**Def** A morphism $f: X \to Y$ is an iso if $\exists f^{-1}: Y \to X$ that's a left inverse $f^{-1} \circ f = 1_X$ & a right inverse $f \circ f^{-1} = 1_Y$

**Prop** In Set, $f: X \to Y$ is a mono iff it has a left inverse, and an epi iff it has a right inverse (using the Axiom of Choice).

Thus $f$ is an iso iff it's a mono & epi.

Assume we have identity.

**Prop** In Ring (rings & ring homomorphisms) $f: \mathbb{Z} \to \mathbb{Q}$

is a mono and an epi but not an iso, in fact it has neither left nor right inverse.

**Pf:**

There's no ring homomorphism $g: \mathbb{Q} \to \mathbb{Z}$ since it would send $\frac{1}{2}$ to some multiplicative inverse of 2.

**Why is $f$ a mono?**

Need: $f \circ g = f \circ h \Rightarrow g = h$  \[ \mathbb{R} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Q} \]

If $(f \circ g)(r) = (f \circ h)(r) \quad \forall r \in \mathbb{R}$,

since $f$ is 1-1 $g(r) = h(r) \quad \forall r$ (as a function)

$\Rightarrow g = h$

**Why is $f$ epi?**

Need: $g \circ f = h \circ f \Rightarrow g = h$

We know $g(p) = h(p) \quad & \quad g\left(\frac{q}{p}\right) = h\left(\frac{q}{p}\right)$

So $g(1) = g\left(\frac{q}{q}\right) = g(q) \cdot g\left(\frac{1}{q}\right)$, so we can write $g\left(\frac{1}{q}\right) = \frac{1}{g(q)}$.

So $g\left(\frac{2}{q}\right) = g(p) \cdot g\left(\frac{1}{q}\right) = \frac{g(p)}{g(q)}$

So $g$ (or similarly $h$) is determined by its value on integers, since they agree on $\mathbb{Z}$ they're equal.

**Puzzle:** In Top, find $f: X \to Y$ that is epic & mono but not an isomorphism.
Limits & Colimits

These are ways of building new objects in a category C from diagrams in C.

\[ \begin{array}{ccc}
   & X & \\
   \downarrow & f & \downarrow \gamma \\
   & Y & \\
\end{array} \]

An example of a limit is:

**Def.** Given objects \( X, Y \in C \), a product of them is an object \( Z \) equipped with morphisms \( \pi_1, \pi_2 : Z \to X, Y \) called projections to \( X, Y \),

\[ \begin{array}{ccc}
   & Z & \\
   \downarrow \pi_1 & \downarrow \pi_2 \\
   X & \downarrow & Y \\
\end{array} \]

s.t. for any candidate \( Q \)

\[ \begin{array}{ccc}
   Q & \downarrow \beta & Z \\
   \downarrow f & \downarrow \beta_1 & \downarrow \pi_1 \\
   X & \downarrow & Y \\
\end{array} \]

there \( \exists ! \gamma : Q \to Z \) s.t.

\[ \begin{array}{ccc}
   Z & \downarrow \gamma & Q \\
   \downarrow f & \downarrow \beta_2 & \downarrow \pi_2 \\
   X & \downarrow & Y \\
\end{array} \]

this diagram commutes

\[ \begin{array}{ccc}
   Z & \downarrow \gamma & Q \\
   \downarrow f & \downarrow \beta_2 & \downarrow \pi_2 \\
   X & \downarrow & Y \\
\end{array} \]

\[ f = \pi_2 \circ \gamma \]

\[ g = \pi_1 \circ \gamma \]

The definition of coproduct is just the same but with all arrows reversed.

**Prop.** In Set, we get a product of \( X, Y \) by taking

\[ X \times Y = \{ (x, y) : x \in X, y \in Y \} \] with \( p(x, y) = x \) and \( q(x, y) = y \).

**Pf.**

\[ \begin{array}{ccc}
   & Q & \\
   \downarrow f & \downarrow \\
   X & \downarrow & Y \\
\end{array} \]

Let \( \quad : Q \to X \times Y \) be \( \quad(\gamma) = (f(\gamma), g(\gamma)) \)

We indeed get \( p \circ \gamma = f \), \( q \circ \gamma = g \) & \( \gamma \) is the unique map obeying these equations.
But we could also take as our product any set $S$ that's isomorphic to $X \times Y$, via some iso $\alpha: S \to X \times Y$

\[
\begin{array}{c}
X \times Y \xrightarrow{\alpha} S \\
\downarrow p_1 \quad \downarrow q_0 \quad \downarrow p_2 \\
X \quad Y
\end{array}
\]

Use $p \circ \alpha$ & $q \circ \alpha$ as projections; then you can check

\[
\begin{array}{c}
p \circ \alpha \quad q \circ \alpha \\
\downarrow \quad \downarrow \\
X \quad Y
\end{array}
\]

is also a product of $X$ & $Y$.

So "any object isomorphic to a product can also be a product."

Prop

Suppose

\[
\begin{array}{c}
W \\
\downarrow p \\
X
\end{array}
\quad \text{and} \quad
\begin{array}{c}
Z \\
\downarrow p' \\
Y
\end{array}
\]

are both a product of $X \times Y$.

Then $W \times Z$ are isomorphic

"Products are unique up to isomorphism."

Pf:

Since $W$ is the product

\[
\begin{array}{c}
W \\
\downarrow q \\
Y
\end{array}
\]

(1) $q \circ p = 1_Y$ \quad $Y: Z \to W$ making this commute

Since $Z$ is the product

\[
\begin{array}{c}
Z \\
\downarrow q' \\
Y
\end{array}
\]

(2) $q' \circ p' = 1_Y$ \quad $Y: W \to Z$ s.t. the diagram commutes

Suffices to show $Y$ & $Y'$ are inverse. Why is $\Psi = \Psi: W \to W$ the identity?

If we can show this, the same argument will show $\Psi \circ \Psi = 1_Z$.

\[
\begin{array}{c}
W \\
\downarrow q \\
Y
\end{array}
\]

There is a unique arrow making this commute since $W$ is the product.

It: $W \to W$ does the job, but also $W$ does the job.
This commutes since \(* + + *\) do

So this commutes:

\[
\begin{array}{c}
W \\ \downarrow p \\
\end{array} \quad \begin{array}{c}
\psi \\
\downarrow q \\
W \\ \downarrow p' \\
\end{array} \quad \begin{array}{c}
W \\
\downarrow q' \\
X \\
\end{array}
\]

So \(\psi, \psi = 1_w\) by uniqueness.
Proposition: If a morphism is an iso, it's both a mono & an epi.

(We've seen the converse is false)

Proof:

If \( f : X \to Y \) has a left inverse \( f^{-1} \), it's a mono:
\[
\forall g, h, \quad f \circ g = f \circ h \Rightarrow g = h
\]

Similarly, if \( f \) has a right inverse \( f^{-1} \), it's an epi:
\[
\forall g, h, \quad g \circ f = h \circ f \Rightarrow g = h
\]

Definition: A morphism with a left inverse is called a split monomorphism.
A morphism with a right inverse is called a split epimorphism.

In Set every mono (or epi) splits, but we saw not in Ring (or Top).

Coproducts

Definition: Given objects \( X \) & \( Y \), a coproduct of \( X \) & \( Y \) is an object \( Z \) equipped with morphisms \( i : X \to Z \) & \( j : Y \to Z \) (where \( i \) & \( j \) are called inclusions).

Which is universal: for any diagram
\[
\begin{array}{ccc}
X & \\ \downarrow \ & \SEarrow \ & \downarrow \Psi \\
Y & \\ \Psi \circ i & \rightarrow & \Psi : Z \rightarrow Q
\end{array}
\]

Making the following diagram commute:
\[
\begin{array}{ccc}
X & \rightarrow & Q \\
\downarrow \ & \ & \downarrow \Psi \\
Y & \rightarrow & Q
\end{array}
\]

Proposition: In Set, a coproduct of \( X \) & \( Y \) is their disjoint union
\[
X + Y = X \times \{0\} \cup Y \times \{1\}
\] with \( i : X \rightarrow X + Y \) & \( j : Y \rightarrow X + Y \)
\[
i : x \rightarrow (x, 0) \quad j : y \rightarrow (y, 1)
\]
PRODUCTS $\times$

Set  

coproduct product $S \times T$

COPRODUCTS $+$

disjoint union $S \sqcup T = S + T$

Top  

coproduct product $X \times Y$ with product topology

disjoint union $X \sqcup Y = X + Y$

Grp  

product of groups $G \times H$

free product $G \ast H$

$\text{Ab Grp}$  

$A \ast B = A \times B$ product of abelian groups $A \ast B$

$\text{Vect}$  

$V \oplus W = V \times W$ direct sum of vector space $V \oplus W$

The free product $G \ast H$ consists of equivalence classes of words $X_1 \cdot X_2 \cdots X_n$ where $X_i \in G \cup H$

where $X_1 \cdot X_2 \cdot \cdots \cdot X_{i-1} \cdot x_i \cdot x_{i+1} \cdots X_n \sim X_1 \cdot X_2 \cdots X_n$ where $\sim$ is the identity in $G \ast H$

and $x_i \cdot x_1 \cdot x_{i+1} \cdots x_n = x_1 \cdot x_2 \cdot y_1 \cdot y_2 \cdots y_n$ if $x_i, x_{i+1} \in G$ or $x_i, y_{i+1} \in H$

and $y = x_{i+1} x_i$

General limits & colimits

Given any diagram $U \rightarrowtail V \twoheadrightarrow X$

in a category $C$:

a cone over the diagram is:

A limit of the diagram is a cone that's universal:

i.e., given any competitor (another candidate), another cone over the same diagram:

$\exists \ Y : Q \rightarrowtail Z$ s.t. all triangles including $Y$ commute:

if $U$ is any object in the diagram and $p : Z \rightarrow U$ is the morphism in the universal cone, and $f : Q \rightarrow U$ is the morphism in the competitor then $f = p \circ Y$. 

\[ U \rightarrow V \rightarrow X \]
A category is like a cone, but with morphisms to Z instead of from:

A **colimit** is the universal cone:

\[ f = \gamma_0 \circ i \]

**Examples of different diagrams**

<table>
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<tr>
<th>Diagrams</th>
<th>LIMITS</th>
<th>COLIMITS</th>
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<tr>
<td><img src="binary-product.png" alt="Diagram" /></td>
<td>(binary) product</td>
<td>(binary) coproduct</td>
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<tr>
<td><img src="equalizer.png" alt="Diagram" /></td>
<td>equalizer</td>
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<td><img src="pullback.png" alt="Diagram" /></td>
<td>pullback</td>
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<td>C</td>
<td>pushout</td>
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<td><img src="terminal.png" alt="Diagram" /></td>
<td>terminal object 1</td>
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<tr>
<td><img src="binary-product.png" alt="Diagram" /></td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td><img src="binary-product.png" alt="Diagram" /></td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>
What's a limit of the empty diagram:

\[ \exists ! \gamma : Z \to \emptyset \]

It's an object \( Z \) s.t. for all objects \( Q \), \( \exists ! \gamma : Q \to Z \).
This is called a terminal object.

In Set, any 1-element set is a terminal object.
In Vectk, any 0-dim vector space is a terminal object.

Similarly, an initial object \( Z \) is one s.t. for any object \( Q \), \( \exists ! \gamma : Z \to Q \).

In Set, the empty set is an initial object.
In Vectk, any 0-dim vector space is an initial object.

In any abelian category, initial objects are terminal & vice versa.