The Canonical 1-form

We've seen that the canonical 1-form

$$\alpha = p_i dq^i$$

on $T^*X$ plays 2 roles in classical mechanics:

1) $d\alpha = \omega$ lets you do Hamiltonian mechanics on the "phase space" $T^*X$.

2) The integral of $\alpha$ along a path

$$\gamma : [t_0, t_1] \rightarrow T^*X$$

is almost the action of that path, which gives Lagrangian mechanics:

$$S(\gamma) = \int_{t_0}^{t_1} (\dot{p}_i \dot{q}^i - H) \, dt$$

$$= \int_{\gamma} \alpha - \int_{t_0}^{t_1} H \, dt$$

$$= \int_{\gamma} \alpha - E(t_1) + E(t_0)$$

where the last step works if

$$\gamma : [t_0, t_1] \rightarrow \{ H(q, p) = E \} \subseteq T^*X.$$ 

Putting these together we can give a physical interpretation.
of $\int_\Sigma \omega$ where $\Sigma \subseteq T^*X$ is a surface with $\partial \Sigma = \gamma$ for some loop $\gamma$.

Stokes' Theorem says

$$\int_\Sigma \omega = \int_{\partial \Sigma} \alpha = \int_{\gamma} \alpha = S(\gamma) + \int_{t_0}^{t_1} H \, dt$$

which is just $S(\gamma)$ in the limit where we let $t_1 \to t_0$, reparameterizing $\gamma$ suitably.

Moral: $\omega$ tells you the action it costs to run around the surface $\Sigma$.

Now let's go to the extended phase space $T^*(X \times \mathbb{R})$.

A point in the extended configurations space $(q,t) \in X \times \mathbb{R}$ says where & when your system is. Similarly, a point $(q, t, p, p_0) \in T^*(X \times \mathbb{R})$ specifies position, time, momentum and energy $E = -p_0$. 
In these terms any path
\[ \gamma : [t_0, t_1] \rightarrow T^*X \]
\[ t \mapsto (q(t), p(t)) \]
gives
\[ \tilde{\gamma} : [t_0, t_1] \rightarrow T^*(X \times \mathbb{R}) \]
\[ t \mapsto (q(t), t, p(t), -H(q(t), p(t))) \]
if we choose a Hamiltonian \( H : T^*X \rightarrow \mathbb{R} \). Now the action is
\[ S(\tilde{\gamma}) = \int_{t_0}^{t_1} \left( p_i \dot{q}^i - H(q(t), p(t)) \right) dt \]
\[ = \int_{t_0}^{t_1} p_i dq^i - H(q(t), p(t)) dt \]
\[ = \int_{\tilde{\gamma}} \tilde{\alpha} \]
where
\[ \tilde{\alpha} = p_i dq^i + p_0 dt \]
is the canonical 1-form on \( T^*(X \times \mathbb{R}) \), since the \( p_0 \) coordinate of the path \( \tilde{\gamma} \) was cleverly chosen to be \(-H(q(t), p(t))\).
Carlo Rovelli (see http://arxiv.org/abs/gr-qc/0207043) has reformulated this idea to apply not just to particles, but strings and higher-dimensional "branes." This requires a canonical 2-form (or higher form) on... something (not the cotangent bundle, of course — there’s no such thing!)

To understand this approach, let’s allow \( \tilde{\gamma} \) to be more general:

\[
\tilde{\gamma} : [s_0, s_1] \longrightarrow T^*(X \times \mathbb{R})
\]

\[
S \longmapsto (q(s), t(s), p(s), -H(q(s), p(s)))
\]

where \( H : T^*X \longrightarrow \mathbb{R} \) is as before. In other words

\[
\tilde{\gamma} : [s_0, s_1] \longrightarrow Y \subseteq T^*(X \times \mathbb{R})
\]

where \( Y \) contains all information about \( H \):

\[
Y = \left\{ (q, p, t, p_0) : p_0 = -H(q, p) \right\}
\]

\( Y \) is a codimension 1 submanifold, therefore not symplectic, but it has a 1-form on it:

\[
\alpha|_Y = i^*\tilde{\alpha}
\]

—i.e., the pullback of \( \tilde{\alpha} \) along the inclusion

\[
i : Y \hookrightarrow T^*(X \times \mathbb{R})
\]
We also have a 2-form on $Y$:
\[ \tilde{w}|_Y = i^* \tilde{w}. \]

Claim: a curve
\[ \tilde{y} : [s_0, s_1] \rightarrow Y \]
gives a solution of Hamilton's equations iff its tangent vector $\tilde{y}'(s)$ satisfies
\[ \tilde{w}|_Y (\tilde{y}'(s), -) = 0. \]

Note: even though $\tilde{w}$ is nondegenerate, $\tilde{w}|_Y$ is not, so this equation has nontrivial solutions.

\[ \tilde{y}'(s) = \frac{dq(s)}{ds} \frac{\partial}{\partial q^i} + \frac{dp_i(s)}{ds} \frac{\partial}{\partial p_i} + \frac{dt}{ds} \frac{\partial}{\partial t} + \frac{dp_o}{ds} \frac{\partial}{\partial p_o}, \]

Now compute
\[ \tilde{w}|_Y (\tilde{y}'(s), -) \]
using $\tilde{w} = dp_i \wedge dq^i + dp_o \wedge dt$

We get
\[ \tilde{w}|_Y (\tilde{y}'(s), -) = - \frac{dq^i}{ds} dp_i + \frac{dp_i}{ds} dq^i - \frac{dt}{ds} dp_o - \left( \frac{\partial H}{\partial q^i} \frac{dq^i}{ds} + \frac{\partial H}{\partial p_i} \frac{dp_i}{ds} \right) dt \]
\[ + \frac{dt}{ds} dH = + \frac{dt}{ds} \left( \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i \right) \]
This being zero is equivalent to

\[ \frac{dp_i}{ds} = \frac{dt}{ds} \frac{\partial H}{\partial q_i} \quad \Rightarrow \quad \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i} \]  

(Hamilton's Equations, if \( dt = 0 \))

\[ \frac{dq_i}{ds} = -\frac{dt}{ds} \frac{\partial H}{\partial p_i} \quad \Rightarrow \quad \frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i} \]

(Conservation of Energy)

\[ \frac{\partial H}{\partial q_i} \frac{dq_i}{ds} + \frac{\partial H}{\partial p_i} \frac{dp_i}{ds} = 0 \quad \Rightarrow \quad \frac{dH}{ds} = 0 \quad \Rightarrow \quad \frac{dH}{dt} = 0 \]