Note: gauge group of Electromagnetism is U(1).

Categorification: "boost up a dimension"

2/10/02

Space = \( \mathbb{R}^3 \cong (x, y, z) = (x_1, x_2, x_3) \)

Spacetime = \( \mathbb{R}^4 \cong (+, x, y, z) = (x_0, x_1, x_2, x_3) \)

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Electromagnetism

- Electric field
  - Time-dependent "\( \mathbf{E} \)"
  - 1-form on space

- Magnetic field
  - Time-dependent "\( \mathbf{B} \)"
  - 2-form on space

Categorified Electromagnetism

- "Pseudo vector field"
  - Time-dependent "\( \mathbf{E} \)"
  - 2-form on space

- "Pseudo scalar field"
  - Time-dependent "\( \mathbf{b} \)"
  - 3-form on space

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\[ \mathbf{F} = \mathbf{E} \, dt + \mathbf{B} \]

- 2-form on spacetime

\[ \mathbf{F} = \mathbf{E} \, dt + \mathbf{B} \]

- 3-form on spacetime

If \( \mathbf{E} = E_i \, dx^i \) (\( i, j = 1, 2, 3 \))

\[ \mathbf{B} = \frac{1}{2} \varepsilon_{ijk} \mathbf{B}^i \, dx^j \, dx^k \]

Then

\[ \mathbf{F} = E_i \, dx^i \, dt + \frac{1}{2} \varepsilon_{ijk} \mathbf{B}^i \, dx^j \, dx^k \]

or

\[ \mathbf{F} = \frac{1}{2} \varepsilon_{ijk} \mathbf{E}^i \, dx^j \, dx^k \, dt + \mathbf{B} \, dx^i \, dx^j \, dx^k \]

or if

\[ \mathbf{F} = \frac{1}{6} \mathbf{F}_{\alpha \beta \gamma} \, dx^\alpha \, dx^\beta \, dx^\gamma \]

\[ \alpha, \beta, \gamma = 0, 1, 2, 3 \]
Electromagnetism

Then \( F = \begin{pmatrix} 0 & -E_z & -E_y & E_x \\ E_x & 0 & B_z & -B_y \\ -E_y & B_z & 0 & B_x \\ E_z & -B_x & -B_y & 0 \end{pmatrix} \) time

space

A is a 2-form in space

\( A = \nabla \times \mathbf{E} + \mathbf{B} \)

electromagnetic field!

Sourcefree Maxwell's eqn.
(don't involve matter or metric)

\[ dF = 0 \text{ iff } d(\mathbf{E}dt + \mathbf{B}) = 0 \]

\[ \text{iff } \int \mathbf{dE} dt + \int d\mathbf{B} = 0 \]

\( \text{iff } d \mathbf{E} \cdot dt + d \mathbf{B} = 0 \)

Categorified Electromagnetism

Sourcefree 2-Maxwell eqn.

\[ dF = 0 \text{ iff } \ldots \text{ some argument holds...} \]

\[ \text{iff } d_s \mathbf{E} \cdot dt + d_s \mathbf{B} + dt \frac{d\mathbf{B}}{dt} = 0 \]

\( \text{iff } (d_s \mathbf{E} - d_s \mathbf{B}) \cdot dt + d_s \mathbf{B} = 0 \) or:

\[ \nabla \cdot \mathbf{E} - \frac{dB}{dt} = 0 \]

Here we switched a 1-form \( \mathbf{E} \) to 2-form,

so we get 2 minus signs.

\[ d_s \mathbf{B} = 0 \]

\( \nabla \cdot \mathbf{B} = 0 \)

B is a 3-form, \( \mathbf{d} \) of

\[ \nabla \times \mathbf{E} + \frac{d\mathbf{B}}{dt} = 0 \text{ or } \nabla \cdot \mathbf{B} = 0 \]

or:

\[ \nabla \times \mathbf{E} + \frac{d\mathbf{B}}{dt} = 0 \text{ or } \nabla \cdot \mathbf{B} = 0 \]
Note: $d$ on $\mathbb{R}^n$ (spacetime) is related to $d$ on $\mathbb{R}^3$ (space) ("$d_s$") as follows:

$$dw = d_s w + dt \frac{dw}{dt}$$

say $w = f dx_1 \ldots dx_n$

$$dw = df dx_1 \ldots dx_n$$

$$= \sum_{i=1}^{n} \frac{df}{dx_i} dx_i dx_1 \ldots dx_n + df \frac{dt}{dt} dx_1 \ldots dx_n$$

$$= d_s w + dt \frac{dw}{dt}$$
**Electromagnetism**  \( E = - \frac{\partial}{\partial t} \phi \)  

**Categorified Electromagnetism**

The potentials:

- \( \phi \) is a 0-form on space "\( \phi \)"
- \( A \) is a 2-form on space "\( A \)"

\[ a = -\phi \, dt + A \]

If \( F = da \) then the boring Maxwell eqn follows from \( d^2 = 0 \).

\[ dF = 0 \quad \text{follows from} \quad d^2 = 0. \]

\[ F = da \quad \text{iff} \quad E \cdot dt + B = dt \, da + d_s A \]

**same argument up until:**

\[ E = \frac{\partial A}{\partial t} - d_s \phi \quad \text{and} \quad B = d_s A \]

\[ E = - \frac{\partial A}{\partial t} - d_s \phi \quad \text{or} \quad B = \nabla \times A \]

\[ \nabla \cdot \mathbf{A} = \mathbf{E} \]
Now talk about eqns that do involve matter

(Hodge + operator)

Hodge - * operator

If V is a real vector space then we can form

\[ T^p V = V \otimes \cdots \otimes V \]

p times

The permutation group \( S_p \) acts on \( T^p V \)

\[ \sigma (v_1 \otimes \cdots \otimes v_p) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)} \]

Define the symmetric tensors:

\[ S^p V = \{ x \in T^p V \mid \sigma x = x \} \text{ don't change} \]

and

antisymmetric tensors:

\[ \Lambda^p V = \{ x \in T^p V \mid \sigma x = (-1)^{\text{sgn} \sigma} x \} \]

A \( p \)-form on \( V \) is the same as a smooth function

\[ \omega: V \rightarrow \Lambda^p V^* \]
If $e_1, \ldots, e_n$ is a basis of $V$ then let
\[ dx_1, \ldots, dx_n \] be the dual basis, and then
$w$ can be written as
\[ \left( \text{sum over } i_1 \ldots i_r \right) \; w_{i_1 \ldots i_r} \; dx^{i_1} \ldots dx^{i_r} \]
is a basis of $\Lambda^r V^*$
IR-valued funtcts on $V$

If $V$ is $n$-dim', then $\Lambda^n V$ is $1$-dim'l
w/ basis $dx, \ldots, dx$,
called a volume form

First note that $\Lambda^n V \cong \mathbb{R}$ a real $n$-space

Any nonzero elt. of $\Lambda^n V$, or "volume form" will
be a basis of $\Lambda^n V$. Given any volume form,
say vol, we get an isomorphism:

\[ \Lambda^p V^* \cong \Lambda^{n-p} V \]
\[ \Lambda^p V \cong \Lambda^{n-p} V^* \quad \text{defined as below:} \]

(Recall - Hodge-star operator takes a $p$-form
(to an $(n-p)$ form.)

This isn't exactly the Hodge-star
operator, but similar.
Multiplication: \( \Lambda^p V \times \Lambda^n V \rightarrow \Lambda^n V \cong \text{IR} \)

\[ \text{vol} \rightarrow 1 \]

gives

\[ \Lambda^p V \rightarrow (\Lambda^{n-p} V)^* \cong \Lambda^{-p} V^* \]

which is an isomorphism.

We eventually want to talk about

\[ *d*F = \mathcal{J} \] (non-trivial Maxwell eqn)