1. Connections as functors.

The functor $S$ must map $\gamma: p \to q$ to $S(\gamma): S(p) \to S(q)$, where $S(p)$ and $S(q)$ are objects in $\mathbb{R}$. But $\mathbb{R}$ is a category with one object only, so $S(p) = S(q) = \bullet$ and the functor $S$ is uniquely determined by

$$S(\gamma) = \int_{\gamma} A.$$ 

To see that this is a functor, one need only recall from differential geometry that

$$\int_{\gamma \circ \gamma'} A = \int_{\gamma} A + \int_{\gamma'} A.$$

[Thanks to Derek for pointing out that the point of this exercise is the triviality of the object map.]

2. The symplectic potential in local coordinates.

I prefer the notation $p_q \in T^*_p M \subseteq T^* M$ rather than $(q, p)$. The coordinate functions $q_i$ and $p^i$ on $T^* M$ are such that

$$q_i(p_q) = x_i(q), \quad \text{and} \quad p^i(p_q) = p_q \left( \frac{\partial}{\partial x^i} \right)_{q}.$$

A tangent vector $v_x \in T(T^* M)$, where $x = p_q \in T^* M$ (beware of the unfortunate clash of notations between $x \in T^* M$ and $x_i: M \to \mathbb{R}$), is of the form

$$v_x = v_i \left. \frac{\partial}{\partial q_i} \right|_{x} + v^j \left. \frac{\partial}{\partial p^j} \right|_{x}.$$

The map $d\pi: T(T^* M) \to TM$ maps $T_x(T^* M) \to T_{\pi(x)} M$, and its differential, and in fact

$$d\pi(v) = v^i \left. \frac{\partial}{\partial x^i} \right|_{\pi(x)}.$$

By definition of $p^i$ and of $v$, then,

$$\alpha(v) = \sum_i p^i v_i = \sum_i p^i dq^i(v).$$

3. The symplectic structure.

The exterior derivative of

$$\alpha = p^i dq_i,$$

is

$$d\alpha = dp^i \wedge dq^i.$$

4. The action as phase-space area.

By the generalized Stokes’ theorem, if $\gamma = \partial D$,

$$S(\gamma) = \oint_{\gamma} \alpha = \oint_{\partial D} \alpha = \int_D d\alpha = \int_D \omega.$$