In this homework we’ll see how a vector bundle $E$ equipped with a connection over a manifold $X$ gives a functor

$$F: \mathcal{P}(X) \to \text{Vect}$$

where $\mathcal{P}(X)$ is the category of paths in $X$, defined in the last homework. Recall that objects of $\mathcal{P}(X)$ are points of $X$, while morphisms are piecewise-smooth paths in $X$. The functor $F$ maps each point $x \in X$ to the fiber of $E$ over the point $x$. Similarly, $F$ maps each path $\gamma: x \to y$ in $X$ to a linear operator

$$F(\gamma): F(x) \to F(y)$$

defined using parallel transport along the path $\gamma$.

To warm up, let’s see how any linear ordinary differential equation gives a functor. I’ll let you use this fact:

**Theorem 1.** Let $\text{End}(\mathbb{R}^n)$ be the algebra of linear operators from $\mathbb{R}^n$ to itself. Suppose $A: [a,b] \to \text{End}(\mathbb{R}^n)$ is any smooth function and $t_0 \in [a,b]$. Given any vector $\psi_0 \in V$, the differential equation

$$\frac{d\psi(t)}{dt} = A(t)\psi(t) \tag{1}$$

has a unique smooth solution $\psi: [a,b] \to \mathbb{R}^n$ with $\psi(t_0) = \psi_0$.

**Sketch of Proof.** With the given initial conditions, Equation (1) is equivalent to the integral equation

$$\psi(t) = \psi_0 + \int_{t_0}^{t} A(s)\psi(s)ds.$$ 

A solution of this equation is none other than a fixed point of the map $T$ sending the function $\psi$ to the function $T\psi$ given by

$$(T\psi)(t) = \psi_0 + \int_{t_0}^{t} A(s)\psi(s)ds.$$ 

$T$ maps the Banach space of continuous $\mathbb{R}^n$-valued functions on $[a,b]$ to itself. If $\int_{a}^{b} \|A(s)\|ds = M$ then

$$\|T(\psi_1) - T(\psi_2)\| \leq M\|\psi_1 - \psi_2\|.$$ 

We call a map with this property a **contraction** if $M < 1$. An easy argument shows that any contraction on a Banach space has a unique fixed point, so our equation has a unique solution. If $M \geq 1$, we can chop the interval $[a,b]$ into smaller intervals for which this bound does hold, and prove the theorem one piece at a time. □

1. Let $K[a,b]$ be the category whose objects are points of the interval $[a,b]$, with exactly one morphism from any object to any other. Given a function $A: [a,b] \to \text{End}(\mathbb{R}^n)$ satisfying the conditions of Theorem 1, use the theorem to prove there is a unique functor

$$F: K[a,b] \to \text{Vect}$$

such that:

- $F$ sends any object to $\mathbb{R}^n$. 

- $F$ sends any morphism $f: t_0 \to t_1$ to the linear operator
  \[ \psi_0 \mapsto \psi(t_1) \]
  where $\psi: [a, b] \to \mathbb{R}^n$ is the unique solution of Equation (1) with $\psi(t_0) = \psi_0$.

More poetically, $K[a, b]$ is the category whose objects are **moments of time** between time $a$ and time $b$. The morphisms in this category are **passages of time**. Applied to the passage of time from $t_0$ to $t_1$, the functor $F$ gives the **time evolution operator** mapping $\psi(t_0)$ to $\psi(t_1)$, where $\psi$ is any solution of
  \[ \frac{d\psi(t)}{dt} = A(t)\psi(t). \]

Next, suppose $X$ is a smooth manifold and $A$ is a smooth $\text{End}(\mathbb{R}^n)$-valued 1-form on $X$. For each point $x \in X$, such a thing gives a linear map
  \[ A_x: T_xX \to \text{End}(\mathbb{R}^n), \]
and $A_x$ varies smoothly as a function of $x$. If we take $n = 1$, $A$ becomes an ordinary 1-form and the following result reduces to a problem in the last homework assignment:

2. Suppose $A$ is a smooth $\text{End}(\mathbb{R}^n)$-valued 1-form on the manifold $X$. Show that there is a unique functor
  \[ F: \mathcal{P}(X) \to \text{Vect} \]
such that
- $F$ sends any point of $X$ to $\mathbb{R}^n$.
- $F$ sends any piecewise-smooth path $\gamma: [0, T] \to X$ to the linear operator
  \[ \psi_0 \mapsto \psi(T) \]
  where $\psi: [0, T] \to \mathbb{R}^n$ is the unique solution of the equation
  \[ \frac{d\psi(t)}{dt} = A_{\gamma(t)}(\gamma'(t)) \psi(t) \]
  with $\psi(0) = \psi_0$.

An $\text{End}(\mathbb{R}^n)$-valued 1-form $A$ is called a **connection** on the trivial vector bundle
  \[ \pi: X \times \mathbb{R}^n \to X. \]
If $\psi(t)$ satisfies Equation (2), we say the vector $\psi(t)$ is **parallel transported** along the path $\gamma$ using the connection $A$. The linear operator $F(\gamma)$ is called the **holonomy** of the connection $A$ along the path $\gamma$.

All this stuff generalizes to the case of a connection on a nontrivial vector bundle
  \[ \pi: E \to X \]
except that now the functor $F$ maps each point $x \in X$ to the **fiber of $E$ over $x$**, namely $E_x = \pi^{-1}(x)$. To handle this case, we choose an open cover of $X$ such that $E$ restricted to each open set is trivializable, and reduce the problem to the case treated above. After huffing and puffing, we get:
Theorem 2. Suppose $A$ is a smooth connection on a smooth vector bundle $\pi: E \to X$ over a smooth manifold $X$. Then there is a unique functor

$$F: \mathcal{P}(X) \to \text{Vect}$$

such that:

- For any object $x$ of $\mathcal{P}(X)$, $F(x)$ is the fiber of $E$ over $x$.
- For any morphism $\gamma: x \to y$ of $\mathcal{P}(X)$, $F(\gamma)$ is the holonomy of $A$ along $\gamma$.

The converse is not true: there are functors $F: \mathcal{P}(X) \to \text{Vect}$ that don’t come from connections on vector bundles! However, we can characterize the functors that do by means of three conditions:

- $F(\gamma_1) = F(\gamma_2)$ when $\gamma_2$ is obtained by reparametrizing $\gamma_1$:
  $$\gamma_2(t) = \gamma_1(f(t))$$
  for any monotone increasing function $f$.
- $F(\gamma_2) = F(\gamma_1)^{-1}$ when $\gamma_2$ is a reversed version of $\gamma_1$:
  $$\gamma_2(t) = \gamma_1(f(t))$$
  for any monotone decreasing function $f$.
- $F(\gamma)$ depends smoothly on $\gamma$ (in a certain precise sense).

For some hints on how to prove this, try:


If we drop the smoothness condition, we call $F$ a generalized connection. These play an important role in loop quantum gravity.