The Bar Construction

Why do adjoint functors

\[ \begin{array}{c}
D \\
\text{"free"} \\
L \\
\downarrow \\
\uparrow \\
R \\
\text{"forgetful"} \\
C
\end{array} \]

give simplicial objects

\[
\begin{array}{c}
\varepsilon_d \\
\bar{d} \\
L R d \\
\Rightarrow \\
\L L (\varepsilon_d) \Rightarrow \\
L R L R d \\
\Rightarrow \\
\ldots..\end{array}
\]

in \( D \) from objects \( d \in D \) ? In other words, why do we get a functor

\[ \bar{d} : \Delta^{op} \longrightarrow D \]

( the bar construction applied to \( d \in D \) ) ?

To answer these questions, we need to learn more about \( \Delta \) & adjoint functors — which are closely connected!
First,

"Δ is the walking monoid."

In other words, Δ's sole purpose in life is to contain a monoid. A monoid, naively speaking, is a set with an associative product and unit. This is just a monoid in Set; more generally, a monoid can be defined in any monoidal category. Given a monoidal category $C$, a monoid in $C$ is an object $a \in C$ together with a product:

$$m : a \otimes a \rightarrow a$$

and unit

$$i : 1 \rightarrow a$$

satisfying the associative law

$$a \otimes (a \otimes b) \cong (a \otimes a) \otimes b$$

(Where for expository purposes, we assume $C$ is strict, so $(a \otimes b) \otimes c = a \otimes (b \otimes c)$)
the left/right unit laws; saying these diagrams commute:

\[
\begin{array}{ccc}
1_a & \downarrow & 1 \otimes 1_a \\
\downarrow & & \downarrow \text{m} \\
a \otimes a & \Longrightarrow & a
\end{array}
\]

\[
\begin{array}{ccc}
a \otimes 1 & \Longrightarrow & a \\
1_a \otimes i & \downarrow & 1_a \\
\downarrow & & \text{m} \\
a \otimes a & \longrightarrow & a
\end{array}
\]

How is \( \Delta \) a monoidal category, and what's the monoid in it? Recall \( \Delta \) has

- finite totally ordered sets as objects
- order-preserving maps as morphisms.

We write the finite totally ordered sets as finite ordinals \( 0, 1, 2, 3, \ldots \). We can make \( \Delta \) into a monoidal category with \( + \) as tensor product:

\[
\begin{array}{ccc}
\cdots + \cdots & = & \cdots \\
3 & 2 & 5
\end{array}
\]

on objects, but also on morphisms. A typical
morphism in $\Delta$ looks like

$$
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xleftarrow{\text{5}}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xrightarrow{f}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xleftarrow{\text{4}}
\end{array}
$$

(f must be order-preserving), and we tensor (+) then as follows

$$
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\oplus
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
= \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
$$

So $\Delta$ is a monoidal category; what's the obvious monoid in it? It's some totally ordered finite set $a$ with order-preserving maps

$$
m: a + a \rightarrow a
$$

$$
i: 0 \rightarrow a
$$

satisfying assoc. & l/r unit laws. We take $a = 1$, take $m: a + a \rightarrow a$ to be the only thing it could be

$$
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xleftarrow{\text{1}}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xrightarrow{1}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xleftarrow{2}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
$$

& take $i: 0 \rightarrow 1$ to be the only thing it could be:

$$
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xleftarrow{0}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\xrightarrow{0}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
$$
These satisfy associativity:
\[
\begin{array}{c}
Y \\
\downarrow \\
1
\end{array}
\begin{array}{c}
Y \\
\downarrow \\
1
\end{array}
\begin{array}{c}
Y
\end{array}
\]

and l/r unit laws:
\[
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
1
\end{array}
\]

So \( \Delta \) is a monoidal category containing a monoid; in fact it's the free monoidal category on a monoid. If we freely generate a monoidal category from a monoid we get a category with morphisms like
\[
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\]

satisfying relations including
\[
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
Y
\end{array}
\]

& monoidal category axioms. This is just \( \Delta \)!
Second:

adjoint functors give monoids.

Given some adjoint functors

\[
\begin{array}{c}
D \\
\downarrow \\
L \\
\downarrow R \\
C
\end{array}
\]

we have the unit

\[\eta_c : c \rightarrow RLC\]  \hspace{1cm} (think "inclusion of the generators")

and counit

\[\epsilon_d : LRD \rightarrow d\]  \hspace{1cm} (think "evaluation of formal expressions")

Note we have:

- categories  \( C, D \)
- functors  \( L, R \)
- natural trans.  \( \eta, \epsilon \)

so we're talking about a situation in \( \text{Cat} \), the 2-category of categories. We can draw what's going on using string diagrams, which apply to 2-categories as well as monoidal categories. We draw the unit

\[\iota : 1_c \rightarrow RL\]
as

![Diagram](image)

(We shade the "D" region, leaving the "C" region white.)

Similarly, we draw the counit $\varepsilon: L R \Rightarrow 1_D$ as:

![Diagram](image)

In fact, whenever we have adjoint functors, the zig-zag equations hold:

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{Diagram 2} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{Diagram 3} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{Diagram 4} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{Diagram 5} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{Diagram 6} \\
\end{array}
\end{align*}
\]
We'll get a monoid with product

\[ \begin{array}{c}
R \\
L \\
R \\
L \\
\end{array} \]

& unit:

\[ \begin{array}{c}
R \\
L \\
\end{array} \]

This kind of monoid is called a monad.