1. To put a $P(k)$-structure on the $n$-element set $S$, we first have to totally order the elements. There are $n!$ ways to do this. Then, each of the numbers $j \in \{1, 2, \ldots, k\}$ must be assigned to one of the $n$ elements of $S$. Since repeats are allowed (more than one $j$ may be assigned to a given point of $S$), there are $n$ possibilities for each $j$. Hence,

$$|P(k)_n| = n!n^k.$$ 

2. $|P(k)|(z) = \sum_{n \geq 0} \frac{|P(k)_n|}{n!} z^n = \sum_{n \geq 0} n^k z^n.$$

3. $|P(0)|(z) = \sum_{n \geq 0} n^0 z^n = \sum_{n \geq 0} z^n = \frac{1}{1-z}.$

4. Define the number operator by $N\Psi = A^*A\Psi$. Then

put a $N\Psi$-structure on a set $S$

= put an $A^*A\Psi$-structure on $S$

$\cong$ choose $x \in S$ and put an $A\Psi$-structure on $S\setminus\{x\}$

$\cong$ choose $x \in S$ and put $\Psi$-structure on $(S\setminus\{x\}) + 1$

This corresponds to ‘pointing’ $S$ (i.e., giving it one distinguished point) and putting a $\Psi$-structure on it.

5. Putting an $NP(k)$ structure on $S$ amounts to pointing $S$, and then totally ordering $S$ and giving it a $k$-pointing. But this is the same as totally ordering $S$, giving it a distinguished point (say we give one point a star), and then $k$-pointing $S$, because surely the order doesn’t matter. Finally, by ‘reindexing’

$$\{\ast\} \cup \{1, 2, \ldots, k\} = \{\ast, 1, 2, \ldots, k\} \cong \{1, 2, \ldots, k+1\}$$

so that we’ve really just put a total ordering on $S$ and then given it a $(k+1)$-pointing, i.e., put a $P(k+1)$-structure on it. Hence,

$$NP(k) \cong P(k+1).$$
6. Decategorifying \( P(k + 1) \cong NP(k) \),

\[
|P(k + 1)|(z) = |NP(k)|(z) \\
= |A^*AP(k)|(z) \\
= z \frac{d}{dz}|P(k)|(z) \quad \text{def of } N
\]

\[
|A^*| = z, |A| = \frac{d}{dz}
\]

7. By the previous problem we know

\[
|P(1)|(z) = z \frac{d}{dz}|P(0)|(z) \\
= z \frac{d}{dz} \left( \frac{1}{1-z} \right) \\
= \frac{z}{(1-z)^2}
\]

8. By the previous formula,

\[
|P(1)|(-1) = \frac{-1}{(1+1)^2} = -\frac{1}{4}.
\]

But since \( |P(1)|(z) = \sum_{n \geq 0} nz^n \) by the definition of \( |P(k)| \) found in 3, so

\[
|P(1)|(-1) = 1 - 2 + 3 - 4 + \ldots = -\frac{1}{4}.
\]

9. Using the definition of Abel summation:

\[
A \sum_n an := \lim_{t \uparrow 1} \sum_n t^n a_n,
\]

we see that

\[
A \sum_{n=1}^{\infty} (-1)^{n+1} n = \lim_{t \uparrow 1} \sum_{n=1}^{\infty} t^n (-1)^{n+1} n \quad \text{by def}
\]

\[
= - \lim_{t \uparrow 1} \sum_{n=1}^{\infty} (-t)^n n \quad \text{collecting}
\]

\[
= - \lim_{t \uparrow 1} |P(1)|(-t) \quad \text{by 2}
\]

\[
= - \lim_{t \uparrow 1} \frac{-t}{(1+t)^2} \quad \text{by 7}
\]

\[
= - \frac{1}{2}\frac{\pi}{2} \quad \text{continuity}
\]

\[
= \frac{1}{4}
\]
10. First we compute:

\[ |P(2)|(z) = z \frac{d}{dz} |P(1)|(z) \]
\[ = z \frac{d}{dz} \left( \frac{z}{(1-z)^2} \right) \]
\[ = z \left( \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} \right) \]
\[ = \frac{z}{(1-z)^2} + \frac{2z^2}{(1-z)^3} \]

Now, to compute the Abel sum of \( 1^2 - 2^2 + 3^2 - 4^2 + \ldots \)

\[ |P(2)|(-1) = A \sum_{n=1}^{\infty} n^2 (-1)^n \]
\[ = \lim_{t \to 1} \sum_{n=1}^{\infty} t^n n^2 (-1)^n \]
\[ = \lim_{t \to 1} \sum_{n=1}^{\infty} n^2 (-t)^n \]
\[ = \lim_{t \to 1} |P(2)|(-t) \]
\[ = \lim_{t \to 1} \frac{-t}{1+t} + \frac{2t^2}{(1+t)^3} \]
\[ = \frac{1}{4} + \frac{2}{8} \]
\[ = 0 \]

Lastly, we wish to find \( \zeta(-2) = 1^2 + 2^2 + 3^2 + 4^2 + \ldots \) From

\[ \zeta(-2) = \sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots \]

we get

\[ 2^2 \zeta(-2) = 2^2 \sum_{n=1}^{\infty} n^2 = \sum_{n=1}^{\infty} (2n)^2 = 2^2 + 4^2 + 6^2 + 8^2 + \ldots \]

Subtracting,

\[ \zeta(-2) - 2 \cdot 2^2 \zeta(-2) = 1^2 - 2^2 + 3^2 - 4^2 + \ldots = 0. \]

Thus,

\[ \zeta(-2) - 2 \cdot 2^2 \zeta(-2) = (1 - 2^3)\zeta(-2) = -7\zeta(-2) = 0 \implies \zeta(-2) = 0, \]

as expected.