[ w = dx \wedge dy. \]
or-(alternate explanation)

Let $x, y$ be the standard basis of $\mathbb{C}^2$ and define

\[
\begin{cases}
    \omega(x, x) = 0 \\
    \omega(x, y) = 1 \\
    \omega(y, x) = -1 \\
    \omega(y, y) = 0
\end{cases}
\]

Check: $\omega(gv, gw) = \text{det} g \cdot \omega(v, w)$ for all $v, w \in \mathbb{C}^2$

so $\omega$ is invariant under $SL(2, \mathbb{C})$.

So: $V \cong V^*$ as a rep of $SL(2, \mathbb{C})$ and thus $SU(2)$.

What about $\overline{V}$?

(Thm: If we have a unitary rep of a group, $V^* \cong \overline{V}$)

So as reps of $SU(2)$, $\overline{V} \cong V^* \cong V$.

But as reps of $SL(2, \mathbb{C})$, $\overline{V} \not\cong V^* \cong V$. 

The rep on \( V \) sends any matrix
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})
\]
to itself, i.e., a linear transformation of \( V = \mathbb{C}^2 \). The rep on \( \overline{V} \) sends the matrix to
\[
\text{SL}(2, \mathbb{C}) \ni \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}
\] (this is what the conjugate rep does = just conjugates each entry)
as a linear transf. on \( \overline{V} \).

Saying \( V \neq \overline{V} \) means there's no invertible \( 2 \times 2 \) matrix \( T \) s.t.
\[
T \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T.
\]
\[\forall \ a, b, c, d \in \mathbb{C} \text{ s.t. } ad - bc = 1\]
meaning complex conjugate of a holomorphic function (constant functions are holomorphic) (analytic)

RHS has matrix entries holomorphic in \( a, b, c, d \),
LHS has entries anti-holomorphic (complex conj. of holomorphic) in \( a, b, c, d \). Only func's that are holo, e.g., anti-holom. are constants, so \( T = 0 \).
So— we have two different "spinor reps" of the double cover of Lorentz group:

left-handed spinors: V
right-handed spinors: \( \bar{V} \)

(neutrinos/antineutrinos)

but these become equivalent when restricted to \( SU(2) \), the double cover of the rotation group.

To get bigger reps, irreps of \( SU(2) \) and \( SL(2, \mathbb{C}) \), we'll use a standard trick:

Suppose \((\rho, V)\) is a rep. of \( G \).

We can form the rep \((\rho^{\otimes n}, V^{\otimes n})\) where

\[
\rho^{\otimes n}(g)(v_1 \otimes \ldots \otimes v_n) = \rho(g)v_1 \otimes \ldots \otimes \rho(g)v_n
\]

\[
V^{\otimes n} = V \otimes \ldots \otimes V
\]

\( n \) times

These are never irreducible (if \( n > 1 \)) because we can define projection operators (square them, we get themself back):

\[
\rho_S, \rho_A : V^{\otimes n} \rightarrow V^{\otimes n}
\]
Ps is "symmetrization":

\[ p_s(v_1 \otimes \ldots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \]

and \( p_A \) is "antisymmetrization":

\[ p_A(v_1 \otimes \ldots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \]

Ex) \( n = 2 \)

\( v \otimes v \) if \( v = w \)

\( 0 \) if \( v = w \)

\[ p_s(v \otimes w) = \frac{1}{2} (v \otimes w + w \otimes v) / p_A(v \otimes w) = \frac{1}{2} (v \otimes w - w \otimes v) \]

If we have \( n \) identical "bosons" (e.g. photons, mesons, etc) each of which has Hilbert space \( H \), the collection of all the them has Hilbert space:

\[ S^n H = \{ p_s \psi : \psi \in H \otimes^n \} \]

for "fermions" (e.g. protons, neutrons, electrons, quarks) we antisymmetrize and instead use:

\[ \Lambda^n H = \{ p_A \psi : \psi \in H \otimes^n \} \]

(Note - can't put 2 electrons in the same state.)

Same works for a \( V \), space \( V \).

We'll look at \( V = C^2 \) and get irreps of \( SU(2) \) and \( SL(2, \mathbb{C}) \) by forming:

\[ S^n V, \Lambda^n V \]