A Pointed Assignment
Jeff Morton

1. \(P(k)_n\) is the number of maps from the \(k\) set (i.e. "the" set with \(k\) elements) to "the" \(n\) set (set with \(n\) elements) multiplied by the number of total orderings on \(n\), since we choose both such a map and a total ordering on \(n\). The number of maps \(f : k \to n\) is just \(n^k\) since each element of \(k\) has \(n\) possible images, and these are all chosen independently (i.e. we allow repetition). The number of total orderings is of course \(n!\). So the product of these gives \(|P(k)_n| = n^k \cdot n!|\).

2. We have in general \(|P(k)|(z) = \sum_{n \geq 0} \frac{|P(k)_n|z^n}{n!}\). In this case, the \(n!\) factors cancel (that is, the total ordering on \(n!\) makes this a case we could think of as an ordinary generating series), so we have:

\[|P(k)|(z) = \sum_{n \geq 0} n^k \frac{z^n}{n!} = \sum_{n \geq 0} n^k z^n\]

3. In the special case where \(k = 0\) the generating function above is:

\[|P(0)|(z) = \sum_{n \geq 0} n^0 z^n = \sum_{n \geq 0} z^n = \frac{1}{1-z}\]

4. To put an \(NP(k)\)-structure on a set \(S\) is to put an \(A^*A\Psi\)-structure on it, and this means, by the definition of the \(A^*\) operator, to choose an element \(x\) of \(S\) and then put an \(A\Psi\)-structure on \(S \setminus \{x\}\). Now, to put an \(A\Psi\)-structure on \(S \setminus \{x\}\) is, by the definition of the \(A\) operator, to put a \(\Psi\)-structure on \((S \setminus \{x\}) + 1\), that is, \(S \setminus \{x\}\) with a single point added to it. So, to put an \(NP\)-structure on \(S\) is to choose a point \(x \in S\), remove it from \(S\), then add a new point to the resulting set, and put a \(\Psi\)-structure on the set thus created. This is equivalent to identifying a special point of \(S\) (since we have a natural isomorphism between \(S\) and the resulting set which sends every element of \(S \setminus x\) to itself, and \(x\) to the new one-point set denoted \(1\)) and then putting a \(\Psi\)-structure on it.

5. To put an \(NP(k)\)-structure on a set \(S\) is to specify a point \(x \in S\) and also put a \(P(k)\)-structure on it. Now, a \(P(k)\)-structure on \(S\) is a \(k\)-pointing - that is, a labelling of \(k\) points (possibly with repetition) of \(S\) by numbers \(1 \ldots k\). On the other hand, a \(P(k + 1)\)-structure is a labelling of \(k + 1\) elements of \(S\) by numbers \(1 \ldots k + 1\). There is a natural way to define an isomorphism between such structures. Given an \(NP(k)\)-structure on \(S\), construct a \(P(k + 1)\)-structure on \(S\) by assigning the numbers \(1 \ldots k\) to the same points in the \((k + 1)\)-pointing as in the \(k\)-pointing, and assigning the number \(k + 1\) to the specially identified point from the \(NP(k)\)-structure. This is clearly reversible, hence an isomorphism. In particular, it is natural since there is a unique natural choice for which element of \(k + 1\) to assign to the special point. Thus, by thinking of one assignment of labels from \(k + 1\) as an assignment of labels in \(k\) to a pointed set, we have

\[NP(k) \cong P(k + 1)\]
6. We have seen previously that the effect of the $A$ and $A^*$ operators on the generating functions corresponding to a structure type $\Psi$ is, respectively, $|A\Psi|(z) = \frac{1}{z} |\Psi|(z)$ and $|A^*\Psi|(z) = z |\Psi|(z)$. So combining these, and the existence of the isomorphism above gives that:

$$|P(k+1)|(z) = |NP(k)|(z) = |A^*AP(k)|(z) = z \frac{d}{dz} |P(k)|(z)$$

7. By part 6, we have that $|P(1)|(z) = z \frac{d}{dz} |P(0)|(z)$, but since by part 3 we know that $|P(0)|(z) = \frac{1}{1-z}$, we find that:

$$|P(1)|(z) = z \frac{d}{dz} \left( \frac{1}{z} \right) = z \left( -\frac{-1}{(1-z)^2} \right) = \frac{z}{(1-z)^2}$$

8. Now we come to the point of this assignment - evaluating divergent sums using our generating function, and using Abel sums. The first says that if we have the expression above equal to $-|P(1)|(z)$, then $|P(1)|(-1) = \frac{-1}{(1-(-1))^2} = -\frac{1}{2^2} = -\frac{1}{4}$. On the other hand, we know from part 2 that

$$|P(1)|(z) = \sum_{n \geq 0} (-1)^n z^n$$

But then, if $z = -1$, this gives that

$$|P(1)|(-1) = \sum_{n \geq 0} n \cdot (-1)^n = -1 + 2 - 3 + \ldots$$

Using these two expressions, we could claim that $-(-1 + 2 - 3 + \ldots) = -\frac{1}{4}$. This is the same as what we want, namely that $1 - 2 + 3 - 4 + \ldots = \frac{1}{4}$.

9. We observe that the sum above does not actually converge, since the point $z = -1$ is not strictly inside the radius of convergence for the function $|P(1)|(z)$ written as a power series expanded about $z = 0$. This is because the function has a pole at $z = 1$, so the radius of convergence is 1, but this function is analytic everywhere else in the complex plane. So the function can be analytically continued to $z = -1$, though the power series does not converge there. This is exactly what the Abel sum:

$$A \sum_{n=1}^{\infty} (-1)^{n+1} n = -\lim_{t \to 1^-} \sum_{n=1}^{\infty} t^n (-1)^n \cdot n$$

is doing: this is an analytic continuation of $|P(1)|(z)$ to $z = -1$ along the negative real axis. This is

$$-\lim_{t \to 1^-} \sum_{n=1}^{\infty} (-t)^n \cdot n = -\lim_{t \to 1^-} |P(1)|(-t) = -\lim_{t \to 1^-} \frac{-t^4}{(1+t)^2} = \frac{1}{4}$$

So in fact the Abel sum of the series in question is indeed the value we found using $|P(1)|(z)$. 


10. We have that $|P(2)|(z) = z \frac{d}{dz} |P(1)|(z) = z \frac{d}{dz} \left( \frac{z}{(1-z)^2} \right)$, using parts 6 and 7 respectively. This means that $|P(2)|(z) = (1-2z+z^2) = 1 = \frac{1+z}{(1-z)^3}$.

But on the other hand, we know by part 2 that $|P(2)|(z) = \sum_{n \geq 0} n^2 z^n$.

If we evaluate this sum at $z = -1$, we get the alternating sum

\[-1 - 2 - 3 - 4 \ldots,\]

so the Abel sum of the series $1^2 - 2^2 + 3^2 - 4^2 + \ldots$ will be the negative of $|P(2)|(-1)$, by the same reasoning as above, namely that $|P(2)|(z)$ as given above is an analytic function on all of $\mathbb{C}$ except for a pole of order 3 at $z = 1$. Thus we can extend analytically in a unique way to $z = -1$, and so:

\[
A \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n^2 = -\lim_{t \to 1} t^n (-1)^n \cdot n^2
\]

\[
= -\lim_{t \to 1} (-t)^n \cdot n^2
\]

\[
= -\lim_{t \to 1} |P(2)|(t)
\]

\[
= -\lim_{t \to 1} \frac{(1-t)}{(1+t)^3}
\]

\[
= 0
\]

Now using Euler’s approach, we would say that $\zeta(-2) = 1^2 + 2^2 + 3^2 + \ldots$ and we can also get that $4\zeta(-2) = 2^2 + 4^2 + 6^2 + \ldots$, since each term here is 4 times the corresponding term in $\zeta(-2)$. Thus, if we subtract twice this second series, we should get the alternating series from above:

\[-7(1^2 + 2^2 + 3^2 + \ldots) = 1^2 - 2^2 + 3^2 - 4^2 \ldots = 0\]

In other words, $\zeta(-2) = 1^2 + 2^2 + 3^2 + \ldots = 0$. 

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