Homework: "A spring in imaginary time!" at http://math.ucr.edu/home/baez/classical/

There's a difference in sign which ultimately comes from $i^2 = -1$.

dynamics of a thrown rock
(minimizes action)

statics of a hung spring
(minimizes energy)

To do this homework, you'll use the Lagrangian approach to classical particle dynamics.

Suppose $X$ is a manifold, the configuration space. We want a law of physics satisfied by paths

$$\gamma: [t_0, t_1] \to X$$

the "Euler-Lagrange equation". To get this, we define

$$P_{x_0, x_1} X = \{ \gamma: [t_0, t_1] \to X : \gamma(t_i) = x_i \}$$

which is an infinite dimensional manifold in its own right (in fact, a Frechet manifold), and we choose a smooth function

$$S: P_{x_0, x_1} X \to \mathbb{R}$$

called the action.
Abstractly, the Euler-Lagrange equation says

\[ dS(\gamma) = 0 \]

where \( dS \in \Omega^1(P_{x_0,x_1}X) \). So, a particle will follow a path that's a critical point of the action. A bit more concretely, we have:

\[
\forall \delta \gamma \quad dS(\gamma)(\delta \gamma) = 0
\]

\[
\begin{align*}
T^*_\gamma(P_{x_0,x_1}X) & \quad T_{\gamma}P_{x_0,x_1}X
\end{align*}
\]

where \( \delta \gamma \) is a "variation" in \( \gamma \), i.e. a tangent vector at \( \gamma \in P_{x_0,x_1}X \).

Note: we can think of \( \delta \gamma \) as a path in \( TX \), with

\[
\delta \gamma(t) \in T_{\gamma(t)}X \subseteq TX
\]
In physics we often have actions of the form

$$ S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t)) \, dt $$

where the Lagrangian is some smooth function

$$ L : TX \to \mathbb{R} $$

(position/velocity pairs)

Let's see what $dS(\gamma) = 0$ says in this case. For all $\delta \gamma \in T_{\gamma} P_{x_0} X$, we have

$$ 0 = dS(\gamma)(\delta \gamma) \\
= \int_{t_0}^{t_1} \frac{dL}{dx} \delta \gamma^i(t) + \frac{dL}{dy} \delta \dot{\gamma}^i(t) \, dt $$

where $\gamma^i$ are local coordinates on $X$ (which is locally $\approx \mathbb{R}^n$), which gives local coordinates $x^i, y^i$ on $TX$ (locally $\approx T\mathbb{R}^n \approx \mathbb{R}^n \oplus \mathbb{R}^n$). Continuing this calculation:

$$ 0 = \int_{t_0}^{t_1} \left( \frac{dL}{dx} \delta \gamma^i(t) + \frac{dL}{dy} \frac{d}{dt} \delta \gamma^i(t) \right) \, dt \\
= \int_{t_0}^{t_1} \left( \frac{dL}{dx} \delta \gamma^i(t) - \left( \frac{d}{dt} \frac{dL}{dy} \right) \delta \gamma^i(t) \right) \, dt $$
The boundary terms in our integration by parts vanish since
\[ \delta g(t_0) = \delta g(t_1) = 0 \]
(see the picture). So:
\[ 0 = \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial \dot{x}^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} \right) \delta g_i(t) \, dt \]
for all \( \delta g \). So we get
\[ \frac{\partial L}{\partial x^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} \]
as the Euler–Lagrange equations. More pedantically:
\[ \frac{\partial L}{\partial x^i} (x(t), \dot{x}(t)) = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} (y(t), \dot{y}(t)) \]
Physicists don’t write \((x^i, y^i)\) as coordinates on \( TX \); they use \((q^i, \dot{q}^i)\) even though "\(\dot{q}^i\)" here is not the time derivative of anything. Also, they write
\[ q : [t_0, t_1] \to X \]
instead of \( y : [t_0, t_1] \to X \). Now \( q^i \) is ambiguous! But tough. The E-L equations look like:
\[ \frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \]
Example: a particle in a potential in $\mathbb{R}^n$. Here $X = \mathbb{R}^n$ and $L : TX \to \mathbb{R}$ is given by:

$$L(q, \dot{q}) = \frac{m}{2} \dot{q}^2 - V(q)$$

The Lagrangian is weird: it's not the sum of energies, but the difference:

$$L = \text{Kinetic} - \text{Potential} = \text{total}$$

(how much is happening)  (how much could be happening)

Nature usually tries to minimize the integral of the Lagrangian (the "happeningness") over time. The E-L equations say:

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}$$

$$-\frac{\partial V}{\partial q^i} = \frac{d}{dt} m \ddot{q}_i$$

$$-\frac{\partial V}{\partial \dot{q}^i} = m \ddot{q}_i$$
or

\[ F = ma \]

where \( F_i = -\frac{\partial V}{\partial q^i} \) is the force & \( a_i = \ddot{q}_i \) is the acceleration.

Quite generally, for any Lagrangian \( L: TX \rightarrow \mathbb{R} \), we can define:

\[ \frac{\partial L}{\partial q^i} = F_i \quad \text{force} \]

\[ \frac{\partial L}{\partial \dot{q}^i} = p_i \quad \text{momentum} \]

so the E-L equations say

\[ \frac{dp_i}{dt} = F_i \, . \]

To relate all this to cohomology, step back: we have derived classical mechanics from the principle of least action, based on

\[ S: PX \rightarrow \mathbb{R} \]
It would be great if there were a 1-form $\alpha \in \Omega^1(X)$ such that

$$S(\gamma) = \int_\gamma \alpha$$

Alas, the action $S$ in our example is not of this form! Why not? Because in our example $S(\gamma)$ depends on the reparameterization of $\gamma$, for obvious physical reasons. But, we can write $S(\gamma)$ as the integral of some 1-form over a path in some other space. What's this other space? For any manifold $M$, $T^*M$ has a God-given 1-form on it, called the canonical 1-form. We'll use this to get the job done, but with

$$M = X \times \mathbb{R}$$

(time space)

—the extended configuration space.