The big picture in quantum mechanics:

**Classical**

**Lagrangian**

\[ Q = \text{config. space} \]
\[ P_{x_0, x_1} = \{ \gamma : [t_0, t_1] \rightarrow Q : \gamma(t_i) = x_i \} \]
\[ L : TQ \rightarrow \mathbb{R} \text{ Lagrangian} \]
\[ S = \int_{t_0}^{t_1} L(\dot{q}(t), \dot{p}(t)) \, dt \]

**Hamiltonian**

\[ \tilde{q}(t) = (q(t), p(t)) \in T^*Q \]
\[ \dot{\tilde{q}}(t) = V_H(\tilde{q}(t)) \]
\[ \partial H = \omega^*(V_H, -) \]

\[ V_H = \text{"Hamiltonian vector field"} \]
\[ \omega^* = \text{"symplectic structure"} \]

\[ X = T^*Q = \text{phase space} \]
\[ H : X \rightarrow \mathbb{R} \text{ Hamiltonian} \]
\[ H = p_i q^i - L \cdot \lambda \]

**Quantum**

\[ \psi_t(x) = \int_Q \psi_{0,t}(x') \exp(iS(x')) \, dx' \]
\[ \psi_t \in L^2(Q) \]

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This chart raises lots of questions:

1) How do you do the "path integral" over $P_{\alpha_0, \alpha_1}$? Apparently, there's no meaning to the "measure" $d\gamma$, but there is to $e^{iS(\gamma)/\hbar}d\gamma$, at least in well-behaved cases, e.g. the case where

$$Q = \text{smooth fin-dim manifold}$$

$$L(q, \dot{q}) = \frac{m}{2} |\dot{q}|^2 - V(q)$$

where $|\dot{q}|$ is defined using a complete Riemannian metric. The completeness assumption is needed to keep our particle from "falling off the edge", & $V: Q \to \mathbb{R}$ should be smooth and bounded below, for the same reason.

For the basic ideas, try Feynman & Hibbs' "Quantum Mechanics and Path Integrals." For mathematical rigor, try Barry Simon's "Functional Integration and Quantum Physics."

2) How do we get the Hamiltonian operator $\hat{H}: L^2(Q) \to L^2(Q)$ from the Hamiltonian function $H: T^*Q \to \mathbb{R}$? In some cases, it's easy to write down $\hat{H}$, e.g. under the same assumptions we wrote down in question #1:

$$H(q, p) = \frac{1}{2m} |p|^2 + V(q)$$

in $Q$ a complete Riemannian manifold & $V: Q \to \mathbb{R}$ smooth and bided below.
In this situation Schrödinger wrote:

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + \text{mult}_{V}$$

where $\nabla^2$ is the Laplacian on $Q$ and $\text{mult}_{V}$ is the operator "multiplication by $V$," (often written simply as "$V".") Schrödinger got this by guessing the quantization rule

$$p \quad \mapsto \quad \frac{\hbar}{i} \nabla$$

Under our assumptions on $Q$ & $V$, Kato & Rellich showed that $\hat{H}$ is self-adjoint, which is precisely what you need to solve Schrödinger's equation. If $A:K \to K$ is a self-adjoint operator on a Hilbert space, $e^{itA}:K \to K$ is well-defined and unitary, & defining

$$\psi_t = e^{itA}\psi_0$$

we get

$$\frac{d}{dt} \psi_t = iA\psi_t.$$ 

But we'd like a much more systematic theory of "quantizing" functions $H: T^*Q \to \mathbb{R}$ to get operators $\hat{H}:L^2(Q) \to \mathbb{R}^2(Q)$. Even better, can we handle the case when the phase space $X$ isn't $T^*Q$? Then we don't even have $L^2(Q)$ at hand.
This leads us to "geometric quantization." For more on this, try:

http://math.ucr.edu/home/baez/quantization.html

Then try Sniatycki's book. A lot of cohomology comes into the game — starting with the fact that $[\alpha] \in H^2(X,\mathbb{R})$ must come from an integral cohomology class

$$H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{R})$$