1. For $e^{ta} : \mathcal{E} \to \mathcal{E}$ defined by
   
   $$e^{ta} = \sum_{k \geq 0} \frac{(ta)^k}{k!},$$

   and $f(z) = \sum_{j \geq 0} c_j z^j$, we have
   
   $$e^{ta} f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(ta)^k}{k!} c_n z^n$$
   
   $$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^k}{k!} n(n-1)(n-2) \cdots (n-k+2)(n-k+1) z^{n-k}$$
   
   $$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^{n-k} t^k$$
   
   $$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} t^k$$
   
   $$= \sum_{n=0}^{\infty} c_n (z+t)^n$$
   
   $$= f(z+t)$$

2. Using 1, we have
   
   $$(e^a - 1) f(z) = (e^a f - f)(z) = e^a f(z) - f(z) = f(z + 1) - f(z) = \Delta f(z).$$

3. Suppose $F \in \mathcal{E}$ and $\Delta F = f$. Then
   
   $$F(n) - F(0) = \underbrace{F(n) - F(n-1)}_{f(n-1)} + \underbrace{F(n-1) - F(n-2)}_{f(n-2)} + \cdots + \underbrace{F(2) - F(1)}_{f(2)} + \underbrace{F(1) - F(0)}_{f(1)}$$
   
   $$= f(n-1) + f(n-2) + \cdots + f(2) + f(1) + f(0)$$
   
   $$= \sum_{j=0}^{n-1} f(j)$$
4. We define $a^{-1} : \mathcal{E} \to \mathcal{E}$ by
\[
(a^{-1})f(z) = \int_0^z f(u)du.
\]
Then
\[
(aa^{-1})f(z) = a(a^{-1}f)(z) = a\left(\int_0^z f(u)du\right) = \frac{d}{dz} \int_0^z f(u)du = f(z),
\]
by the FTOC. However, suppose $f(z) = 1, \forall z$. Then $f \in \mathcal{E}$ but
\[
a^{-1}af(z) = a^{-1}\left(\frac{d}{dz} 1\right) = a^{-1}(0) = \int_0^z 0 du = 0 \neq 1 = f(z).
\]

5. Since we have
\[
\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!},
\]
we can multiply both sides by $e^x - 1$ to get
\[
x = (e^x - 1) \sum_{k \geq 0} B_k \frac{x^k}{k!}.
\]
Then
\[
e^x = \sum_{j \geq 0} \frac{x^j}{j!} \implies e^x - 1 = \sum_{j \geq 1} \frac{x^j}{j!},
\]
and we can write
\[
x = \left(\sum_{j \geq 1} \frac{x^j}{j!}\right) \left(\sum_{k \geq 0} B_k \frac{x^k}{k!}\right).
\]
This term on the right is a power series. It looks like two, but after multiplying them as a Cauchy product, it is clearly just one power series (see 7). Since we may always define an operator by a power series, this allows us to write
\[
a = \left(\sum_{j \geq 1} \frac{a^j}{j!}\right) \left(\sum_{k \geq 0} B_k \frac{a^k}{k!}\right). \tag{1}
\]
Now since
\[
\sum_{j \geq 1} \frac{a^j}{j!} = e^{ta} - 1 = \Delta
\]
by 2, and
\[
\sum_{k \geq 0} B_k \frac{a^k}{k!} = \frac{a}{e^a - 1}
\]
by definition, we can rewrite (1) as
\[
a = \Delta \frac{a}{e^a - 1}.
\]
6. Using $\Delta^{-1} = \frac{a}{e^a - 1} a^{-1}$, we have

$$\Delta \Delta^{-1} f = \Delta \left( \frac{a}{e^a - 1} a^{-1} \right) f$$

def of $\Delta^{-1}$

$$= \left( \Delta \frac{a}{e^a - 1} \right) a^{-1} f$$

operator associativity

$$= aa^{-1} f$$

by 5

$$= f$$

by 4

However, we don’t necessarily have $\Delta^{-1} \Delta f = f$. For example, suppose $f(z) = e^{2\pi iz}$. Then $f \in E$, but

$$\Delta^{-1} \Delta f = \Delta^{-1}(\Delta f)$$

$$= \Delta^{-1}(e^{2\pi i(z+1)} - e^{2\pi iz})$$

$$= \Delta^{-1}(e^{2\pi iz} e^{2\pi i} - e^{2\pi iz})$$

$$= \Delta^{-1}(e^{2\pi iz} - e^{2\pi iz})$$

$$= \Delta^{-1}(0)$$

$$= 0$$

$$\neq e^{2\pi iz} = f(z).$$

7. From 5 we have

$$x = \sum_{j \geq 1} \frac{x^j}{j!} \sum_{k \geq 0} B_k \frac{x^k}{k!}$$

$$= \left( x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right) \left( B_0 \frac{1}{0!} + B_1 \frac{x^1}{1!} + B_2 \frac{x^2}{2!} + B_3 \frac{x^3}{3!} + \ldots \right)$$

$$= B_0 x + B_1 \frac{x^1}{1!} + B_2 \frac{x^2}{2!} + B_3 \frac{x^3}{3!} + B_4 \frac{x^4}{4!} + B_5 \frac{x^5}{5!} + \ldots$$

$$= \sum_{n \geq 1} \sum_{k=0}^{n-1} B_k \frac{x^k}{k!} \frac{x^{n-k}}{(n-k)!}$$

$$= \sum_{n \geq 1} \sum_{k=0}^{n-1} B_k \frac{x^n}{n!} \frac{n!}{k! (n-k)!}$$
\[ x = \sum_{n \geq 1} x^n \sum_{k=0}^{n-1} \binom{n}{k} B_k \]  

Then taking the coefficients of \( x^{i+1} \) for \( i \geq 1 \),

\[
0 = \frac{1}{(i+1)!} \sum_{k=0}^{i} \binom{i+1}{k} B_k \\
= \frac{1}{(i+1)!} \left( B_0 \frac{(i+1)!}{0!(i+1)!} + B_1 \frac{(i+1)!}{1!(i)!} + \ldots + B_i \frac{(i+1)!}{(i)!i!} \right) \\
= \frac{B_0}{0!(i+1)!} + \frac{B_1}{1!(i)!} + \frac{B_2}{2!(i-1)!} + \ldots + \frac{B_i}{(i)!i!}.
\]

8. From (2) above, we can collect coefficients for all terms and immediately obtain

\[
1 = 1B_0 \\
0 = 1B_0 + 2B_1 \\
0 = 1B_0 + 3B_1 + 3B_2 \\
0 = 1B_0 + 4B_1 + 6B_2 + 4B_3 \\
0 = 1B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 \\
0 = 1B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4 + 6B_5
\]

9. From the first coefficient comparison above, we see \( B_0 = 1 \). Plugging this into the equations from 8, we obtain:

\[
B_0 = 1 \quad B_1 = -\frac{1}{2} \quad B_2 = \frac{1}{6} \quad B_3 = 0 \quad B_4 = -\frac{1}{30} \quad B_5 = 0
\]

10. Using the previously obtained formula

\[
\sum_{i=1}^{n} i^{p} = \frac{1}{p+1} \sum_{k=0}^{p+1} B_k \binom{p+1}{k} (n+1)^{p+1-k}
\]
with $p = 4$, we obtain

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{5} \sum_{k=0}^{5} B_k \binom{5}{k} (n+1)^{5-k}$$

$$= \frac{1}{5} \left( B_0 (n+1)^5 + 5B_1(n+1)^4 + 10B_2(n+1)^3 + 10B_3(n+1)^2 + 5B_4(n+1)^1 + B_5 \right)$$

$$= \frac{1}{5} B_0(n+1)^5 + B_1(n+1)^4 + 2B_2(n+1)^3 + 2B_3(n+1)^2 + B_4(n+1)^1 + \frac{1}{5}B_5$$

$$= \frac{1}{5} (n+1)^5 - \frac{1}{2}(n+1)^4 + \frac{1}{3}(n+1)^3 - \frac{1}{30}(n+1)$$

11. The formula

$$1^p + 2^p + \cdots + n^p = \frac{(B + n + 1)^{p+1} - B^{p+1}}{p + 1}$$

is just a nifty way of writing

$$\sum_{i=1}^{n} i^p = \frac{1}{p + 1} \sum_{k=0}^{p+1} B_k \binom{p+1}{k} (n+1)^{p+1-k}$$

When $n = 0$ it reduces to

$$(B + 1)^{p+1} = B^{p+1}$$

which is just a nifty way of writing the recursive formula for Bernoulli numbers in terms of Pascal’s triangle, which we have already seen in problem 8.