GAUGE THEORY ON A GRAPH

For us, a graph $\mathcal{G}$ is a finite set $E$ of edges, a finite set $V$ of vertices, and maps $s, t: E \rightarrow V$.

\begin{align*}
  & s(e) \quad e \quad t(e) \\
\end{align*}

Fixing a group $G$, a connection on $\mathcal{G}$ is a map

\[ A: E \rightarrow G \]

assigning to each edge $e$ a holonomy $A_e \in G$.

A gauge transformation on $\mathcal{G}$ is a map

\[ g: V \rightarrow G \]

Gauge transformations act on connections via:

\[ (gA)_e = g_{t(e)} A_{s(e)} g_{s(e)}^{-1} \]
Let
\[ \mathcal{A} = \{ \text{connections on } \mathcal{Y} \} = G^E \]
\[ \Omega^*_G = \{ \text{gauge transformations on } \mathcal{Y} \} = G^V \]

If \( G \) has a left- and right-invariant measure (Haar measure), \( \mathcal{A} \) gets a \( \Omega^*_G \)-invariant measure, so \( \Omega^*_G \) acts as unitary operators on \( L^2(\mathcal{A}) \).

Via
\[ \mathcal{A} \]
\[ \downarrow \Omega^*_G \]

the measure on \( \mathcal{A} \) pushes forward to a measure on \( \mathcal{A}/\Omega^*_G \), and:
\[ L^2(\mathcal{A}/\Omega^*_G) \cong \{ \psi \in L^2(\mathcal{A}) : g\psi = \psi \} \]
If \( G \) is compact, it has a unique Borel measure \( \mu \) that is left- and right-invariant and has \( \int_G \mu = 1 \) — normalized Haar measure. Then we can describe \( L^2(\mathbb{A}) \) using the Peter-Weyl theorem:

\[
L^2(G) \cong \bigoplus_{\rho \in \text{Irrep}(G)} \rho \otimes \rho^*
\]

so

\[
L^2(\mathbb{A}) \cong L^2(G^E)
\]

\[
\cong \bigotimes_{e \in E} L^2(G)
\]

\[
\cong \bigotimes_{e \in E} \bigoplus_{\rho \in \text{Irrep}(G)} \rho \otimes \rho^*
\]

\[
\cong \bigoplus_{\rho : E \to \text{Irrep}(G)} \bigotimes_{e \in E} \rho_e \otimes \rho_e^*
\]

\[
\cong \bigoplus_{\rho : E \to \text{Irrep}(G)} \bigotimes_{e \in V} \rho_{e:t(e)} \otimes \rho_{e:s(e)}
\]
$G \times G$ acts on $L^2(G^-)$ by left/right translations; Peter-Weyl says how:

$$L^2(G^-) \cong \bigoplus_{\rho \in \text{irrep}(G)} \rho \otimes \rho^*$$

This says how $G^- \times G$ acts on $L^2(G^-)$:

$$L^2(G^-) \cong \bigoplus_{\rho : E \rightarrow \text{irrep}(G)} \bigotimes_{e \in V} \left[ \bigotimes_{e : t(e) = v} \rho_e \otimes \bigotimes_{e : s(e) = v} \rho_e^* \right]$$

If $g \in G^- \times G$, $g_v$ acts on this factor in the obvious way! Thus:

$$L^2(G^-)^{G^-} \cong \bigoplus_{\rho : E \rightarrow \text{irrep}(G)} \bigotimes_{e \in V} \text{Inv} \left[ \bigotimes_{e : t(e) = v} \rho_e \otimes \bigotimes_{e : s(e) = v} \rho_e^* \right]$$

$$\cong \bigoplus_{\rho : E \rightarrow \text{irrep}(G)} \bigotimes_{e \in V} \text{Hom} \left( \bigotimes_{e : s(e) = v} \rho_e, \bigotimes_{e : t(e) = v} \rho_e \right)$$

intertwiners
SPIN NETWORKS

Theorem — If $\mathcal{V}$ is a graph and $G$ is a compact group, an orthonormal basis for $L^2(\mathcal{V}/G)$ is given by all ways of labelling edges of $\mathcal{V}$ by irreps $\rho_e \in \text{Irrep}(G)$ and vertices by intertwiners $\mathcal{2}_v : \bigotimes_{e} \rho_e \rightarrow \bigotimes_{e} \rho_e$

running over any orthonormal basis of such intertwiners.

$\mathcal{V} = (\mathcal{V}, \rho, \mathcal{2})$ is called a spin network.

Proof —

$L^2(\mathcal{V}/G) = \bigoplus_{\rho : E \rightarrow \text{Irrep}(G)} \bigotimes_{V \in V} \text{Hom}(\bigotimes_{e} \rho_e, \bigotimes_{e} \rho_e)_{\rho(t(e) = v)}$
EXAMPLES:

\[ G = U(1) \quad - \quad \text{electromagnetism!} \]

Irrep \((G) \cong \mathbb{Z} \quad - \quad \text{"charges"} \]

All irreps are 1-dimensional: \[ \rho_n(e^{i\theta}) = e^{i \theta}, \; n \in \mathbb{Z}. \]

\[ \rho_n \otimes \rho_m = \rho_{n+m} \]

\[ \rho_n^* \cong \rho_{-n} \]

Space of intertwiners:

\[ \text{is 1-dimensional if } n_1 + \cdots + n_k = m_1 + \cdots + m_\ell, \]

0-dimensional otherwise.

\( L^2(G/0) \) has basis of "flux networks":

Faraday's electric field lines! \[ \nabla \cdot \vec{E} = 0! \]
\( G = SU(2) \) — weak force, gravity!

Irrep \( \rho(G) \approx \frac{N}{2} \) — "spins" \( j = 0, \frac{1}{2}, 1, \ldots \)

\[ \rho_j \otimes \rho_k \cong \rho_{|j-k|} \oplus \cdots \oplus \rho_{j+k} \]

\[ \rho_j^* \cong \rho_j \]

Space of intertwiners:

\[ j \quad \downarrow \quad \lambda \quad \downarrow \quad k \]

is 1d if \( j + k + \lambda \in \mathbb{N} \) & \( |j-k| \leq \lambda \leq j+k \),

0d otherwise.

Basis of intertwiners given by "virtual trivalent trees" with edges labelled by spins satisfying above constraints.

\( L^2(\Omega G) \) has basis of spin networks:
Given any graph $\gamma$ with edges real-analytically embedded in $M$, let

$$\mathcal{D}_\gamma = \{ \text{connections on } \gamma \}$$

$$\mathcal{G}_\gamma = \{ \text{gauge transformations on } \gamma \}$$

defined as before!

$$\mathcal{M}_\gamma = \mathcal{G}_\gamma \text{-invariant measure on } \mathcal{D}_\gamma$$

with $$\int_{\mathcal{D}_\gamma} \mathcal{M}_\gamma = 1$$

If we arbitrarily trivialize $P$ at each point of $M$, we get onto maps:

$$\begin{array}{ccc}
\mathcal{D}_\gamma & \xrightarrow{\gamma} & \mathcal{G}_\gamma \\
\downarrow & & \downarrow \\
\mathcal{D}_\gamma & & \mathcal{G}_\gamma
\end{array}$$

Idea: form $\overline{\mathcal{G}}_\gamma$, $\overline{\mathcal{G}}_\gamma$ as an inverse limit of these $\mathcal{D}_\gamma$, $\mathcal{G}_\gamma$. 

G-AUGE THEORY ON A
REAL-ANALYTIC MANIFOLD

Let

\( M \) be a real-analytic manifold
\( G \) a compact connected Lie group
\( P \rightarrow M \) a smooth principal \( G \)-bundle
\( \mathcal{A} = \{ \text{smooth connections on } P \} \)
\( \mathcal{G} = \{ \text{smooth gauge transformations of } P \} \)

We want to define \( L^2(\mathcal{A}/\mathcal{G}) \) but there's no good measure. So: define \( L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}}) \)
where

\( \mathcal{A} \xrightarrow{\text{dense}} \overline{\mathcal{A}} \)
\( \mathcal{G} \xrightarrow{\text{dense}} \overline{\mathcal{G}} \)

and \( \overline{\mathcal{A}} \) has a \( \overline{\mathcal{G}} \)-invariant measure.

HOW?
Graphs real-analytically embedded in $M$ form a category:

\[ \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\boxed{\text{f}} & & \\
\bullet & \rightarrow & \bullet
\end{array} \]

where morphisms are:

- adding new edges & vertices
- subdividing edges with new vertices
- reversing orientation of edges

Any morphism $f : \gamma \rightarrow \gamma'$ induces:

\[ f^* : \mathcal{A}_\gamma \rightarrow \mathcal{A}_{\gamma'} \]

\[ \text{measure-preserving continuous onto map:} \]

\[ f^* \mathcal{M}_{\gamma'} = \mathcal{M}_\gamma \]

\[ f^* : \sigma_{\gamma'} \rightarrow \sigma_\gamma \]

\[ \text{onto homomorphism of Lie groups} \]

Let

\[ \overline{a} = \lim_{\gamma} \mathcal{A}_\gamma \]

\[ \overline{\sigma} = \lim_{\gamma} \sigma_\gamma \]

& also for $\sigma'$
Theorem - $\overline{A}$ is a compact Hausdorff space; $\overline{G}$ is a compact Hausdorff group acting continuously on $\overline{A}$. We have inclusions:

$$A \subset_{\text{dense}} \overline{A}$$

$$A \subset_{\text{dense}} \overline{G}$$

$\overline{A}$ has a $\overline{G}$-invariant Borel measure $\mu$ with $\int_{\overline{A}} \mu = 1$. $A \subset \overline{A}$ is contained in a set of measure zero.

Key lemma: given $\gamma, \gamma'$ there exists $\gamma''$ with

$$f \mapsto f'$$

$\gamma \Rightarrow \gamma''$

Not true in smooth category: 

Nonetheless this theorem is true in smooth category.
Corollary: \( L^2(\bar{G} / \bar{G}) \) has an orthonormal basis given by spin networks \( \psi = (\chi, \rho, z) \), where:

- \( \chi \) ranges over graphs analytically embedded in \( M \), without lone vertices:
  - unnecessary vertices:
  - or redundancies:
    - pick one
    - pick one

- \( \rho \) ranges over labellings of edges by nontrivial irreps of \( G \)

- \( z \) ranges over labellings of vertices by intertwiners chosen from an orthonormal basis.