Smooth Functors
And Beyond

We've seen that if $M$ is a smooth space (e.g. a manifold) then there is a smooth category $PM$ where:

- objects are points $x \in M$
- morphisms $Y: x \to y$ are thin homotopy classes of smooth maps $f: [0,1] \to M$ with $f(0) = x$, $f(1) = y$
  if $f$ constant near $0$ and $1$.

We think of $PM$ as a category of "configurations" and "processes" for some physical system. So to formulate the Lagrangian approach to the physics of this system, we need a smooth functor...
\[ S: \mathbb{P}M \to \mathbb{R} \]

describing the "action" of any process.

Last time we saw:

**Theorem** - There's a 1-1 correspondence between smooth functions

\[ S: \mathbb{P}M \to \mathbb{R} \]

1-forms \( A \) on \( M \), given by:

\[ S(\gamma) = \int_{\gamma} A \]

where we pick a representative (path) for \( \gamma \) to define the integral.

Alas, this isn't general enough... as we'll soon see.

To do quantum physics, what matters is not

\[ S: \mathbb{P}M \to \mathbb{R} \]
but the phase
\[ e^{iS} : PM \rightarrow U(1) \]
which has less information since
\[ \exp: \mathbb{R} \rightarrow U(1) \]
is many-to-one. In fact, \( e^{iS} \) is also sufficient to do classical physics!

\[ \frac{d}{ds} S(\gamma_s) = 0 \]

If we seek critical points of the action (instead of minima), we can work with \( e^{iS} \) instead of \( S \):
\[ \frac{d}{ds} e^{iS(\gamma_s)} = 0 \]
(for all smooth homotopies \( \gamma_s \) of \( \gamma \) holding endpoints fixed).
The critical points of $e^S$ are the same as those of $S$, so this doesn't seem like a big deal.

**Theorem** — There's a 1-1 correspondence between smooth functors

$$P : PM \rightarrow U(1)$$

i.e., forms $\alpha$ on $M$, given by:

$$P(\alpha) = e^{i\theta} \alpha$$

Here $P$ stands for "phase". So far the picture looks like:

![Diagram](image-url)
For each point \( x \in M \) we have a circle of possible phases for the system in configuration \( x \), so we have a "trivial principal \( U(1) \) bundle":

\[
\begin{align*}
M \times U(1) & \rightarrow (x, \alpha) \\
\downarrow \pi_1 & \\
M & \rightarrow x
\end{align*}
\]

Sitting over \( x \in M \) we have a fiber

\[
\Pi_1^{-1}(x) \subseteq M \times U(1)
\]

which is a circle - the set of possible phases our system could have at \( x \).

This example is called "trivial" because each fiber is \( U(1) \) - or is canonically isomorphic to \( U(1) \).
$\Pi_1^{-1}(x) = \{ (x, \alpha) : \alpha \in U(1) \}$

where the isomorphism sends $(x, \alpha)$ to $\alpha$.

More interesting are the nontrivial principal $U(1)$ bundles.

For example, let the fiber over $x$ be the set of points in $S^2$ that are 1 to $x$.

$M = S^2 = CP^1$
We can't smoothly identify all the fibers with $U(1)$ since that would produce a nowhere vanishing smooth vector field on $S^2$:

![Diagram](attachment:diagram.png)

Now let's get a bit more formal.

What's the difference between a circle and the circle? The circle is $U(1) \subseteq \mathbb{C}$.

A circle is a "$U(1)$-torsor" - a copy of $U(1)$ that's forgotten what the element $1$ is.

Def. - For any group $G$, a $G$-torsor is a set $X$ equipped with an action (a right action) of $G$:

$$\alpha : X \times G \to X$$

$$(x, g) \mapsto xg$$
\[ x1 = x \]
\[ (xg)h = x(gh) \]

such that \( X \) is isomorphic to \( G \) as a space with right \( G \)-action: there's a bijection

\[ \beta : X \rightarrow G \]

s.t.

\[ \beta(xg) = \beta(x)g \]

If \( G = U(1) \), the difference between right and left actions is inessential since \( U(1) \) is abelian. More importantly, any circle equipped with the ability to rotate it by any phase \( \phi \in U(1) \) is a \( U(1) \)-torsor.
E.g.

If a point in $P$ is a point $x \in S^1$ together with a point in the circle $\mathbb{S}^1$ to $x$, i.e. $\pi : P \to M$ is the obvious map, then $\pi^{-1}(x)$ is a $(U(1))$-torsor.

More precisely, $\pi^{-1}(x)$ becomes a $(U(1))$-torsor after we pick a "right" or left-hand "rule" for rotating $y \in \pi^{-1}(x)$ by a phase $\varphi \in U(1)$.

A principal $U(1)$ bundle is (among other things) a smooth space $P$ with a smooth map
such that each fiber $\pi^{-1}(x)$ ($x \in M$) is equipped with the structure of being a U(1) - torsor.

Next time we'll really define "principal U(1) - bundle" & include a clause saying that the U(1) - torsor structure on $\pi^{-1}(x)$ varies smoothly with $x$. 