1 + 2 + 3 + ⋯ = -\frac{1}{12}

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1. Totally ordered, k-pointed sets.

   There are $n!$ ways to totally order the set $n$, and $n^k$ functions $f: k \to n$, so
   
   $|P(k)_n| = n!n^k$.

2. Generating function

   $|P(k)| = \sum_{n \geq 0} \frac{|P(k)_n|}{n!} z^n = \sum_{n \geq 0} n^k z^n$.

3. Totally ordered sets.

   Since 0-pointing amounts to doing nothing (in one way), $P(0)$ is isomorphic to the structure type “being a totally ordered finite set”, and that has generating function
   
   $|P(0)(z)| = \sum_{n \geq 0} z^n = \frac{1}{1-z}$.

4,5. $N\Phi$-structures.

   To put a $N\Phi$-structure on a set $S$ means to choose $s \in S$ and to put an $A\Phi$-structure on $S - \{s\}$ or, equivalently, to choose $s \in S$ and to put a $\Phi$-structure on $S$. This is equivalent to pointing $S$ and putting a $\Phi$-structure on it. Hence, an $N\Phi$-structure is a pointed $\Phi$-structure.

6,7. $k$-pointed $\Phi$-structures.

   It is obvious that an $N^k\Phi$-structure is isomorphic to a $k$-pointed $\Phi$-structure, so a $P(k)$ structure isomorphic to an $N^kP(0)$-structure, and so $NP(k) \simeq P(k+1)$.

   The decategorification of this isomorphism is
   
   $z \frac{d}{dz} |P(k)|(z) = |P(k+1)|(z),$

   so
   
   $|P(1)|(z) = z \frac{d}{dz} |P(0)|(z) = \frac{z}{(1-z)^2}$.

8,9,10. Euler’s trick.

   $1 - 2 + 3 - 4 + 5 - 6 + \cdots = |P(1)|(-1) = -\frac{1}{4}$.

   We can prove this rigorously using Abel sums like this:

   $A \sum_{n \geq 0} (-1)^{n+1} n = \lim_{t \to 1} \sum_{n \geq 0} (-t)^n n = \lim_{t \to 1} -\sum_{n \geq 0} t^n n = \lim_{t \to 1} \sum_{n \geq 0} t^n \frac{d}{dt} t^n = \lim_{t \to 1} -\sum_{n \geq 0} t^n \frac{d}{dt} \frac{1}{1-t} = \lim_{t \to 1} t \frac{d}{dt} \frac{1}{1-t} = \frac{1}{4}$.

   Finally, since
   
   $|P(2)|(z) = z \frac{d}{dz} |P(1)|(z) = z \left[ \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} \right] = \frac{z(1+z)}{(1-z)^3}$

   we have
   
   $(1 - 2^3)\zeta(-2) = |P(2)|(-1) = 0$,

   so $\zeta(-2) = 0$. 