In class we studied the structure type $G$ for which:

A $G$-structure on a finite set $S$ is a way of chopping $S$ into two parts, ordering the first part and chopping it into blocks of length 1, and ordering the second part and chopping it into blocks of length 2.

Translating this description into an equation, we saw

$$G = \frac{1}{1-Z} \frac{1}{1-Z^2}$$

which gives the generating function

$$|G|(z) = \frac{1}{1-z} \frac{1}{1-z^2} = (1 + z + z^2 + \cdots)(1 + z^2 + z^6 + \cdots) = 1 + z^2 + 2z^3 + 3z^4 + 3z^5 + \cdots$$

If we write

$$|G|(z) = \sum_{n \geq 0} g_n z^n$$

then $g_n$ is the number of ways of writing the number $n$ as a sum of 1’s and 2’s, where we don’t care about the order — or equivalently, where we write all the 1’s first:

$$0 =$$
$$1 = 1$$
$$2 = 1 + 1, 2$$
$$3 = 1 + 1 + 1, 1 + 2,$$
$$4 = 1 + 1 + 1 + 1, 1 + 1 + 2, 2 + 2$$
$$5 = 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 2 + 2$$

and so on. The reason we don’t see an $n!$ in the denominator of this generating function is that putting a $G$-structure on the set $n$ secretly involves choosing a total order on the elements of $n$, and there are $n!$ ways to make this choice.

(When the factors of $n!$ cancel like this, people often call the generating function an ordinary generating function. When they don’t, people call it an exponential generating functions. But, I’m trying to emphasize that ordinary generating functions are just exponential generating functions where the factors of $n!$ have cancelled.)

Now let’s generalize this idea in various directions!

1. Let $a_n$ be the number of ways to write $n$ as a sum of 1’s, 5’s, and 10’s, where we don’t care about the order. Write down a closed-form expression for

$$f(z) = \sum_{n \geq 0} a_n z^n$$

2. How many ways are there to make change for a hundred dollars in pennies, nickels and dimes? (Hint: in Mathematica, Series[$f(z)$, $z$, $n$] computes the first $n$ terms in the Taylor expansion of the function $f(z)$.)