1. k-coloring a finite set.

A $k$-colouring of the set $n$ is a function $f: n \to k$. For each element of $n$ there are $k$ choices of the image of $f$, and the choices for each of the elements are all independent, so there are $k^n$ possible $f: n \to k$. Hence, $|C(k)_n| = k^n$.

2. The generating function of $k$-colourings.

$$|C(k)|(z) = \sum_{n \geq 0} \frac{k^n}{n!} z^n = e^{kz}.$$ 

3, 4. Annihilation.

Trivially,

$$a|C(k)|(z) = \frac{d}{dz} e^{kz} = ke^{kz} = k|C(k)|(z).$$

Suppose that $|C(w)|(z)$ is an eigenvector of $a$ with eigenvalue $w$, that is,

$$\frac{d}{dz}|C(w)|(z) = w|C(w)|(z).$$

Letting

$$|C(w)|(z) = \sum_{n \geq 0} \frac{|C(w)_n|}{n!} z^n,$$

the eigenvalue equation becomes

$$\sum_{n \geq 1} \frac{|C(w)_n|}{(n-1)!} z^{n-1} = \sum_{n \geq 0} \frac{w|C(w)_n|}{n!} z^n$$

so

$$\sum_{n \geq 0} \frac{|C(w)_{n+1}|}{n!} z^n = \sum_{n \geq 0} \frac{w|C(w)_n|}{n!} z^n$$

and $|C(w)_{n+1}| = w|C(w)_n|$. It follows that $|C(w)_n| = w^n|C(w)_0|$, and that

$$|C(w)|(z) = |C(w)_0|e^{wz},$$

so for each complex number $w$ there is a one-dimensional space of eigenvectors of the annihilation operator with eigenvalue $w$.

5. Categorified eigenvalue problem.

Let $K$ be the structure type such that putting it on a set $S$ is “picking a colour out of $k$ and $S$ is empty”. It follows that $|K|(z) = k$.

We seek a structure type $T_k$ such that

$$AT_k \simeq K \times T_k.$$ 

Observe that “putting an $AT_k$ structure on a set $S$” is “putting a $T_k$ structure on the set $S + 1$”, and “putting a $K \times T_k$ structure on the set $S$” is the same as “picking a colour out of $k$ and putting a $T_k$ structure on $S$. That is,
putting a $T_k$ structure on $S + 1$ is the same as picking a colour out of $k$ and putting a $T_k$ structure on $S$.

It easily follows by induction on $|S|$ that a $T_k$ structure on the set $S$ is a $T_k$ structure on the empty set, and a $k$-colouring of $S$. How many ways there are to put a $T_k$ structure on the empty set is undetermined, but it is basically the structure type $K'$ for some integer $k'$. It follows that

$$T_k \simeq K' \times C(k).$$

6. Inner product on Fock space.

Assume, without loss of generality, that $n \geq m$. Then,

$$\langle z^n, z^m \rangle = \langle (a^*)^n 1, z^m \rangle = \langle 1, a^n z^m \rangle = \langle 1, \frac{d^n z^m}{dz^n} \rangle = \delta_{n,m} n! = \delta_{n,m} n!.$$

7. Normalizing coherent states.

Let $\psi_w = e^{wz}$. Then,

$$\langle \psi_w, \psi_w \rangle = \sum_{n \geq 0} \frac{w^n}{n!} z^n = \sum_{n,m \geq 0} \frac{w^n w^m}{n! m!} \langle z^n, z^m \rangle = \sum_{n \geq 0} \frac{|w|^{2n}}{n!} = e^{|w|^2}.$$

Hence, the normalized coherent state is

$$\tilde{\psi}_w = e^{-\frac{|w|^2}{2}} + wz.$$

While we’re at it, we are going to need $\|z\psi_w\|^2$ later:

$$\langle z\psi_w, z\psi_w \rangle = e^{-|w|^2} \sum_{n \geq 0} \frac{|w|^{2n} (n + 1)}{n!} = |w|^2 + 1.$$

8,9,10,11. Heisenberg uncertainty for coherent states.

We use the facts that $q = \frac{1}{\sqrt{2}} (a + a^*)$ and that $p = \frac{1}{\sqrt{2}i} (a - a^*)$. Then,

$$\langle \psi_w, q\psi_w \rangle = \frac{1}{\sqrt{2}} \left( \langle \psi_w, a\psi_w \rangle + \langle a\psi_w, \psi_w \rangle \right) = \frac{w + w^*}{\sqrt{2}}$$

$$\langle \psi_w, p\psi_w \rangle = \frac{w - w^*}{\sqrt{2i}}.$$

Similarly,

$$\langle \psi_w, q^2\psi_w \rangle = \langle q\psi_w, q\psi_w \rangle = \frac{1}{2} \left( \langle a\psi_w, a\psi_w \rangle + \langle a^2 \psi_w, \psi_w \rangle + \langle \psi_w, a^2 \psi_w \rangle + \langle a^* \psi_w, a^* \psi_w \rangle \right)$$

$$= \frac{1}{2} \left( |w|^2 + w^2 + w^2 + |w|^2 + 1 \right) = \langle \psi_w, q\psi_w \rangle^2 + \frac{1}{2}$$

$$\langle \psi_w, p^2\psi_w \rangle = \frac{1}{2} \left( |w|^2 - w^2 - w^2 + |w|^2 + 1 \right) = \langle \psi_w, p\psi_w \rangle^2 + \frac{1}{2}.$$