1. We have
\[ !(n) = n! - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots \]
where $A_i$ is the set of all permutations that fix $i$ (the $i^{th}$ element). Thus, we can equivalently consider $A_i$ to be the set of permutations on the $(n-1)$-element set $\{n \setminus \{i\}\}$. So $|A_i| = (n-1)!$.

But which of the original $n$ elements gets to play the role of $i$? There are $\binom{n}{1}$ possibilities in total. Since we are summing over all $i$,
\[ \sum_i |A_i| = \binom{n}{1} (n-1)!. \]

BSA, $A_i \cap A_j$ corresponds to those permutations fixing both $i$ and $j$. Thus $|A_i \cap A_j| = (n-2)!$ and since there are $\binom{n}{2}$ ways to choose $i$ and $j$, we have
\[ \sum_{i<j} |A_i \cap A_j| = \binom{n}{2} (n-2)!. \]

Continuing in this vein,
\[ !(n) = n! - \binom{n}{1} (n-1)! + \binom{n}{2} (n-2)! - \ldots + (-1)^n \binom{n}{n} (n-n)! \]

2. Since $\left( k = \frac{n!}{k!(n-k)!} \right)$ the above formula for $!(n)$ simplifies readily as
\[ !(n) = n! - \frac{n!}{1!(n-1)!} (n-1)! + \frac{n!}{2!(n-2)!} (n-2)! - \ldots + (-1)^n \frac{n!}{n!(n-n)!} (n-n)! \]
\[ = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!} \right). \]

3. The probability that nobody receives the correct coat is given by (number of derangements)/(number of permutations, i.e., the previous formula gives
\[ \frac{!(n)}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \]

Thus,
\[ \lim_{n \to \infty} \left( \frac{!(n)}{n!} \right) = \lim_{n \to \infty} \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} \]
4. Now \( \lim_{n \to \infty} \left( \frac{n!}{e^n} \right) = 1 \) is equivalent to

\[
!n \sim \frac{n!}{e}.
\]

In other words,

\[
\forall \varepsilon, \exists N \text{ s.t. } n \geq N \implies \left| !n - \frac{n!}{e} \right| < \varepsilon.
\]

In particular, we can choose \( \varepsilon = \frac{1}{2} \). Then \( N = 1 \). Since \( !n \) is always an integer, and \( \left| !n - \frac{n!}{e} \right| \) for \( n \geq 1 \), this shows \( !n \) is the closest integer to \( \frac{n!}{e} \).

Actually, that doesn’t quite work because we need some info about the monotonicity of \( \left| !n - \frac{n!}{e} \right| \), so let’s break out the big guns:

We have

\[
\frac{n!}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!} = 1 - \sum_{k>n} \frac{(-1)^n}{n!}.
\]

So

\[
!n = \frac{n!}{e} - n! \sum_{k>n} \frac{(-1)^n}{n!},
\]

and we just need to show

\[
\left| n! \sum_{k>n} \frac{(-1)^n}{n!} \right| < \frac{1}{2}.
\]

This is a rapidly convergent alternating series, so the sum is trapped between any two consecutive partial sums:

\[
n! \sum_{k=n+1}^{N} \frac{(-1)^k}{k!} \leq n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \leq n! \sum_{k=n+1}^{N+1} \frac{(-1)^k}{k!}.
\]

In particular, it’s trapped between the second and third:

\[
n! \left( \frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} \right) \leq n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \leq n! \left( \frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + \frac{(-1)^{n+3}}{(n+3)!} \right)
\]

\[
\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{(n+1)(n+2)} \leq n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \leq \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{(n+1)(n+2)} + \frac{(-1)^{n+3}}{(n+1)(n+2)(n+3)}
\]

Now we can take the absolute value of the left-hand side\(^1\):

\[
\left| \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{(n+1)(n+2)} \right| = \left| \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} \right| = \left| \frac{n+1}{(n+1)(n+2)} \right| = \left| \frac{1}{n+2} \right| \leq \frac{1}{3}, \forall n \geq 1.
\]

\(^1\)Since one of \( \{(-1)^{n+1}, (-1)^{n+2}\} \) is 1 and the other is \(-1\), and since we are taking the absolute value, we can arbitrarily let one be 1 and the other be \(-1\). Hence the first equality.
Similarly for the right-hand side:
\[
\left| \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{(n+1)(n+2)} + \frac{(-1)^{n+3}}{(n+1)(n+2)(n+3)} \right| = \left| \frac{(n+2)(n+3)+1}{(n+1)(n+2)(n+3)} - \frac{n+3}{(n+1)(n+2)(n+3)} \right|
\]
\[
= \left| \frac{n^2+4n+4}{(n+1)(n+2)(n+3)} \right|
\]
\[
= \left| \frac{n+2}{(n+1)(n+3)} \right|
\]
\[
\leq \frac{3}{8}, \forall n \geq 1
\]

Now the sum in question is trapped between two quantities of absolute value less than \(\frac{1}{2}\) (and note that any two consecutive partial sums are less than \(\frac{1}{24}\) apart), we have
\[
\left| n! \sum_{k>n} \frac{(-1)^n}{n!} \right| < \frac{1}{2},
\]
and hence
\[
\left| n! - \frac{n!}{e} \right| < \frac{1}{2} \quad \implies \quad n! = \left[ \frac{n!}{e} \right].
\]

5. We construct an isomorphism \( P \cong E^Z D \).

If \( f \) is a permutation on \( S \), then let
\[
A = \{ x \in S : f(x) = x \}, \quad B = S \setminus A.
\]
Now \( f(b) \neq b, \forall b \in B \), by definition of \( B \), so \( f \) is a derangement of \( B \) and the identity on \( A \). In more categorical terms, \( f \) induces a splitting of \( S \) into two parts such that one part is left untouched and the other part is deranged. I.e., the first part is simply given the structure of a finite set, while the second part is given a derangement. This process of splitting a set into two pieces and putting different structures on each piece corresponds to multiplication of structure types. Since \( P \) is the structure type of “being permuted” and \( E^Z \) is the structure type of “being a finite set” and \( D \) is the structure type of “being deranged”, we have
\[
P \cong E^Z D.
\]

6. Decategorifying the above isomorphism, we obtain
\[
\frac{1}{1-z} = e^z|D|.
\]
The left side comes from
\[
|P|(z) = \frac{p_0}{0!} + \frac{p_1}{1!}z^1 + \frac{p_2}{2!}z^2 + \frac{p_3}{3!}z^3 + \ldots
\]
\[
= \frac{0}{0!} + \frac{1}{1!}z^1 + \frac{2}{2!}z^2 + \frac{3}{3!}z^3 + \ldots
\]
\[
= 1 + z + z^2 + z^3 + \ldots
\]
\[
= \frac{1}{1-z},
\]
where the second line follows because there are \( n! \) permutations of the \( n \)-element set, and the last line follows as a geometric series. Multiplying both sides by \( e^{-z} \) gives a formula for \(|D|\):

\[
|D|(z) = \frac{e^{-z}}{1 - z}.
\]

7. If we differentiate the above formula, the quotient rule yields

\[
\frac{d}{dz}|D|(z) = \frac{d}{dz} \left( \frac{e^{-z}}{1 - z} \right) = \frac{-(1 - z)e^{-z} - e^{-z}(-1)}{(1 - z)^2}
= e^{-z} \frac{1 - (1 - z)}{(1 - z)^2}
= e^{-z} \frac{z}{(1 - z)^2}
\]

Thus, \((1 - z) \frac{d}{dz}|D|(z) = e^{-z} \frac{z}{1 - z}\). On the other hand,

\[
|D|(z) - e^{-z} = \frac{e^{-z}}{1 - z} - \frac{(1 - z)e^{-z}}{1 - z} = e^{-z} \frac{1 - (1 - z)}{1 - z} = e^{-z} \frac{z}{1 - z},
\]

showing that

\[(1 - z) \frac{d}{dz}|D|(z) = |D|(z) - e^{-z}.
\]

8. Since the number of derangements of the \( n \)-element set is \(!n\), we have

\[
|D|(z) = \sum_{n=0}^{\infty} \frac{!n}{n!} z^n.
\]

Also, from 7 we have

\[(1 - z) \frac{d}{dz}|D|(z) = |D|(z) - e^{-z}.
\]

Now, throwing caution to the wind and differentiating infinite sums term-by-term,

\[
(1 - z) \frac{d}{dz}|D|(z) = (1 - z) \sum_{n=1}^{\infty} \frac{!n}{n!} nz^{n-1}
= \sum_{n=1}^{\infty} \frac{!n}{n!} nz^{n-1} - \sum_{n=1}^{\infty} \frac{!n}{n!} nz^n
= \sum_{n=0}^{\infty} \frac{!(n+1)}{(n+1)!} (n+1)z^n - \sum_{n=1}^{\infty} \frac{!n}{n!} nz^n
= \sum_{n=1}^{\infty} \left( \frac{!(n+1)(n+1)}{(n+1)!} - \frac{!(n)n}{n!} \right) z^n
= \sum_{n=1}^{\infty} \left( \frac{!(n+1)(n+1)}{(n+1)!} - \frac{!(n)n}{n!} \right) z^n
= 0
\]
Now we manipulate the other side of the equation:

\[
|D|(z) - e^{-z} = \sum_{n=0}^{\infty} \frac{!n}{n!} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n \quad \text{by above}
\]

\[
= \sum_{n=0}^{\infty} \frac{!n - (-1)^n}{n!} z^n \quad \text{combine}
\]

\[
= \sum_{n=1}^{\infty} \frac{!n - (-1)^n}{n!} z^n \quad 0 - (-1)^0 = 0
\]

Combining these equalities gives

\[
\sum_{n=1}^{\infty} \left( \frac{!(n+1)(n+1)}{(n+1)!} - \frac{!(n)n}{n!} \right) z^n = \sum_{n=1}^{\infty} \frac{!n - (-1)^n}{n!} z^n,
\]

so equating the coefficients gives

\[
\frac{!(n+1)(n+1)}{(n+1)!} - \frac{!(n)n}{n!} = \frac{!n - (-1)^n}{n!}.
\]

Multiplying by \( n! \) and cancelling the \( n + 1 \), we get

\[
!(n + 1) - !(n)n = !n - (-1)^n
\]

\[
!(n + 1) = !(n)n + !n - (-1)^n
\]

\[
!(n + 1) = !n(n + 1) + (-1)^{n+1}
\]

There is another way to obtain the same result using just combinatorics. It is much more basic, but avoids possible irksome analysis technicalities. Note that an \( n \)-derangement can be derived from its predecessors in just one of two ways:

case i) Take a derangement of the first \( n - 1 \) elements, then swap the \( n^{\text{th}} \) with one of them.

case ii) Derange \( n - 2 \) of the first \( n - 1 \) elements, then swap the \( n^{\text{th}} \) with the one that has remained hitherto fixed.

A moment’s reflection shows that these are all the \( n \)-derangements, and one produced one way cannot be produced the other way. Since there are \( n - 1 \) ways to do each of these things,

\[
!n = (n - 1)!(n - 1) + (n - 1)!(n - 2)
\]

\[
= (n - 1)!(n - 1) + !(n - 2)
\]

\[
= n!(n - 1) - !(n - 1) - (n - 1)!(n - 2)
\]

\[
!n - n!(n - 1) = -!(n - 1) - (n - 1)!(n - 2)
\]
Note that if the left side of this last equation were denoted $L_n$, then the right side would be $-L_{n-1}$. This leads to a bizarre but simple reductio ad iteratum:

$$!n - n!n(n - 1) = -(-(n - 2)! + (n - 3))$$

$$= (-1)^k(n - k) + (n - k)(n - k - 1))$$

$$= (-1)^{n-2}((2 + 2!1))$$

$$= (-1)^{n}(1 + 0)$$

(continuing)

Finally, adding back the $n!n(n - 1)$ gives

$$!n = n!n(n - 1) + (-1)^n.$$