The concept of 'dual vector space' has a massive generalization in terms of category theory. It goes like this....

Suppose $C$ is a monoidal category. An adjunction in $C$ is a quadruple $(x, x^*, i, e)$ where:

- $x$ and $x^*$ are objects in $C$.
- $i: 1 \to x \otimes x^*$ and $e: x^* \otimes x \to 1$ are morphisms in $C$ (called the unit and counit of the adjunction, respectively).
- The following diagrams commute:

$$
1 \otimes x \xrightarrow{i \otimes 1} (x \otimes x^*) \otimes x \xrightarrow{a_{x,x^*,x}} x \otimes (x^* \otimes x)
$$

$$
x \xrightarrow{\eta_x} x \otimes 1 \xrightarrow{1 \otimes e} x \otimes 1
$$

$$
x^* \otimes 1 \xrightarrow{i \otimes 1} x^* \otimes (x \otimes x^*) \xrightarrow{a_{x^*,x,x^*}^{-1}} (x^* \otimes x) \otimes x^*
$$

$$
x^* \xrightarrow{\epsilon_x^{-1}} x \xleftarrow{r_x} 1 \otimes x^*
$$

Aaron Lauda has dubbed the above commutative diagrams the zig-zag identities. Why? The string diagram for the unit $i: 1 \to x \otimes x^*$ looks like this:

$$
\begin{aligned}
  i \\
  \downarrow \\
  x
\end{aligned}
$$

where it is understood that the downward pointing arrow corresponds to $x$ and the upward pointing arrow to $x^*$. Similarly, the counit $e: x^* \otimes x \to 1$ looks like this:

$$
\begin{aligned}
  e \\
  \uparrow \\
  x
\end{aligned}
$$

These string diagrams are reminiscent of the Feynman diagrams for the creation and annihilation of particle/antiparticle pairs! In this notation, the zig-zag identities simply say that we can straighten a zig-zag in a piece of string:

$$
\begin{aligned}
  i_x \quad = \quad x \quad = \quad x \\
  \epsilon_x \quad i_x \\
  \longdownarrow \quad \quad \longdownarrow \\
  r_x \quad \quad e_x
\end{aligned}
$$
1. The category \( \text{Vect}_k \) has finite-dimensional vector spaces over a fixed field \( k \) as its objects and linear maps between these as its morphisms. \( \text{Vect} \) becomes a monoidal category with the usual tensor product of vector spaces and with the unit object \( 1 = k \).

Suppose \( V \in \text{Vect}_k \) and \( V^* \) is its dual, i.e. the space of all linear maps \( f: V \to k \). Define \( i_V: k \to V \otimes V^* \cong \text{End}(V) \) by
\[
i_V(\alpha) = \alpha 1_V
\]
and define \( e_V: V^* \otimes V \to k \) by
\[
e_V(f \otimes v) = f(v).
\]

a. Show that \( (V, V^*, i_V, e_V) \) is an adjunction.

b. What goes wrong when \( V \) is infinite-dimensional?

In the above situation we often call \( V^* \) ‘the’ dual of \( V \), but one should be a bit careful. After all, the precise definition of ‘linear map’ depends on the definition of ‘function’, and different people use slightly different definitions of ‘function’ — for example, by saying a function is a set of ordered pairs, but using different definitions of ‘ordered pair’, such as Norbert Wiener’s original 1914 definition \((x, y) = \{\{x\}, \emptyset\}\}, Kazimierz Kuratowski’s more efficient 1921 definition \((x, y) = \{\{x\}, \{x, y\}\} \), or his brother Zygmunt’s 1922 definition \((x, y) = \{\{y\}, \{y, x\}\} \). (Tragically, Kazimierz and Zygmunt killed each other in a foolish retinue over this issue in 1923.)

So, if we were being incredibly nitpicky, we might call \( V^* \) ‘a’ dual of \( V \). The concept of adjunction makes this more precise, by saying exactly what a dual should be like — at least in the finite-dimensional case. And the really nice thing is that we can prove that any two duals of the same object are isomorphic in a god-given way:

2. Suppose \( x \) is an object in the monoidal category \( C \) and \( (x, y, i, e) \) and \( (x, y', i', e') \) are adjunctions.

a. Construct an isomorphism \( f: y \to y' \).

b. Describe the sense in which the isomorphism \( f: y \to y' \) makes \( (x, y, i, e) \) and \( (x, y', i', e') \) into isomorphic adjunctions.

(Hint: it’s easiest to do these using string diagrams.)

This result means we’re allowed to speak of ‘the dual’ of \( x \) as long as we use the word ‘the’ in its official category-theoretic sense. In set theory, we’re allowed to speak of the element with some property whenever such an object exists and any two elements with this property are equal. In category theory, we’re allowed to speak of the object equipped with some stuff whenever such an object exists and any two objects equipped with this stuff are isomorphic in a specified way.

Finally, let’s show that monoidal functors automatically preserve duals of objects:

3. Suppose \( C \) and \( D \) are monoidal categories and \( F: C \to D \) is a monoidal functor. Show that if \( (x, y, i, e) \) is an adjunction in \( C \), there is an adjunction in \( D \) making \( F(y) \) into the dual of \( F(x) \).

(Hint: when \( F \) is a strict monoidal functor this adjunction in \( D \) is just \((F(x), F(y), F(i), F(e))\), but in general we need to keep track of the fact that \( F \) preserves the tensor product and unit object only up to specified isomorphisms.)