Math 260: Categorified inner products
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We will consistently use skeleta. The skeleton of FinSet is $E$, the category with $\mathbb{N}$ as its set of objects, no morphism other that automorphisms, and $\text{Aut}(n) = \mathbb{Z}_n$.

1. $\langle Z^n/n!, Z^n/n! \rangle$.

The structure type $Z^n/n!$, being an $n$-element set, has for its total groupoid the groupoid of $n$-element sets with bijections, which is a full subcategory of $E$, and whose skeleton consists of an $n$-element set as its single object and the group $n!$ of permutations of its elements as morphisms. In other words,

$$Z^n/n!: 1/n! \to E$$

is the inclusion. Now, the pull-back

$$\langle Z^n/n!, Z^n/n! \rangle \to 1/n!$$

has objects of the form

$$n^\bullet \xrightarrow{\alpha} n^\bullet,$$

where $n \in E$, $\bullet \in 1/n!$ and $\alpha : n \to n$. Morphisms are pairs of morphisms

$$\begin{array}{ccc}
\bullet & \xrightarrow{\gamma} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{\gamma'} & \bullet
\end{array}$$

such that

$$\begin{array}{ccc}
n^\bullet & \xrightarrow{\alpha} & n^\bullet \\
\downarrow & & \downarrow \\
(Z^n/n!)(\gamma) & \xrightarrow{(Z^n/n!)(\gamma')} & (Z^n/n!)(\gamma')
\end{array}$$

commutes,

where $\gamma : \bullet \to \bullet$ can be chosen freely, and determines $\gamma'$ uniquely. Hence, all $n^\bullet \xrightarrow{\alpha} n^\bullet$ are isomorphic and have automorphism groups isomorphic to $n!$. It follows that

$$\langle Z^n/n!, Z^n/n! \rangle \simeq 1/n!.$$

2. $\langle Z^n, Z^n/n! \rangle$.

The skeleton of the groupoid of totally-ordered $n$-element sets with order-preserving bijections consists of a single $n$-element set as object, and the identity permutation as morphism, since no other permutations preserve a total order. This is equivalent to the groupoid $1$.

The structure type “being a totally-ordered $n$-element set” is the inclusion functor $Z^n : 1 \to E$ and the pull-back

$$\langle Z^n, Z^n/n! \rangle \to 1/n!$$

again has objects of the form

$$n^\bullet \xrightarrow{\alpha} n^\bullet$$

with $\alpha \in E$,

with morphisms being of the form

$$\begin{array}{ccc}
\bullet & \xrightarrow{\gamma} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{(Z^n/n!)(\gamma)} & \bullet
\end{array}$$

such that

$$\begin{array}{ccc}
n^\bullet & \xrightarrow{\alpha} & n^\bullet \\
\downarrow & & \downarrow \\
Z^n(1) & \xrightarrow{(Z^n/n!)(\gamma)} & n^\bullet
\end{array}$$

commutes,
so \((Z^n \! / \! n!)(\gamma) = \alpha^{-1}\beta\) is determined uniquely by \(\alpha\) and \(\beta\) and, since \(Z^n \! / \! n!\) is faithful, there is a unique such \(\gamma\) in \(Z^n \! / \! n!\) for each pair \(\alpha, \beta\). Hence, all \(n^* \xrightarrow{\alpha} n^*\) are isomorphic in exactly one way, and
\[
\langle Z^n, Z^n \! / \! n! \rangle \simeq 1.
\]

3. \(\langle Z^n, E^{KZ} \rangle\).

The proper generalization of \(K\)-colouring and colour-preserving bijections (when \(K\) is a set) to the case when \(K\) is a groupoid is “\(K\)-flavourings with flavour-preserving, colour-changing bijections” (flavour-compatible bijections for short). To see why, consider a skeleton of \(K\), whose only morphisms are automorphisms. We can visualize the skeleton as a collection of “flavoured objects” each of which with “internal (colour) degrees of freedom”, and “colour-changing, flavour-preserving morphisms”. This sounds a lot like quantum chromodynamics!

The stuff type \(E^{KZ}\), “being a \(K\)-flavoured finite set”, has for its total groupoid the groupoid of \(K\)-flavoured finite sets with flavour-compatible bijections, denoted \(E^K\). In symbols,
\[
E^{KZ} : E^K \to E.
\]

To fix the notation, an object of \(E^K\) is an \(n\)-tuple \(k = (k_1, \ldots, k_n)\) of objects of \(K\), for some \(n \in E\); and a morphism \(\kappa : k \Rightarrow k'\), where \(k' = (k_1, \ldots, k_n)\) for the same \(n\), consists of a bijection \(\kappa_0 : \{1, \ldots, n\} \to \{1, \ldots, n\}\) and morphisms \(\kappa_i : k_i \Rightarrow k'_i\). In other words, \(k, k'\) are objects of \(K^n\) and \((\kappa_1, \ldots, \kappa_n)\) is a morphism of \(K^n\).

The pull-back \(\langle Z^n, E^{KZ} \rangle \to X^\omega\) has objects of the form \(n^* \xrightarrow{\alpha} n^k\), with \(\alpha \in n!\) and \(k \in K^n\).

and morphisms of the form

\[
\begin{array}{ccc}
\bullet & k & \xrightarrow{\kappa} & k' \\
1 & \downarrow & \downarrow & \downarrow \\
\bullet & Z^n(1) & \xrightarrow{E^{KZ}(\kappa)} & n^k & \xrightarrow{\beta} & n^k'
\end{array}
\]

that is, \(\beta = \alpha \kappa_0\). Given \(\alpha\) and \(\beta\), \(\kappa_0\) is uniquely determined, and \(\kappa\) is a morphism of \(K^n\) such that \(\kappa_0 = \alpha^{-1}\beta\). There may be any number (including none) of \(\kappa\) with that \(\kappa_0\). Since there are no other restrictions on \(k\) or \(\kappa\), it is clear that
\[
\langle Z^n, E^{KZ} \rangle \simeq K^n.
\]

4. \(\langle Z^n \!/ n!, E^{KZ} \rangle\).

By definition of \(X_n\) as the pull-back
\[
\begin{array}{ccc}
X_n & \xrightarrow{X} & X \\
\downarrow & \downarrow & \downarrow \\
1 \!/ n! & \xrightarrow{Z^n \!/ n!} & E
\end{array}
\]

we have that
\[
\langle Z^n \!/ n!, F \rangle = X_n.
\]

so
\[
\langle Z^n \!/ n!, E^{KZ} \rangle \simeq K^n \!/ n!.
\]
5. $\langle e^{KZ}, e^{KZ} \rangle$.

The pull-back

\[
\begin{array}{ccc}
\langle E^{KZ}, E^{KZ} \rangle & \longrightarrow & E^K \\
\downarrow & & \downarrow \\
E^K & \longrightarrow & E^{KZ}
\end{array}
\]

has objects of the form

\[
\eta^k \xrightarrow{\alpha} \eta^{k'}, \quad \text{with } \alpha \in n! \quad \text{and} \quad k, k' \in K^n,
\]

and morphisms of the form

\[
\begin{array}{ccc}
k & \xrightarrow{\kappa} & k' \\
\downarrow \kappa & & \downarrow \kappa' \\
k'' & \xrightarrow{E^{KZ}(\kappa)} & k'''
\end{array}
\]

such that

\[
\eta^k \xrightarrow{\alpha} \eta^{k'}
\]

commutes.

In other words, $\alpha \kappa'_0 = \kappa_0 \alpha'$. In this case, once $\kappa_0$ is (freely) specified, then $\kappa'_0$ is determined. Other than that, as long as $\kappa$ and $\kappa'$ preserve the flavouring, they can be freely specified, as can $k, k'$. This corresponds to a $K^2$-flavouring of the $n$-element set, with morphisms $\kappa = (\kappa_0, \kappa', \kappa'')$, where $\kappa_0: n \to n$ is a flavour-preserving permutation, and $\kappa', \kappa''$ are colour-changing morphisms in $K^n$. Hence, $\kappa$ is a morphism of $(K^2)^n \!// n!$, and

\[
\langle E^{KZ}, E^{KZ} \rangle \simeq E^{K^2}.
\]