SPIN FOAM MODELS

We've seen a bit of how spin networks can describe the quantum geometry of space:

How can we understand the quantum geometry of spacetime? The problem of time makes this tough in the canonical (Hamiltonian) approach. It's natural to try the path-integral (Lagrangian) approach.

This leads to spin foams.
It's easiest to define "closed" spin networks & spin foams. Given a group $G$,

A **closed spin network** $\Psi = (\gamma, \rho, \mathcal{Z})$ is a graph $\gamma$ with:

1) edges $e$ labelled by unitary irreps $\rho_e$ of $G$.

2) vertices $v$ labelled by intertwiners

$$\mathcal{Z}_v : \rho_{e_1} \otimes \cdots \otimes \rho_{e_n} \rightarrow \rho_{e_1'} \otimes \cdots \otimes \rho_{e_m'}$$

A **closed spin foam** $F = (K, \rho, \mathcal{Z})$ is an oriented 2d PL-CW complex $K$ with:

1) faces $F$ labelled by unitary irreps $\rho_f$ of $G$.

2) edges $e$ labelled by intertwiners

$$\mathcal{Z}_e : \rho_{f_1} \otimes \cdots \otimes \rho_{f_n} \rightarrow \rho_{f_1'} \otimes \cdots \otimes \rho_{f_m'}$$
A spin foam should be something going from one spin network to another.

Or: a generic "slice" of a spin foam should be a spin network:

Spin foam faces → spin network edges
Spin foam edges → spin network vertices

Spin foam vertices correspond to "interactions" where a spin network changes topology.
We can also define "open" spin networks. These are the morphisms in a category:

\[ \psi \xrightarrow{x} \phi \xrightarrow{y} \]

When \( G \) is the Poincaré group or a compact Lie group, these are just Feynman diagrams!

Similarly, "open spin foams" are the 2-morphisms in a certain 2-category:

\[ \psi \xrightarrow{F} \psi' \]
A spin foam model assigns an amplitude $Z(F) \in \mathbb{C}$ to each spin foam $F$, computed as a product of:

- vertex amplitudes \[ \text{"interactions"}\]
- edge amplitudes \[ \text{"propagators"}\]
- face amplitudes

(A categorified analogue of Feynman diagrams.)

One can study spin foams in a real-analytic manifold, or not embedded, but let's consider them in a triangulated manifold, living in the 2-skeleton of the dual complex:
Given a triangulated \((n-1)\)-manifold \(S\) representing "space", define the kinematical Hilbert space \(Z(S)\) by
\[
Z(S) = L^2(\Omega_g/\Gamma_g)
\]
where the graph \(\Gamma\) is the dual 1-skeleton:

Given a triangulated cobordism \(M: S \to S'\) representing "spacetime", let
\[
Z(M): Z(S) \to Z(S')
\]
be given by
\[
\langle \psi', Z(M)\psi \rangle = \sum_{F: \psi \to \psi'} \chi(F)
\]
where \(F\) is in the dual 2-skeleton of \(M\). If convergent & triangulation-independent,
we can often promote \(Z\) to a TQFT.
Ponzano-Regge Model

~1968

This is a theory of Riemannian quantum gravity in 3d spacetime, with $G = SU(2)$.

- **Vertex amplitude** is "tet net" or "$6j$ symbols"
- **Edge amplitude** is reciprocal of "theta net"
- **Face amplitude** is "loop"

Problem: sum over spin foams diverges, corresponding to infrared divergence in path integral.
Turayev-Viro Model
~1992

The divergences in the Ponzano-Regge model were cured by replacing SU(2) by the corresponding quantum group, with $q$ a suitable root of unity. Now there are finitely many “spins” to sum over & we get a TQFT.

This corresponds to introducing a cosmological constant $\Lambda$ in Riemannian general relativity: $q = e^{\frac{2\pi i}{k+2}}$, $k = \frac{4\pi}{\sqrt{\Lambda}}$.

Positive curvature eliminates infrared divergences!

Same idea works for other quantum groups.
OOGURI MODEL
\sim 1992

\[ G = SU(2) \] but now in 4d spacetime:

\[ S = \int_M \text{tr} (E \wedge F) \]

Corresponds to "topological gravity" with curvature of SU(2) connection Ad(P)-valued 2-form

Again suffers from divergences.
Replacing $SU(2)$ by the corresponding quantum group in the Ooguri model, the sum over spins becomes finite & we get a 4d TQFT.

We believe this is the quantization of the field theory with

$$S = \int_M \text{tr} \left( E^a F_a + \frac{\lambda}{12} E^a E^a \right),$$

but this needs more work!

We get similar TQFTs from other quantum groups.
RIEMANNIAN BARRETT–CRANE MODEL
~1997

4d Riemannian general relativity can be expressed as a theory with

\[ G = \tilde{SO}(4) \cong SU(2) \times SU(2) \]

and

\[ S = \int_M \text{tr} (E \wedge F) \]

but with extra constraints saying

\[ E = e \wedge e \]

\[ e : TM \rightarrow \Omega^1 \cong TM \]

with metric, so

\[ e ne \in \Omega^2(M, \Lambda^2 \Omega) \]

Imposing a version of this in corresponding Ooguri-like spin foam model, we get a convergent theory w/o needing quantum group technology!

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LORENTZIAN BARRETT-CRANE MODEL
1999

A similar model works in the Lorentzian case, where

$$G = \widetilde{SO}(3,1) \cong SL(2,\mathbb{C})$$

Here the relevant unitary irreps are infinite-dimensional so the vertex amplitude is not obviously convergent... but it does converge (gr-qc/0101107) ... and even better, the integral over labellings converges (gr-qc/0104057)! Stay tuned....