Last time we saw that if $C$ has products, the functor $x: C^2 \to C$

is a right adjoint to the
diagonal functor $\Delta: C \to C^2$

$\Delta(c) = (c, c)$

& similarly $\otimes: C^2 \to C$, if $C$ has coproducts, is a left adjoint to $\Delta$.

(Thus $\otimes: \text{Vect}_k \to \text{Vect}_k$ is both left & right adjoint to $\Delta: \text{Vect}_k \to \text{Vect}_k^2$.)

In fact if a category has limits, these limits give a right adjoint to some functor:

"limits are right adjoints". Similarly "colimits are left adjoints".

We often think about the limit of a diagram in a category $C$. What's a

"diagram in $C$", really?

\[ \begin{array}{ccc} & c' & \\
\downarrow & & \downarrow \\
c & c'' & \\
\end{array} \]

Namely, it's a collection of objects & morphisms between them.

We can make it into a category:

\[ \begin{array}{ccc} & c' & \\
\downarrow & & \downarrow \\
c & c'' & \\
\end{array} \]

Now it's a subcategory of $C$.

We're often interested in diagrams of some shape, like pullbacks

These "shapes" can be interpreted as categories:
Let $D$ be any category, we'll take this as our "diagram shape". What is a $D$-shaped diagram in some category $C$?

It's a functor $F: D \to C$:

```
\begin{tikzpicture}
  \node (A) at (0,0) {$D$};
  \node (B) at (1,1) {$\bullet$};
  \node (C) at (2,2) {$\bullet$};
  \node (D) at (3,3) {$\bullet$};
  \node (E) at (4,4) {$\bullet$};
  \node (F) at (5,5) {$\bullet$};
  \node (G) at (6,6) {$\bullet$};
  \node (H) at (7,7) {$\bullet$};
  \node (I) at (8,8) {$\bullet$};
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (C) -- (D);
  \draw[->] (D) -- (E);
  \draw[->] (E) -- (F);
  \draw[->] (F) -- (G);
  \draw[->] (G) -- (H);
  \draw[->] (H) -- (I);
  \draw[->] (I) -- (A);
  \node at (1.5,1.5) {$F$};
\end{tikzpicture}
```

When we take the limit of this diagram, we get an object (defined up to isomorphism) $\operatorname{lim} F \in C$.

What's the process that takes us from $F: D \to C$ to $\operatorname{lim} F \in C$?

The key: there's a category $C^D$ with:

- objects being functors $F: D \to C$
- morphisms being natural transformations $D \xrightarrow{\eta} C$

These morphisms look like:

```
\begin{tikzpicture}
  \node (A) at (0,0) {$D$};
  \node (B) at (1,1) {$\bullet$};
  \node (C) at (2,2) {$\bullet$};
  \node (D) at (3,3) {$\bullet$};
  \node (E) at (4,4) {$\bullet$};
  \node (F) at (5,5) {$\bullet$};
  \node (G) at (6,6) {$\bullet$};
  \node (H) at (7,7) {$\bullet$};
  \node (I) at (8,8) {$\bullet$};
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (C) -- (D);
  \draw[->] (D) -- (E);
  \draw[->] (E) -- (F);
  \draw[->] (F) -- (G);
  \draw[->] (G) -- (H);
  \draw[->] (H) -- (I);
  \draw[->] (I) -- (A);
  \node at (1.5,1.5) {$G$};
  \node at (3.5,3.5) {$H$};
  \node at (5.5,5.5) {$I$};
\end{tikzpicture}
```

Where all the squares commute.

When we take a limit of $F: C \to D$, we study cones over $F$:

```
\begin{tikzpicture}
  \node (A) at (0,0) {$\bullet$};
  \node (B) at (1,1) {$\bullet$};
  \node (C) at (2,2) {$\bullet$};
  \node (D) at (3,3) {$\bullet$};
  \node (E) at (4,4) {$\bullet$};
  \node (F) at (5,5) {$\bullet$};
  \node (G) at (6,6) {$\bullet$};
  \node (H) at (7,7) {$\bullet$};
  \node (I) at (8,8) {$\bullet$};
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (C) -- (D);
  \draw[->] (D) -- (E);
  \draw[->] (E) -- (F);
  \draw[->] (F) -- (G);
  \draw[->] (G) -- (H);
  \draw[->] (H) -- (I);
  \draw[->] (I) -- (A);
  \node at (1.5,1.5) {$\alpha$};
\end{tikzpicture}
```

A cone over $F$ is a natural transformation $\alpha: G \Rightarrow F$ where $G$ sends every object of $D$ to some object of $C$ & $G$ sends every morphism of $D$ to the identity morphism of that object.
So this recipe should be a functor $\Delta_b : C \to C^0$.

$\Delta_b(a)$ is the diagram

$\Delta_b(a)$ is the diagram

$\Delta_b(a)$ is the diagram

Here $b = \Delta_b(a)$.

So: a cone over $F$ with apex $q \in C$ is a natural transformation $\alpha : \Delta_b(a) \to F$.

What's the limit of a diagram? If $F \in C^0$

It's a universal cone over that diagram.

Remember $U$ is the right adjoint of $F$ if:

$\text{hom}(Fx, y) \cong \text{hom}(x, Uy)$

So adjoint functors are about converting one kind of morphism into another in a bijective way, & that's what we're doing when we're stating the universal property:

- morphisms $Y : q \to \text{lim } F$ in $C$
- cones over $F$ with apex $q$, i.e. natural transformations $\alpha : \Delta_b(a) \to F$ morphisms $\alpha$ from $\Delta_b(a)$ to $F$ in $C^0$.

So: $\text{hom}(\Delta_b(a), F) \cong \text{hom}(q, \text{lim } F)$
So it looks like we have
\[ \lim: C^D \to C \]
which is right adjoint to
\[ \Delta_D: C \to C^D \]

This is true - you need to check that
\[ \text{hom}(\Delta_D(q), F) \cong \text{hom}(q, \lim F) \]
is a natural bijection to finish the proof of:

**Thm.** If \( C \) has all limits for \( D \)-shaped diagrams, then we have a functor
\[ \lim: C^D \to C \]
\[ F \mapsto \lim F \]
which is right adjoint to \( \Delta_D: C \to C^D \).

The converse is true too: if \( \Delta_D: C \to C^D \) has a right adjoint, then this gives limits of \( D \)-shaped diagrams in \( C \).

What choice of \( D \) gives the case of binary products (a special case of limits)?

Here \( D \) has 2 objects & only identity morphisms, so we could call it \( 2 \),
so \( C^D = C^2 \) & \( x: C^2 \to C \) is right adjoint to \( \Delta_2 = \Delta: C \to C^2 \).

Similarly,

**Thm.** If a category \( C \) has colimits of all \( D \)-shaped diagrams, there's a functor
\[ \text{colim}: C^D \to C \]
left adjoint to \( \Delta_D: C \to C^D \) & conversely.
So \( \text{hom}((\text{colim} F), q) \cong \text{hom}(F, \Delta_D q) \).
Note

$\alpha \in \text{hom}(F, D_{\mathfrak{q}})$ is a cone.
Theorem: Left adjoints preserve colimits; right adjoints preserve limits.

Pf: (sketch)

Let's show that if \( F: C \to D \) is a left adjoint to \( U: D \to C \), then \( F \) preserves colimits.

For concreteness, let's show \( F \) preserves pushouts — general case is analogous.

So suppose we have a pushout in \( C \):

\[
\begin{array}{ccc}
\bullet & \xleftarrow{f} & \bullet \\
\downarrow{g} & & \downarrow{h} \\
\bullet & \xrightarrow{i} & \bullet
\end{array}
\]

Here \( x \) is the apex of a cocone on the diagram we're taking a colimit of, & the universal property holds.

The claim is that applying \( F \) to this universal cocone gives a universal cocone in \( D \):

\[
\begin{array}{ccc}
\bullet & \xleftarrow{F(g)} & \bullet \\
\downarrow{F(f)} & & \downarrow{F(h)} \\
\bullet & \xrightarrow{F(i)} & \bullet
\end{array}
\]

Choose a competitor cocone with apex \( Q \). Need to show \( \exists! \Psi: F(x) \to Q \) making the newly formed triangle commute.

We can look at \( U(Q) \in C \)

Note \( \text{hom}(F(x), Q) \cong \text{hom}(x, U(Q)) \)

So to get \( \Psi: F(x) \to Q \), let's find \( \Psi: x \to U(Q) \).

\( U(Q) \) becomes a competitor due to the adjointness of \( F \) & \( U \), e.g. \( \text{hom}(F(x), Q) \cong \text{hom}(x, U(Q)) \)

For some reason, the triangles involving \( U(Q) \) commute since those involving \( Q \) commute.

So \( U(Q) \) is a competitor.

Thus \( \exists! \Psi: x \to U(Q) \) making the newly formed triangles commute.

This gives us \( \Psi: F(x) \to Q \), check it makes its newly formed triangle commute & is unique (since \( \Psi \) is).
Ex: $F: \text{Set} \to \text{Grp}$ preserves colimits, e.g. coproducts, so

$$F(S + T) \cong F(S) + F(T)$$

Here, $S + T$ is the disjoint union of $S$ and $T$. $F(S + T)$ is the free group with elements of $S + T$ as generators, and $F(S) + F(T) = F(S) * F(T)$ is the "free product" of $F(S)$ and $F(T)$.

Ex: $U: \text{Grp} \to \text{Set}$ preserves limits, e.g. products:

$$U(a \times H) \cong U(a) \times U(H)$$

where $a \times H$ is the usual product of groups $a$ and $H$.

Thin: The composite of left adjoints is a left adjoint. The composite of right adjoints is a right adjoint.

Pf: Suppose we have functors $C \xrightarrow{F} D \xrightarrow{F'} E$ and $F$ and $F'$ are left adjoints of functors $U, U'$. Then $\text{Colim}_C F \cong \text{Lim}_D F'$. We'll show that $F' \circ F: C \to E$ is the left adjoint of $U \circ U': E \to C$.

Want a natural isomorphism $\text{hom}(F' \circ F(c), e) \cong \text{hom}(c, U' \circ U(e))$.

Here's how we get it:

$$\text{hom}(F' \circ F(c), e) \cong \text{hom}(F(c), U'(e))$$

since $F'$ is left adjoint to $U'$

$$\cong \text{hom}(c, U \circ U'(e))$$

since $F$ is left adjoint to $U$.

Ex: $\text{Ring} \xrightarrow{F} \text{Ab}$

$F' \circ F$ is left adjoint to the forgetful functor $U \circ U'$ from $\text{Ring}$ to $\text{Set}$.

Starting from $\emptyset$ (the initial set) we get $F(\emptyset) = \{0\}$ (the trivial abelian group, which is the initial abelian group) and then $F'(F(\emptyset)) = \mathbb{Z}$ (the ring of integers, which is the initial ring).

Starting from a one-element set $\{x\}$, we get

$$F(\{x\}) = \{\ldots, -x, 0, x, x + x, \ldots\} \cong \mathbb{Z}$$

and then $F'(F(\{x\})) = \mathbb{Z}[x]$ the ring of polynomials in $x$ with integer coefficients.
Units and counits of adjunctions (= pair of adjoint functors)
Suppose we have \( F(D) \rightleftharpoons C \) with \( F \) left adjoint to \( U \).

\[ \text{hom}(F(c), d) \cong \text{hom}(c, Ud) \quad \forall \ c \in C, \forall d \in D \]

We can apply this bijection to an identity morphism & get something interesting. We can do this if \( d = Fc \).

\[ \psi \]

\[ \psi(Fc) \]

\( \psi(Fc) \) is called the unit, \( \eta_c : c \to UFc \)

We can also apply \( \psi^{-1} \) to an identity if \( d = Ud \).

\[ \text{hom}(F Ud, d) \overset{\psi^{-1}}{\sim} \text{hom}(Ud, Ud) \]

\[ \psi^{-1}(Ud) \]

\( \psi^{-1}(Ud) \) is called the counit, \( \varepsilon_d : \varepsilon_d : F Ud \to d \)

These give various famous morphisms.

**Ex:**

\( F : \text{Set} \to \text{Grp} \)

\( U : \text{Grp} \to \text{Set} \)

Given any set \( S \), we get a unit: \( \varepsilon_S : S \to UF S \)

This is the "inclusion of the generators": elements of \( S \) are generators of \( FS \).

*Given a group \( G \), get: \( \varepsilon_a : FUG \to G \)

\[ \begin{align*}
    g_1 \cdot g_2 \cdot \ldots \cdot g_n &\mapsto g_1 g_2 \ldots g_n \in G \\
    \text{"formal product"} &\quad \text{"actual product"} \\
    \end{align*} \]

The counits "convert" formal expressions into actual ones."