The Bar Construction

Suppose we have an adjunction:

\[
\begin{array}{ccc}
& D & \\
L & \Downarrow & R \\
& C & \\
\end{array}
\]

We get a unit \( \eta \) and counit \( \varepsilon \):

\[
\begin{array}{ccc}
\phantom{\text{R}} & \phantom{\text{L}} & \phantom{\text{R}} \\
\text{R} & \phantom{\text{L}} & \phantom{\text{R}} \\
\phantom{\text{R}} & \phantom{\text{L}} & \phantom{\text{R}} \\
\end{array}
\quad
\begin{array}{ccc}
\phantom{\text{L}} & \phantom{\text{R}} & \phantom{\text{L}} \\
\text{L} & \phantom{\text{R}} & \phantom{\text{L}} \\
\phantom{\text{L}} & \phantom{\text{R}} & \phantom{\text{L}} \\
\end{array}
\]

unit \( \eta : 1_c \Rightarrow RL \)  

counit \( \varepsilon : LR \Rightarrow 1_d \)

satisfying zig-zag identities. Last time we saw this gives a monad on \( C \), i.e. a monoid in \( \text{End}(C) \), namely \( RL \in \text{End}(C) \).
This is indeed a monoid:

\[
\begin{align*}
\text{multiplication} \quad RLRL & \Rightarrow RL \\
\text{unit} \quad 1_c & \Rightarrow RL
\end{align*}
\]

Satisfying associativity:

\[
Y = Y
\]

\[
\frac{Y}{r} \text{ unit laws} : \quad Y = 1 = Y
\]

We also get a \underline{comonad} on \(D\), i.e. a comonoid in \(\text{End}(D)\), namely \(LR \in \text{End}(D)\).

A "comonoid" is just like a monoid, but upside down:

\[
\begin{align*}
\text{comultiplication} & \quad LRRLR \\
\text{counit} & \quad LR \Rightarrow 1_D
\end{align*}
\]
These satisfy coassociativity: \( \triangle = \triangle \)

\( \triangleright \) l/r counit laws: \( \triangle = 1 = \Delta \)

More tersely, if \( M \) is a monoidal category, \( M^{\mathsf{op}} \) is also monoidal with same \( \otimes \), and:

**Def.** A **comonoid** in \( M \) is a monoid in \( M^{\mathsf{op}} \).

**E.g.** - An **algebra** is a monoid in \( \mathsf{Vect} \),
    a **coalgebra** is a comonoid in \( \mathsf{Vect} \),
    or monoid in \( \mathsf{Vect}^{\mathsf{op}} \).

So: our adjunction gives a comonad \( \Delta \mathsf{R} \),
which is a monoid in \( \mathsf{End}(\Delta)^{\mathsf{op}} \).

Since \( \Delta \) is the free monoidal category
on a monoid, this gives a monoidal functor
\[
\Delta : \Delta \to \mathsf{End}(\Delta)^{\mathsf{op}}
\]
i.e. a monoidal functor

\[ \Delta^{op} \xrightarrow{\alpha} \text{End}(D) \]

and thus a simplicial object in \( \text{End}(D) \)!

Taking a specific object \( d \in D \), we get:

\[ e_{vd}: \text{End}(D) \rightarrow D \]

\[ \begin{array}{c}
F \quad \xrightarrow{1} \quad Fd
\end{array} \]

so we get a simplicial object in \( D \):

\[ \Delta^{op} \xrightarrow{\alpha} \text{End}(D) \xrightarrow{e_{vd}} D \]

This simplicial object in \( D \) is called \( d: \Delta^{op} \rightarrow D \); we call this the bar construction.

Moral: given an adjunction \( \text{L} \dashv \text{R} \), any object \( d \in D \) gives a simplicial object \( d \) in \( D \).
Example: The Cohomology of Groups

Here we take a group $G$, get an adjunction

$$\begin{align*}
G - \text{Set} \\
\downarrow L \\
\downarrow R \\
\text{Set}
\end{align*}$$

where $G - \text{Set}$ is the category of sets w. left $G$-action.

So, given a $G$-Set $X$, we get:

$$X = \cdots \xleftarrow{LRX} \xrightarrow{LRXLX} \cdots$$

a simplicial $G$-set!

What's a 1-simplex, or 2-simplex, in this simplicial $G$-set like?
Given our \( G \)-set \( X \), what's \( LRX \)? \( RX \) is usually just called "\( X \)" - the underlying set of our \( G \)-set \( X \). \( LRX \) has elements "\( gx \)" for each \( g \in G \) and \( x \in X \). Really "\( gx \)" is just \((g, x)\), so \( LRX = G \times X \), which is a \( G \)-set with

\[
g \cdot (g_2, x) = (g, g_2 \cdot x).
\]

So

\[
\begin{array}{ccc}
\text{(-1)-simplices} & \text{0-simplices} & \text{1-simplices} \\
\downarrow & \downarrow & \downarrow \\
X & G \times X & G \times G \times X
\end{array}
\]

\[
\varepsilon_x \leftrightarrow (g, x) \leftrightarrow (g, g_2, x) \leftrightarrow \varepsilon_{LRX} \leftrightarrow (g, g_2, x) \leftrightarrow LR(\varepsilon_x) \leftrightarrow (g, g_2, x)
\]

So, a typical 1-simplex in \( X \) looks like:

\[
(g, g_2 x) \rightarrow (g, g_2, x) \rightarrow (g, g_2 x)
\]
Note: both 0-simplices here have as face the \((-1)\)-simplex \(g, g_2 x\). So this 1-simplex is a \underline{proof} that \(g_1(g_2 x) = (g, g_2) x\) — the 2 formal expressions \((g_1, g_2 x)\) and \((g, g_2, x)\) evaluate via \(\varepsilon_x\) to the same element of \(X\), namely \(g, g_2 x\).

How about a 2-simplex in \(X\)?

Here we see 2 proofs that \(g_1(g_2 g_3 x) = (g, g_2 g_3) x\):
using one step or 2. The triangle is a "metaproof" or "syzygy" - a "homotopy between proofs".

The simplicial G-set X is called EG when X = *. In general X has contractible components, one for each element of X - these are the 1-simplicies of X!

So EG has one contractible component. It's like a "puffed-up point" - a contractible space on which G acts freely.