Hamiltonian Mechanics & Symplectic Geometry

We've seen that any Lagrangian $L: TX \rightarrow \mathbb{R}$ gives Euler-Lagrange equations describing the flow on $TX$ — "time evolution":

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

We also get a Legendre transform

$$\lambda: TM \rightarrow T^*M$$

$$(q, \dot{q}) \mapsto (q, p)$$

where $p_i = \frac{\partial L}{\partial \dot{q}_i}$. Assume $L$ is strongly regular, i.e., $\lambda$ is a diffeomorphism. Then we get a flow on $T^*M$ describing time evolution of position and momentum, which satisfies Hamilton's equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

where $H: T^*M \rightarrow \mathbb{R}$ is the Hamiltonian given by

$$H(q, p) = p_i \dot{q}_i - L(q, \dot{q})$$

where $(q, \dot{q}) = \lambda^{-1}(q, p)$. 
Example: If \((X, g)\) is a Riemannian manifold and \(V: X \to \mathbb{R}\) is the potential energy, then the Hamiltonian for a particle of mass \(m\) on \(X\) is

\[
H(q, p) = \frac{1}{2m} p^2 + V(q)
\]

Here Hamilton's equations say, first:

\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \text{ or } \dot{q}^i = \frac{p^i}{m}
\]

where \(p^i = g^{ij} p_j\) (i.e. the metric \(g: T_2X \times T_2X \to \mathbb{R}\) gives \(b: T_2X \to T_2^*X\) by \(v \mapsto g(v, \cdot)\), and we use this to turn the cotangent vector \(p\) into the tangent vector \(b^{-1}(p) = \#(p)\), with components \(p^i\).)

This equation \(p^i = m q^i\) is boring; the interesting equation is Hamilton's 2nd eqn:

\[
\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \text{ or } \dot{p}_i = -\frac{\partial V}{\partial q^i}.
\]

I.e. \(F = ma\).

Let's seek a coordinate-free formulation of Hamilton's equations. Hamilton's eqns give a vector field \(v_H\) on \(T^*X\) describing how \((q(t), p(t))\) moves under the Hamiltonian flow. This vector field \(v_H\) is
called the Hamiltonian vector field.

\[
\frac{d}{dt}(q(t), p(t)) = v_H(q(t), p(t)) \in T(T^*X)
\]

where

\[
v_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} + \frac{-\partial H}{\partial q^i} \frac{\partial}{\partial p_i}
\]

where \( \frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i} \) is a local coordinate basis of vector fields on \( T^*X \).

Example: If \( X = \mathbb{R} \) then \( T^*X = \mathbb{R}^2 \)

Here \( dH \) is really a 1-form, but in \( \mathbb{R}^2 \) we can use the metric to turn it into a vector \( \tilde{\nabla}H \) and rotate this \( 90^\circ \) to get \( v_H \). So any solution of Hamilton's equations moves along the level curves of \( H \); this is conservation of energy.
What's the coordinate-free description of how to turn the 1-form $dH$ into the vector field $v_H$? One way to turn 1-forms into vector fields is to use a (Riemannian) metric:

$$b : TM \to T^*M$$

$$v \mapsto g(v, -)$$

has an inverse when $g$ is nondegenerate, so we get

$$\#: T^*M \to TM$$

But, $\#dH = \nabla H$ is perpendicular to the level curves of $H$, not tangent. Instead of a metric, let's use an antisymmetric nondegenerate bilinear form

$$\omega : TM \times TM \to \mathbb{R}$$

(where in our example $M = T^*X$). In our example

$$\omega = dp_1 \wedge dp_i$$

Homework: Show that, with this 2-form on $T^*X$ we get

$$\omega(v_H, -) = dH$$

I.e. now we define $b : TM \to T^*M$

$$v \mapsto \omega(v, -)$$

and if $b$ is an isomorphism we say $\omega$ is nondegenerate and we set $v_H = \#(dH)$. 
In fact, the 2-form
\[ \omega = dq^i \wedge dp_i \]
on $T^*X$ can be defined without using coordinates or any metric on $X$. How? Notice that
\[ \omega = -d\alpha \]
where
\[ \alpha = p_i dq^i. \]

We'll see in fact that this $\alpha$ is a 1-form on $T^*X$ — the canonical 1-form on $T^*X$ — which can be defined without coordinates. How do we define $\alpha$ without coordinates? $\alpha$ likes to eat tangent vectors to $T^*X$, e.g.
\[ v \in T_{(q,p)} T^*X. \]

The projection
\[ \pi : T^*X \rightarrow X \]
\[ (q, p) \mapsto q \]
gives
\[ d\pi : T(T^*X) \rightarrow TX \]

So, we can form
\[ d\pi(v) \in T_q X \]
but \( p \in T_q^*X \), so we get \( p(d\pi(v)) \in \mathbb{R} \).

So: define

\[
\alpha(v) = p(d\pi(v)) \quad \text{for } v \in T_{(q,p)}T^*X.
\]

Next time, we'll see this agrees with

\[
\alpha = p_i dq^i.
\]