A strict skeletal 2-group $C$ amounts to:

- a group $G$
- an abelian group $A$
- an action $\rho$ of $G$ on $A$

A weak skeletal 2-group (w/ associator instead of strict associativity, but w/ left & right unit laws strict, and inverses strict i.e. $gg^{-1} = id$) amounts to:

Having the same stuff as before, but now also with:

$$\alpha : G^3 \rightarrow A$$

$$(fg)h$$

$$\alpha_{f,g,h}$$

$$(fgh)$$

(called 3-cocycle cond.) satisfying the pentagon eqn. (and more)

So - we'll define a Lie 2-group to be one of the above (either weak or strict)

where:

- $G$ is a Lie group
- $A$ is an abelian Lie group
- $\rho$ is smooth
- $\alpha$ is smooth
Lie group has a Lie algebra — tangent space @ identity, so we should have a similar notion for our Lie-2 group.

So, a Lie 2-algebra will (probably) amount to:

- a Lie algebra \( g \)
- a vector space \( \mathfrak{a} \) (an abelian Lie alg)
- a representation \( \rho \) of \( g \) on \( \mathfrak{a} \)
- a linear map \( \alpha : g^3 \to \mathfrak{a} \)

satisfying some equation coming from differentiating the pentagon eqn.

**Gauge Theory**

If \( G \) is a group and \( M \) is a smooth manifold (spacetime), a generalized connection is a functor:

\[
\Pi_1(\text{thin})(M) \to G
\]

"thin fundamental groupoid; and morphisms = groupoid; and composition of morphisms is mult. in group"
Recall: fundamental groupoid: make a category
objects = pts
morphisms = paths w/ equiv. relation

\[ \overline{\Pi} \]

So -
\[ \overline{\Pi}_1^{(\text{thin})}(M) \]
has \( \{ \text{objects} \} = M \)
and morphisms are "thin homotopy classes" of paths in \( M \). A homotopy between paths
\[ H : [0, 1]^2 \rightarrow M \]
is thin if it's smooth and its differential (linear map between tangent spaces) \( dH \) has rank \( \leq 2 \) everywhere.

\[ \text{of this path} \]
A thin homotopy - slide path along itself (just re-parametrize curve)

Note - we can turn a corner smoothly - use functions whose
deriv = 0 (slow down)
stop at corner, then start up smoothly again.
$\tilde{\pi}_1$ thin $(M)$ is something like a smooth manifold and if $G$ is a Lie group and $F$ is smooth, we call $F$ a smooth connection—equivalent to the usual definition.

If $F$ is smooth, we can differentiate it:

Form a 1-parameter family of paths starting with the constant path at $p$ and stretching out in a direction as time passes.

$F$ assigns to each path a group element

$F: \tilde{\pi}_1$ thin $(M) \rightarrow G$.

If $F$ is smooth, we can differentiate it and differentiating $F$ gives us a tangent vector—i.e.

an element of the Lie algebra.

A tangent vector $v$ at $p \in M$ determines a Lie algebra element in $g$.

We get a map

$T_p M \rightarrow g \oplus p$

and thus a $g$-valued 1-form

$A$—what physicists call the connection
From we get:

- curvature \( F = dA + \frac{1}{2} [A, A] \)

- Bianchi identities \( dF + [A, F] = 0 \)
  (Maxwell's eqns are 2 of these)

Hodge-* \( \cdot \) Yang-Mills eqns \( d^* F + [A, +F] = 0 \).

\( \cdot \) describes electroweak \( \cdot \) strong forces (w/o matter)

Now categorify all of this

Fix a 2-group \( C \) and a manifold \( M \):

A "generalized 2-connection" is a 2-functor

\[
F : \Pi^2_{\text{thin}}(M) \to C
\]

a 2-category (weak or strong)

2-category

A 2-functor sends \( \text{obj} \to \text{obj} \)

\( \text{morph} \to \text{morph} \)

\( \text{2-morph} \to \text{2-morph} \)

sends source/target \( \to F(\text{source/target}) \)

everything holds on the nose
where $T^\text{thin}_2(M)$ has:

\{objects\} = \text{pts in } M

\{morphisms\} = \text{paths in } M \quad \text{(not equiv. classes)}

\{2-morphisms\} = \text{thin homotopy classes of paths of paths}

$H: C_0, \mathbb{R}^3 \to M$

d$H$ has rank $< 3$.

and $C$ is a Lie 2-group

If $F$ is smooth, we can differentiate it and get:

\[ \gamma: \mathbb{R} \to \text{family of paths} \]

\[ \downarrow \quad \text{gives} \]

a $g$-valued 1-form $A$

(we can also see what $F$ does to a 1-parameter family of 2-morphisms)
Differentiate $F$, apply to this 1-parameter family of 2-morphisms gives an $\mathfrak{g}$-valued 2-form $B$.

(in string theory goes by name of Neveu–Schwarz field)

So — a "smooth 2-connection" boils down to a pair $w = (A, B)$: $\mathfrak{g}$-valued 1-form and $\mathfrak{g}$-valued 2-form

What if $f$ is a weak 2-functor? Is there more?

Want to understand:
- Curvature of $w$: $(F, G)\omega$

\[
F = dA + \frac{1}{2} [A, A] \quad G = dB + \rho(A) \wedge B
\]
Simple examples:

- $G = 1$, $g = 40^4$, $A = U(1)$, $a = u(1) \cong \mathbb{R}$

Here - all we get is a 2-form $B$ - the Neveu-Schwarz field in string theory?

Curvature = $G = dB$ (a 3-form)

Bianchi identities: $dG = ddB = 0$

Yang-Mills eqns: $d + G = 0$

"2-form electromagnetism"

\[\begin{align*}
\text{EM} & \quad \text{Categorified EM} \\
1 \text{ form } A & \quad 2 \text{ - form } B \\
\text{vector potential} & \\
2 \text{ - form } F = dA & \quad 3 \text{ - form } G = dB \\
\text{electric } & \quad \text{magnetic fields} \\
dF = 0 & \quad \text{Maxwell's eqns.} \\
d*F = 0 & \quad dG = 0 \\
d^*G = 0 &
\end{align*}\]

Point

particle $\int_A \in \mathbb{R}$

$\chi \in U(1)$

two-forms integrated over surfaces